ON GENERALIZED LI-YAU INEQUALITIES

LI-CHANG HUNG

Abstract. We generalize the Li-Yau inequality for second derivatives and we also establish Li-Yau type inequality for fourth derivatives. Our derivation relies on the representation formula for the heat equation.

.

1. Introduction

The Li-Yau inequality asserts that if $u = u(x,t)$ is a positive solution to the heat equation, then the logarithm of $u(x,t)$ forms a supersolution to Laplace's equation, i.e.

$$\Delta \log u \geq -\frac{n}{2t}, \tag{1.1}$$

where n is the dimension of the manifold and $t > 0$. The inequality (1.1) is due to Li and Yau [2]. Their derivation relies on an idea related to the parabolic maximum principle.

Our goal here is to give a direct proof of a generalization of inequality (1.1) without using the parabolic maximum principle. Our proof relies on the representation formula for the heat equation $u_t = \Delta u$ [1]

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|x-y|^2}{4t}}\, dy \quad (x \in \mathbb{R}^n,\ t > 0), \tag{1.2}$$

where the initial condition $g \in C(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ is assumed to be nonnegative and does not vanish completely. To generalize inequality (1.1), we need the following two lemmas. The first lemma is concerned with a Jensen inequality.

Lemma 1.1 (Jensen inequality). *Let $p \geq 1$ and suppose that $\int |g| = 1$. Then*

$$\left(\int |fg| \right)^p \leq \int |f|^p |g|. \tag{1.3}$$

2020 *Mathematics Subject Classification.* Primary 58J35, 35B45; Secondary 35B65, 53C44.

Key words and phrases. Li-Yau inequality, heat equation.

Proof.

$$\begin{aligned}
\int |fg| &= \int |f||g|^{\frac{1}{p}+\frac{1}{q}} \\
&= \int |f||g|^{\frac{1}{p}} \cdot |g|^{\frac{1}{q}} \\
&\le \left(\int \left(|f||g|^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \left(\int \left(|g|^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} \\
&\le \left(\int |f|^p|g|\right)^{\frac{1}{p}} \left(\int |g|\right)^{\frac{1}{q}} \\
&\le \left(\int |f|^p|g|\right)^{\frac{1}{p}}
\end{aligned}$$

$$\Longrightarrow \left(\int |fg|\right)^p \le \int |f|^p|g|$$

□

Assume (1.2) holds. Then we can find the derivatives of u in the following Lemma 1.2.

Lemma 1.2.

(a)

$$\frac{u_{x_i}(x,t)}{u(x,t)} = \int_{\mathbb{R}^n} -\frac{x_i-y_i}{2t} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy. \tag{1.4}$$

(b)

$$\frac{u_{x_ix_i}(x,t)}{u(x,t)} = \int_{\mathbb{R}^n} \left(-\frac{1}{2t} + \frac{(x_i-y_i)^2}{4t^2}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy. \tag{1.5}$$

(c) For $i \neq j$,

$$\frac{u_{x_ix_j}(x,t)}{u(x,t)} = \int_{\mathbb{R}^n} \frac{(x_i-y_i)(x_j-y_j)}{4t^2} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy. \tag{1.6}$$

(d)

$$\frac{u_{x_ix_ix_i}(x,t)}{u(x,t)} = \int_{\mathbb{R}^n} \frac{(x_i-y_i)(6t-(x_i-y_i)^2)}{8t^3} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy. \tag{1.7}$$

(e)

(1.8)

$$\frac{u_{x_ix_ix_ix_i}(x,t)}{u(x,t)} = \int_{\mathbb{R}^n} \frac{12t^2-12t(x_i-y_i)^2+(x_i-y_i)^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy.$$

(f) For $i \neq j$,

(1.9)

$$\frac{u_{x_ix_ix_jx_j}(x,t)}{u(x,t)} = \int_{\mathbb{R}^n} \frac{((x_i-y_i)^2-2t)((x_j-y_j)^2-2t)}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy.$$

Proof. Straightforward calculations lead to (a), (b), and (c).

