A postulate-free treatment of Lorentz boosts in Minkowski space

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Abstract

Fundamental results of special relativity, such as the linear transformation for Lorentz boosts, and the invariance of the spacetime interval, are derived from a system of differential equations. The method so used dispenses with the need to make any physical assumption about the nature of spacetime.

1 Introduction

In his original work, A. Einstein [1] derived the Lorentz transformation based on the postulates of the principle of relativity and the invariance of the speed of light c. The transformation in $(1+1)$-dimensional Minkowski space (in units where $c = 1$) is given by

$$\bar{t} = t\cosh\phi - x\sinh\phi, \tag{1a}$$

$$\bar{x} = x\cosh\phi - t\sinh\phi, \tag{1b}$$

where ϕ is the rapidity. A central assumption in special relativity is the invariance of the spacetime interval under any transformation between inertial frames. This is instated (for the interval from the origin) by the relation

$$\bar{t}^2 - \bar{x}^2 = t^2 - x^2. \tag{2}$$

In $(3+1)$-dimensional Minkowski space, the transformation for a boost along the direction specified by the unit vector (n_1, n_2, n_3) is

$$\begin{bmatrix}\bar{t}\\ \bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix} = \begin{bmatrix}\cosh\phi & -n_1\sinh\phi & -n_2\sinh\phi & -n_3\sinh\phi\\ -n_1\sinh\phi & 1+n_1^2(\cosh\phi-1) & n_1n_2(\cosh\phi-1) & n_1n_3(\cosh\phi-1)\\ -n_2\sinh\phi & n_1n_2(\cosh\phi-1) & 1+n_2^2(\cosh\phi-1) & n_2n_3(\cosh\phi-1)\\ -n_3\sinh\phi & n_1n_3(\cosh\phi-1) & n_2n_3(\cosh\phi-1) & 1+n_3^2(\cosh\phi-1)\end{bmatrix}\begin{bmatrix}t\\ x\\ y\\ z\end{bmatrix}, \tag{3}$$

and the preservation of the spacetime interval is expressed by stating that

$$\bar{t}^2 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2 = t^2 - x^2 - y^2 - z^2. \tag{4}$$

In more recent approaches [2, 3, 4], only Einstein's first postulate is used, and in addition, spacetime is assumed to be homogeneous and

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isotropic in nature. In this work, these assumptions are relaxed, and the four spacetime coordinates t, x, y and z are taken to be functions of the rapidity ϕ. In this process, the physical assumptions regarding spacetime are replaced by mathematical conditions. An event in an inertial frame (in $(3+1)$-dimensional Minkowski space) is then given by $(t(\phi), x(\phi), y(\phi), z(\phi))$. The choice of ϕ for the initial frame is arbitrary (in this work, it is chosen to be 0), since there exists no preferred frame. Since distinct values of ϕ result in boosted frames with distinct rapidities, ϕ may be said to "label" an inertial frame.

2 Boosts in $(1+1)$-dimensional Minkowski space

Consider the system of equations (with primes denoting differentiation with respect to ϕ)

$$t' = -x, \tag{5a}$$

$$x' = -t. \tag{5b}$$

Successive differentiation yields

$$t'' = t, \tag{6a}$$

$$x'' = x. \tag{6b}$$

The solutions for t and x are

$$t(\phi) = t(0)\cosh\phi + t'(0)\sinh\phi, \tag{7a}$$

$$x(\phi) = x(0)\cosh\phi + x'(0)\sinh\phi. \tag{7b}$$

Noting that $t'(0) = -x(0)$ and $x'(0) = -t(0)$, we get

$$t(\phi) = t(0)\cosh\phi - x(0)\sinh\phi, \tag{8a}$$

$$x(\phi) = x(0)\cosh\phi - t(0)\sinh\phi. \tag{8b}$$

Equations (8) are strikingly similar to (1), and describe a boost of rapidity ϕ, relative to the frame $\phi = 0$.

We also obtain from (5)

$$tt' - xx' = 0, \tag{9}$$

which on integrating gives

$$t(\phi)^2 - x(\phi)^2 = \text{constant}. \tag{10}$$

This proves the invariance of the (squared) spacetime interval under boosts.

Setting $x(\phi) = 0$, we get from (10)

$$t(\phi)^2 = t(0)^2 - x(0)^2. \tag{11}$$

Defining $\beta(\phi) = \tanh\phi$, we have from (8b)

$$x(0) = \beta(\phi)t(0). \tag{12}$$

$\beta(\phi)$ is thus identified as the velocity of the frame ϕ with respect to the frame $\phi = 0$. Substituting (12) in (11) and taking the positive square root, we obtain

$$t(\phi) = t(0)\sqrt{1 - \beta(\phi)^2}. \tag{13}$$

Equation (13) describes the time dilation of the frame ϕ with respect to the frame $\phi = 0$. The quantity $t(\phi)$ is identified as the proper time of the frame ϕ.

