On Uniformly-accelerated Motion in an Expanding Universe

Alireza Jamali*

Independent Researcher†

alireza.jamali.mp@gmail.com

December 22, 2023

Abstract

Null geodesics of a spacetime are a key factor in determining dynamics of particles. In this paper, it is argued that, within the scope of validity of Cosmological Principle where FLRW model can be safely employed, expansion of the Universe causes the null geodesics to accelerate, providing us with a universal acceleration scale $a_0 = cH_0$. Since acceleration of null rays of spacetime corresponds to null rays of velocity space, demanding the invariance of acceleration of light a_0 yields a new metric for the velocity space which introduces time as a dimension of the velocity space. Being part of the configuration space, modification of distance measurements in velocity space alters the Euler-Lagrange equation and from there the equation of motion, Newton's Second Law. It is then seen that the resulting modification eliminates the need for Dark matter in clusters of galaxies and yields MOND as an approximation.

Keywords— Hubble constant, acceleration of light, velocity space

Introduction

Einstein's theory of relativity is a pillar of modern physics. Not only did it predict novel phenomena, it also gave rise to modern cosmology, providing us with great tools for exploring the cosmos[1, 2, 3]. As elegant and successful as the theory is, it leaves us pondering some of the fundamental questions it aimed to answer. General Relativity (hereafter GR) grew out of the attempt to expand the domain of application of Special Relativity (hereafter SR) to accelerated motion.
Based on his principle of equivalence

$$\mathbf{a} = \mathbf{g},$$

*Corresponding author

†3rd Floor - Block No. 6 - Akbari Alley - After Dardasht Intersection - Janbazané Sharghi - Tehran - Iran

Einstein argued that investigating uniformly-accelerated motion is the same as knowing gravity. Thanks to this principle, via the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}=0,$$

he was able to identify gravitational field with a geometric entity, the connexion Γ. From here Einstein's focus was directed toward gravity and the original question of accelerated motion left largely unattended.

The task of studying uniformly-accelerated motion was taken up by Rindler[4], whose seminal work studies the hyperbolic motion that *one* uniformly-accelerated particle performs. However, if we are to expand *relativity* to uniformly-accelerated motion, the most important question is that of *relative* acceleration of *two* observers, something which Rindler's treatment lacks. This is because in SR, acceleration occupies a 'special' place, as it appears in the equation of motion and according to the principle of relativity, all inertial observers would agree about the *form* of law of motion, no matter how different the contents (phenomena) might appear. But as the law of motion

$$f^\mu = mA^\mu$$

is expressed explicitly using acceleration, this means that acceleration is still 'special' in comparison to velocity or higher order derivatives.

In SR there are no transformations for the relative motion of two uniformly-accelerated observers. Although according to SR acceleration does transform as a vector, the transformation law is a function of relative *velocity* only, not acceleration as well.

In this regard GR does not improve the situation considerably, as it reduces to SR infinitesimally:

$$dx'^\mu=\frac{\partial x'^\mu}{\partial x^\nu}dx^\nu.$$

This means that GR 'inherits' the shortcoming. This can be better seen by writing the vector transformation law as

$$B'^\rho = D^\rho_\sigma B^\sigma,$$

where

$$D^\rho_\sigma=\frac{\partial x'^\rho}{\partial x^\sigma}.$$

This has the same linear *form* as the transformation law of SR

$$B'^\rho=\Lambda^\rho_\sigma B^\sigma.$$

We thus see that GR does not *entirely* solve the problem of accelerated motion.

There is more to accelerated motion than gravity. Our whole notion of 'force' is based on acceleration, and gravity is not the only force in Nature. This is the first reason for studying accelerated motion *per se*.

The second reason to study relatively-accelerated motion in itself is empirical: standard general relativity has not been able to explain certain observational data by itself, most notably galaxy rotation curves[5, 6], without resorting to Aether-like matter. This suggests the possibility that relativity is not the last word on the laws of motion.

In this letter, we study the relative uniformly-accelerated motion of two observers in an expanding Universe, in and for itself, i.e. without focusing on gravity and without bringing GR in. We show that a consequence of kinematics in an expanding Universe (modelled by the FLRW metric), is that light gains acceleration, even in absence of matter. We shall argue that the existence of acceleration of light requires a modification of Newton's Second Law. The result of such a modification would be that the amount of matter in clusters of galaxies is naturally increased and the need for Dark matter (within the scope of validity of FLRW) evaporated.

Velocity Space

Study of accelerated motion is notoriously difficult for it is

- intertwined with gravity, and
- non-linear *in spacetime*,

among other things.
To avoid the first difficulty, in this letter we focus on an infinitesimal region of spacetime where effects of curvature are negligible.
To avoid other difficulties, notably non-linearity, we hereby introduce the notion of *velocity space*, which enables us to investigate relative accelerated motion without having to deal with non-linearities, and which will prove to be a useful conceptual tool in analysing accelerated motion.