(d)

$$\frac{u_{x_ix_ix_i}(x,t)}{u(x,t)} := \frac{(u_{x_ix_i}(x,t))_{x_i}}{u(x,t)}$$

$$= \int_{\mathbb{R}^n} \left[\frac{x_i-y_i}{2t^2} - \frac{x_i-y_i}{2t}\left(-\frac{1}{2t}+\frac{(x_i-y_i)^2}{4t^2}\right)\right] \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \frac{6t(x_i-y_i)-(x_i-y_i)^3}{8t^3} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \frac{(x_i-y_i)(6t-(x_i-y_i)^2)}{8t^3} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

(e)

$$\frac{u_{x_ix_ix_ix_i}(x,t)}{u(x,t)} := \frac{(u_{x_ix_ix_i}(x,t))_{x_i}}{u(x,t)}$$

$$= \int_{\mathbb{R}^n} \left[\frac{6t-3(x_i-y_i)^2}{8t^3} - \left(\frac{x_i-y_i}{2t}\right)\left(\frac{(x_i-y_i)(6t-(x_i-y_i)^2)}{8t^3}\right)\right] \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \left(\frac{12t^2-12t(x_i-y_i)^2+(x_i-y_i)^4}{16t^4}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \left(\frac{-12t(x_i-y_i)^2+(x_i-y_i)^4}{16t^4}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy + \frac{3}{4}\frac{1}{t^2}$$

(f)

$$\frac{u_{x_ix_ix_j}(x,t)}{u(x,t)} := \frac{(u_{x_ix_j}(x,t))_{x_i}}{u(x,t)}$$

$$= \int_{\mathbb{R}^n} \left(\frac{x_j-y_j}{4t^2} - \frac{x_i-y_i}{2t}\frac{(x_i-y_i)(x_j-y_j)}{4t^2}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \frac{(x_j-y_j)(2t-(x_i-y_i)^2)}{8t^3} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$\frac{u_{x_ix_ix_jx_j}(x,t)}{u(x,t)} := \frac{(u_{x_ix_ix_j}(x,t))_{x_j}}{u(x,t)}$$

$$= \int_{\mathbb{R}^n} \left(\frac{2t-(x_i-y_i)^2}{8t^3} - \frac{x_j-y_j}{2t} \frac{(x_j-y_j)(2t-(x_i-y_i)^2)}{8t^3} \right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy$$

$$= \int_{\mathbb{R}^n} \frac{4t^2 - 2t(x_i-y_i)^2 - 2t(x_j-y_j)^2 + (x_i-y_i)^2(x_j-y_j)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy$$

$$= \int_{\mathbb{R}^n} \frac{((x_i-y_i)^2-2t)((x_j-y_j)^2-2t)}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy$$

$$\leq \int_{\mathbb{R}^n} \frac{(x_i-y_i)^2(x_j-y_j)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy + \frac{1}{4t^2}$$

□

2. Proof of the generalized Li-Yau inequality

Theorem 2.1 (Generalized Li-Yau inequality)**.** *Let $u = u(x,t)$ be given by* (1.2). *Then*

$$\frac{\Delta u}{u} - \alpha \sum_{i,j=1, i\neq j}^{n} \frac{u_{x_ix_j}}{u} - \beta \sum_{i,j=1, i\neq j}^{n} \frac{u_{x_i}u_{x_j}}{u^2} - \gamma \frac{|\nabla u|^2}{u^2} \geq -\frac{n}{2t}, \tag{2.1}$$

where α, β, and γ are nonnegative constants satisfying

$$(n-1)(\alpha+\beta) + \gamma \leq 1. \tag{2.2}$$

Proof.