Now, on setting $x(0) = 0$, we get from (8)

$$t(\phi) = t(0)\cosh\phi, \tag{14a}$$

$$x(\phi) = -t(0)\sinh\phi, \tag{14b}$$

which results in

$$x(\phi) = -\beta(\phi)t(\phi). \tag{15}$$

Here, $-\beta(\phi)$ is the velocity of the frame $\phi = 0$ with respect to the frame ϕ.

Now, let $t(\phi) = 0$. Returning to (10), we obtain

$$-x(\phi)^2 = t(0)^2 - x(0)^2, \tag{16}$$

where from (8a),

$$t(0) = \beta(\phi)x(0). \tag{17}$$

Substituting in (16) and taking the positive square root again, we have

$$x(\phi) = x(0)\sqrt{1 - \beta(\phi)^2}. \tag{18}$$

Equation (18) describes the spatial contraction of the frame ϕ with respect to the frame $\phi = 0$. The quantity $x(0)$ is identified as the proper length of the frame $\phi = 0$.

Now, let the event $(t(0), x(0))$ be simultaneous with the origin. In that case, $t(0) = 0$ and we get from (8)

$$t(\phi) = -x(0)\sinh\phi, \tag{19a}$$

$$x(\phi) = x(0)\cosh\phi, \tag{19b}$$

from which

$$t(\phi) = -\beta(\phi)x(\phi). \tag{20}$$

Equation (20) expresses the relativity of simultaneity of the two events.

3 Boosts in (3+1)-dimensional Minkowski space

Let n_1, n_2 and n_3 be real numbers such that $n_1^2 + n_2^2 + n_3^2 = 1$. Consider now the system of equations

$$t' = -n_1 x - n_2 y - n_3 z, \tag{21a}$$

$$x' = -n_1 t, \tag{21b}$$

$$y' = -n_2 t, \tag{21c}$$

$$z' = -n_3 t. \tag{21d}$$

Successive differentiation yields

$$t'' = t, \quad x''' = x', \quad y''' = y', \quad z''' = z'. \tag{22}$$

The solution for t is

$$t(\phi) = A_1 \cosh\phi + A_2 \sinh\phi, \tag{23}$$

where

$$A_1 = t(0), \tag{24a}$$
$$A_2 = t'(0) = -n_1 x(0) - n_2 y(0) - n_3 z(0). \tag{24b}$$

The solution for x is

$$x(\phi) = B_1 \cosh\phi + B_2 \sinh\phi + B_3, \tag{25}$$

where

$$B_1 + B_3 = x(0), \tag{26a}$$
$$B_2 = x'(0) = -n_1 t(0), \tag{26b}$$
$$B_1 = x''(0) = n_1^2 x(0) + n_1 n_2 y(0) + n_1 n_3 z(0). \tag{26c}$$

The solutions for y and z may be obtained in a way similar to that of x. The solution to (21) is

$$\begin{bmatrix} t(\phi) \\ x(\phi) \\ y(\phi) \\ z(\phi) \end{bmatrix} = \begin{bmatrix} \cosh\phi & -n_1 \sinh\phi & -n_2 \sinh\phi & -n_3 \sinh\phi \\ -n_1 \sinh\phi & 1 + n_1^2(\cosh\phi - 1) & n_1 n_2(\cosh\phi - 1) & n_1 n_3(\cosh\phi - 1) \\ -n_2 \sinh\phi & n_1 n_2(\cosh\phi - 1) & 1 + n_2^2(\cosh\phi - 1) & n_2 n_3(\cosh\phi - 1) \\ -n_3 \sinh\phi & n_1 n_3(\cosh\phi - 1) & n_2 n_3(\cosh\phi - 1) & 1 + n_3^2(\cosh\phi - 1) \end{bmatrix} \begin{bmatrix} t(0) \\ x(0) \\ y(0) \\ z(0) \end{bmatrix}. \tag{27}$$

Equation (27) describes a boost of rapidity ϕ in the direction specified by the unit vector (n_1, n_2, n_3) (note the similarity with (3)). It may be checked that (21) reduces to (5) for $(n_1, n_2, n_3) = (1, 0, 0)$. Eliminating n_1, n_2 and n_3 from (21) results in

$$tt' - xx' - yy' - zz' = 0, \tag{28}$$

which on integrating gives

$$t(\phi)^2 - x(\phi)^2 - y(\phi)^2 - z(\phi)^2 = \text{constant}. \tag{29}$$

This proves the invariance of the (squared) spacetime interval under boosts.

References

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