Considered as a three-vector, velocity $\vec{v} = (v_x, v_y, v_z)$ can be represented as a point in a three-dimensional *velocity space.* According to special relativity this space possesses hyperbolic geometry[3], manifested by Einstein velocity-addition rule.
As our focus is on accelerated motion and Einstein velocity-addition rule is limited to motions without acceleration, hence receives no contribution from the relative acceleration of two observers, we shall hereafter approximate this space by a Euclidean one in the limit of small angles where the Pythagorean theorem holds:

$$du^2 := dv_x^2 + dv_y^2 + dv_z^2. \tag{1}$$

The main advantage of velocity space is that it enables us to study uniformly-accelerated motion in spacetime, by merely considering linear motion (which we know well enough) in velocity space instead.

Kinematics

We choose to model the kinematics of our Universe based on the following foundations:

1. We assume that the *Cosmological Principle* holds well in its domain of application, thus
2. focus on scales where the dynamics of point particles can be safely applied to galaxies.
   On this scale the global geometry of the Universe is modelled by the FLRW metric, in which, in accord with the recent observational data[7],
3. we assume the spatial curvature of the Universe to be zero.

Then

$$ds^2 = -c^2dt^2 + \lambda^2(t)d\mathbf{x}^2. \tag{2}$$

Since no study of motion can claim universality without having something to say about light, we focus on the speed of light in such universe, null geodesics, which are equivalently given by the length of the velocity vector

$$\|\mathbf{U}\| = \frac{ic}{\lambda(t)};$$

then

$$\frac{dv}{dt} = -\frac{icH(t)}{\lambda(t)}, \tag{3}$$

meaning *in the current epoch $t = t_0$ of our expanding Universe, light accelerates by the magnitude cH_0.*
This fact is the foundation of this letter.

A critical question arises immediately after our conclusion from (3): with respect to *what* frame of reference does light accelerate?
Recall that the speed of light, which is given by $\|\mathbf{U}\|$, is an invariant, yet a function of cosmic time. This suggests that we answer the question in Einstein' manner and propose the *principle of invariance of the acceleration of light* (hereafter PIA): all uniformly-accelerated observers in vacuum would agree on acceleration of light.

Along with the *generalized principle of relativity* (hereafter GPR), which now requires that our resulting coordinates transformations *in velocity space* be linear, PIA demands the invariance of acceleration of light, $dv/dt = -icH/\lambda$. Applying these two principles, we arrive at the following transformations in the velocity-*time* space

$$v' = \eta(v - a\tau), \tag{4}$$

$$\tau' = \eta(\tau - \frac{av}{(cH_0)^2}), \tag{5}$$

where

$$\eta := \frac{1}{\sqrt{1 + (\frac{a}{cH_0})^2}}, \tag{6}$$

and

$$\tau := it, \tag{7}$$

which follows from (3).
The transformation (5) means that, to satisfy PIA *time has to be added to the velocity space.* Addition of time as a new dimension of the velocity space is the main result of this letter.
It is readily seen that the (non-relativistic) invariant interval of the velocity space is

$$ds^2 = (cH_0)^2 dt^2 + dv^2. \tag{8}$$

Dynamics

The most important consequence of (8) is that the differential of velocity is changed, so as to leave the equations invariant. This change takes effect by

$$\frac{d}{dv} \to \frac{d}{dv}\frac{dv}{du}. \tag{9}$$

In particular, this necessitates the modification of canonical momentum, from

$$p = \frac{\partial \mathcal{L}}{\partial v},$$

to

$$p = \frac{\partial \mathcal{L}}{\partial v}\frac{dv}{du} = \frac{\partial \mathcal{L}}{\partial v}\frac{dv}{dt}\frac{dt}{du} = \frac{\partial \mathcal{L}}{\partial v}\frac{a}{\sqrt{a^2 + (cH_0)^2}}. \tag{10}$$

Applying the law of motion (Euler-Lagrange equation)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v}\frac{dv}{du}\right) = \frac{\partial \mathcal{L}}{\partial x},$$

yields the modified Newton's Second Law

$$F = \frac{ma^2}{\sqrt{a^2 + (cH_0)^2}} + \frac{mvj}{\sqrt{a^2 + (cH_0)^2}} + \frac{mva}{\sqrt{a^2 + (cH_0)^2}}\frac{d}{dt}\frac{1}{\sqrt{a^2 + (cH_0)^2}}. \tag{11}$$

We see that the first term of this modification

$$F \approx \frac{ma^2}{\sqrt{a^2 + (cH_0)^2}}, \tag{12}$$

is in fact the Modified Newtonian Dynamics (MOND) of Milgrom[8, 9], and as such shares all the successes of MOND, like the Baryonic Tully-Fisher law[10]

$$v^4 = GMa_0,$$

without having to account for all its shortcomings, since our proposed model does *not* claim to be a complete cosmological *alternative* to ΛCDM.

It has been shown [11, 12] that this modification does eliminate the need for Dark matter in clusters of galaxies.

Conclusion

In this letter we showed the existence of a universal acceleration scale $a_0 = cH_0$, which is a direct consequence of the expansion of the Universe. By proposing two natural principles, namely invariance of acceleration of light for all uniformly-accelerated observers, and a generalization of the principle of relativity, we demonstrated that acceleration of light in an expanding Universe causes Newton's Second law to be modified. This modification is responsible for the missing mass in clusters of galaxies.

Acknowledgement

I would like to thank my loyal friend, Dr. Mohamad Emami, who was instrumental in the process of preparing this paper.
I also thank Professors McGaugh and Boehmer for patiently answering my questions, in particular on MOND and modified theories of gravity.

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