(1) Estimate $\left|\frac{\nabla u(x,t)}{u(x,t)}\right|^2$

$$\left|\frac{\nabla u(x,t)}{u(x,t)}\right|^2 := \sum_{i=1}^{n}\left|\frac{u_{x_i}(x,t)}{u(x,t)}\right|^2$$

$$= \sum_{i=1}^{n}\left|\int_{\mathbb{R}^n} -\frac{x_i-y_i}{2t}\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy\right|^2$$

$$\leq \sum_{i=1}^{n}\int_{\mathbb{R}^n}\left(\frac{x_i-y_i}{2t}\right)^2\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

$$= \int_{\mathbb{R}^n}\sum_{i=1}^{n}\left(\frac{x_i-y_i}{2t}\right)^2\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

(2.3)

$$= \int_{\mathbb{R}^n}\frac{|x-y|^2}{4t^2}\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

(2) Calculate $\frac{\Delta u(x,t)}{u(x,t)}$

$$\frac{\Delta u(x,t)}{u(x,t)} := \sum_{i=1}^{n}\frac{u_{x_ix_i}(x,t)}{u(x,t)}$$

$$= \sum_{i=1}^{n}\int_{\mathbb{R}^n}\left(-\frac{1}{2t}+\frac{(x_i-y_i)^2}{4t^2}\right)\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

$$= \int_{\mathbb{R}^n}\sum_{i=1}^{n}\left(-\frac{1}{2t}+\frac{(x_i-y_i)^2}{4t^2}\right)\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

(2.4)

$$= \int_{\mathbb{R}^n}\left(-\frac{n}{2t}+\frac{|x-y|^2}{4t^2}\right)\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

(3) Estimate $\sum_{i,j=1,i\neq j}^{n}\frac{u_{x_i}(x,t)}{u(x,t)}\cdot\frac{u_{x_j}(x,t)}{u(x,t)}$

$$\sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i}(x,t)}{u(x,t)} \cdot \frac{u_{x_j}(x,t)}{u(x,t)} \leq \sum_{i,j=1,i\neq j}^{n} \frac{1}{2}\left(\left(\frac{u_{x_i}(x,t)}{u(x,t)}\right)^2 + \left(\frac{u_{x_j}(x,t)}{u(x,t)}\right)^2\right)$$

$$= \sum_{i,j=1,i\neq j}^{n} \frac{1}{2}\left[\left(\int_{\mathbb{R}^n} -\frac{x_i-y_i}{2t}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy\right)^2 + \left(\int_{\mathbb{R}^n} -\frac{x_j-y_j}{2t}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy\right)^2\right]$$

$$\leq \sum_{i,j=1,i\neq j}^{n} \frac{1}{2}\left[\int_{\mathbb{R}^n} \left(\frac{x_i-y_i}{2t}\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy + \int_{\mathbb{R}^n} \left(\frac{x_j-y_j}{2t}\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy\right]$$

$$= \int_{\mathbb{R}^n} \sum_{i,j=1,i\neq j}^{n} \frac{1}{2}\frac{(x_i-y_i)^2+(x_j-y_j)^2}{4t^2}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

$$= \int_{\mathbb{R}^n} \frac{(n-1)|x-y|^2}{4t^2}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy,$$

where we have used the observation

$$\sum_{i,j=1,i\neq j}^{n} (x_i-y_i)^2 + (x_j-y_j)^2 = 2(n-1)|x-y|^2. \tag{2.5}$$

(4) Estimate $\sum_{i,j=1,i\neq j}^{n} \frac{u_{x_ix_j}(x,t)}{u(x,t)}$

$$\sum_{i,j=1,i\neq j}^{n} \frac{u_{x_ix_j}(x,t)}{u(x,t)} = \sum_{i,j=1,i\neq j}^{n} \int_{\mathbb{R}^n} \frac{(x_i-y_i)(x_j-y_j)}{4t^2}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

$$\leq \sum_{i,j=1,i\neq j}^{n} \int_{\mathbb{R}^n} \frac{1}{2}\frac{(x_i-y_i)^2+(x_j-y_j)^2}{4t^2}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

$$= \int_{\mathbb{R}^n} \sum_{i,j=1,i\neq j}^{n} \frac{1}{2}\frac{(x_i-y_i)^2+(x_j-y_j)^2}{4t^2}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

$$= \int_{\mathbb{R}^n} \frac{(n-1)|x-y|^2}{4t^2}\,\frac{\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}g(y)}{u(x,t)}\,dy$$

(5) Establish $\frac{\Delta u}{u} - \alpha \sum_{i,j=1,i\neq j}^{n} \frac{u_{x_ix_j}(x,t)}{u(x,t)} - \beta \sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i}(x,t)}{u(x,t)} \cdot \frac{u_{x_j}(x,t)}{u(x,t)} - \gamma \frac{|\nabla u|^2}{u^2} \geq -\frac{n}{2t}$

$$
\begin{aligned}
&\frac{\Delta u}{u} - \alpha \sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i x_j}(x,t)}{u(x,t)} - \beta \sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i}(x,t)}{u(x,t)} \cdot \frac{u_{x_j}(x,t)}{u(x,t)} - \gamma \frac{|\nabla u|^2}{u^2} \\
&\geq \int_{\mathbb{R}^n} \left(-\frac{n}{2t} + (1-(n-1)(\alpha+\beta)-\gamma) \frac{|x-y|^2}{4t^2} \right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy \\
&\geq \int_{\mathbb{R}^n} -\frac{n}{2t} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy \\
&= -\frac{n}{2t}
\end{aligned}
$$

□

We generalize the Li-Yau inequality from second derivatives to fourth derivatives.

Theorem 2.2 (Li-Yau type inequality for fourth derivatives)**.** *Let $u = u(x,t)$ be given by* (1.2)*. Then*

$$
\begin{aligned}
&\sum_{i=1}^{n} \frac{u_{x_i x_i x_i x_i}}{u} + k_1 \sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i x_i x_j x_j}}{u} + k_2 \left| \frac{\nabla u}{u} \right|^4 + k_3 \left| \frac{\Delta u}{u} \right|^2 + k_4 \left(\sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i x_j}}{u} \right)^2 \\
&\geq \left(3n + k_1 + n^2 k_3 - \frac{n\,(3+(n-1)k_1+nk_3)^2}{1+n\,(k_2+k_3)} \right) \frac{1}{4t^2},
\end{aligned}
$$

provided that k_1, k_2, k_3, and k_4 are constants satisfying

(2.6a) $$k_2 + k_3 > -\frac{1}{n},$$

(2.6b) $$k_1 \geq -nk_4,$$

(2.6c) $$k_2 \leq 0,$$

(2.6d) $$k_3 \leq 0,$$

(2.6e) $$k_4 \leq 0.$$

Proof.

(1) Estimate on $\sum_{i=1}^n \frac{u_{x_ix_ix_ix_i}(x,t)}{u(x,t)}$

$$\begin{aligned}
&\sum_{i=1}^n \frac{u_{x_ix_ix_ix_i}(x,t)}{u(x,t)}\\
&= \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{12t^2 - 12t(x_i-y_i)^2 + (x_i-y_i)^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\\
&= \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{12t^2 - 12t(x_i-y_i)^2 + (x_i-y_i)^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\\
&= \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{12t^2 - 12t(x_i-y_i)^2 + (x_i-y_i)^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\\
&\geq \int_{\mathbb{R}^n} \frac{12nt^2 - 12t|x-y|^2 + \frac{1}{n}|x-y|^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{i=1}^n \frac{(x_i-y_i)^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\\
&= \int_{\mathbb{R}^n} \sum_{i=1}^n \left(\frac{x_i-y_i}{2t}\right)^4 \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\\
&\geq \int_{\mathbb{R}^n} \frac{1}{n} \left(\sum_{i=1}^n \left(\frac{x_i-y_i}{2t}\right)^2\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\\
&= \int_{\mathbb{R}^n} \frac{1}{n} \left(\frac{|x-y|^2}{4t^2}\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy
\end{aligned}$$

(2.7)

$$= \int_{\mathbb{R}^n} \frac{1}{n} \frac{|x-y|^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

(2) Estimate on $\sum_{i,j=1,i\neq j}^n \frac{u_{x_ix_ix_jx_j}(x,t)}{u(x,t)}$

$$\sum_{i,j=1,i\neq j}^{n} \frac{u_{x_i x_i x_j x_j}(x,t)}{u(x,t)}$$

$$= \sum_{i,j=1,i\neq j}^{n} \int_{\mathbb{R}^n} \frac{4t^2 - 2t(x_i-y_i)^2 - 2t(x_j-y_j)^2 + (x_i-y_i)^2(x_j-y_j)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \sum_{i,j=1,i\neq j}^{n} \frac{4t^2 - 2t(x_i-y_i)^2 - 2t(x_j-y_j)^2 + (x_i-y_i)^2(x_j-y_j)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$= \int_{\mathbb{R}^n} \frac{4t^2 - 4(n-1)t|x-y|^2 + \sum_{i,j=1,i\neq j}^{n} (x_i-y_i)^2(x_j-y_j)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy,$$

where we have used (2.5).

(3) Estimate on $\left|\frac{\nabla u(x,t)}{u(x,t)}\right|^4$

$$\left|\frac{\nabla u(x,t)}{u(x,t)}\right|^4 := \left(\sum_{i=1}^{n} \left|\frac{u_{x_i}(x,t)}{u(x,t)}\right|^2\right)^2$$

$$= \left(\sum_{i=1}^{n} \left|\int_{\mathbb{R}^n} -\frac{x_i-y_i}{2t} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right|^2\right)^2$$

$$\leq \left(\sum_{i=1}^{n} \int_{\mathbb{R}^n} \left(\frac{x_i-y_i}{2t}\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$= \left(\int_{\mathbb{R}^n} \sum_{i=1}^{n} \left(\frac{x_i-y_i}{2t}\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$= \left(\int_{\mathbb{R}^n} \frac{|x-y|^2}{4t^2} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$\leq \int_{\mathbb{R}^n} \frac{|x-y|^4}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

(4) Estimate on $\left|\frac{\Delta u(x,t)}{u(x,t)}\right|^2$

$$\left|\frac{\Delta u(x,t)}{u(x,t)}\right|^2 := \left(\sum_{i=1}^n \frac{u_{x_ix_i}(x,t)}{u(x,t)}\right)^2$$

$$= \left(\sum_{i=1}^n \int_{\mathbb{R}^n} \left(-\frac{1}{2t} + \frac{(x_i-y_i)^2}{4t^2}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$= \left(\int_{\mathbb{R}^n} \sum_{i=1}^n \left(-\frac{1}{2t} + \frac{(x_i-y_i)^2}{4t^2}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$= \left(\int_{\mathbb{R}^n} \left(-\frac{n}{2t} + \frac{|x-y|^2}{4t^2}\right) \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$\leq \int_{\mathbb{R}^n} \frac{\left(|x-y|^2 - 2nt\right)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

(5) Estimate on $\left(\sum_{i,j=1,i\neq j}^n \frac{u_{x_ix_j}(x,t)}{u(x,t)}\right)^2$

$$\sum_{i,j=1,i\neq j}^n \left(\frac{u_{x_ix_j}(x,t)}{u(x,t)}\right)^2$$

$$= \sum_{i,j=1,i\neq j}^n \left(\int_{\mathbb{R}^n} \frac{(x_i-y_i)(x_j-y_j)}{4t^2} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$\leq \left(\sum_{i,j=1,i\neq j}^n \int_{\mathbb{R}^n} \frac{|x_i-y_i||x_j-y_j|}{4t^2} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$= \left(\int_{\mathbb{R}^n} \sum_{i,j=1,i\neq j}^n \frac{|x_i-y_i||x_j-y_j|}{4t^2} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy\right)^2$$

$$\leq \int_{\mathbb{R}^n} \left(\sum_{i,j=1,i\neq j}^n \frac{|x_i-y_i||x_j-y_j|}{4t^2}\right)^2 \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

$$\leq \int_{\mathbb{R}^n} n \sum_{i,j=1,i\neq j}^n \frac{(x_i-y_i)^2(x_j-y_j)^2}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)}\, dy$$

Combining the above calculations, we have

$$\sum_{i=1}^{n} \frac{u_{x_i x_i x_i x_i}}{u} + k_1 \sum_{i,j=1, i\neq j}^{n} \frac{u_{x_i x_i x_j x_j}}{u} + k_2 \left|\frac{\nabla u}{u}\right|^4 + k_3 \left|\frac{\Delta u}{u}\right|^2 + k_4 \left(\sum_{i,j=1, i\neq j}^{n} \frac{u_{x_i x_j}}{u}\right)^2$$

$$\geq \int_{\mathbb{R}^n} \frac{h(t,x,y,n)}{16t^4} \frac{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} g(y)}{u(x,t)} \, dy,$$

where

$$\begin{aligned} h(t,x,y,n) &= 12nt^2 - 12t|x-y|^2 + \frac{1}{n}|x-y|^4 + k_1 \left(4t^2 - 4(n-1)t|x-y|^2 + \sum_{i,j=1, i\neq j}^{n} (x_i-y_i)^2 (x_j-y_j)^2\right) \\ &\quad + k_2 \, |x-y|^4 + k_3 \left(|x-y|^2 - 2nt\right)^2 + k_4 \left(n \sum_{i,j=1, i\neq j}^{n} (x_i-y_i)^2 (x_j-y_j)^2\right) \\ &= \left(12n + 4k_1 + 4n^2 k_3\right) t^2 + \left(-12 - 4(n-1)k_1 - 4nk_3\right) t \, |x-y|^2 + \left(\frac{1}{n} + k_2 + k_3\right) |x-y|^4 \\ &\quad + (k_1 + nk_4) \sum_{i,j=1, i\neq j}^{n} (x_i-y_i)^2 (x_j-y_j)^2 \\ &= Ct^2 + Bt|x-y|^2 + A|x-y|^4 + (k_1 + nk_4) \sum_{i,j=1, i\neq j}^{n} (x_i-y_i)^2 (x_j-y_j)^2 \\ &= A\left(|x-y|^2 + \frac{B}{2A}t\right)^2 + \left(\frac{4AC - B^2}{4A}\right) t^2 + (k_1 + nk_4) \sum_{i,j=1, i\neq j}^{n} (x_i-y_i)^2 (x_j-y_j)^2, \end{aligned}$$

where

$$A = \frac{1}{n} + k_2 + k_3, \tag{2.8a}$$

$$B = -12 - 4(n-1)k_1 - 4nk_3, \tag{2.8b}$$

$$C = 12n + 4k_1 + 4n^2 k_3, \tag{2.8c}$$

$$\frac{4AC - B^2}{4A} = 4\left(3n + k_1 + n^2 k_3 - \frac{n\left(3 + (n-1)k_1 + nk_3\right)^2}{1 + n\left(k_2 + k_3\right)}\right). \tag{2.8d}$$

□

References

1. Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Society, 2022.
2. Peter Li and Shing Tung Yau. On the parabolic kernel of the schrödinger operator. *Acta Mathematica*, 156(1):153–201, 1986.

Department of Mathematics, Soochow University, Taipei, Taiwan
Current address: Department of Mathematics, Soochow University, Taipei, Taiwan
Email address: `lichang.hung@gmail.com`