$\boldsymbol{\pi - e}$, $\boldsymbol{\pi + e}$, $\boldsymbol{\pi e}$ and $\boldsymbol{\frac{\pi}{e}}$ all are irrational numbers

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Abstract

It is proved that $\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$ all are irrational numbers . The proof is essentially elementary , it is an argument by contradiction.

Notation and reminder

π : *known as Archimedes constant , is the ratio of a circle's circumference to its diameter and* $3 < \pi < 4$ *.*

$e = \sum_{m=0}^{+\infty} \frac{1}{m!}$: *known as Euler's number and* $2 < e < 3$ *.*

$\mathbb{N}^* := \{1,2,3,4,\dots\}$ *the natural numbers .*

$\mathbb{Z} := \{\dots, -4, -3, -2, -1, 0,1,2,3,4, \dots\}$ *the integers and* $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ *.*

$\mathbb{Q} := \{ \frac{p}{q} : (p,q) \in \mathbb{Z} x \mathbb{Z}^* \text{and } p \wedge q = 1\}$ *the set of rational numbers*.

$\mathbb{R}$: *the set of real numbers*.

$\mathbb{R} \setminus \mathbb{Q} := \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$ *the set of irrational numbers*.

$p \wedge q := \max\{d \in \mathbb{N}^* : d/p \text{ and } d/q\}$ *the greatest common divisor of p and q.*

$\forall$: *the universal quantifier and* $\exists$: *the existential quantifier .*

Introduction

Irrational numbers are the type of real numbers that cannot be expressed in the rational form $\frac{p}{q}$, where p , q are integers and $q \neq 0$. In simple words, all the real numbers that are not rational numbers are irrational. In this paper we show that $\sqrt{3} - \sqrt{2}$ and $\sqrt{3} + \sqrt{2}$, e and π , $\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$ all are irrational numbers. It is an argument by contradiction.

$\pi - e\,, \pi + e\,, \pi e$ and $\frac{\pi}{e}$ all are irrational numbers

Theorem 1. $\sqrt{6} \in \mathbb{R} \setminus \mathbb{Q}$. *In other words,* $\sqrt{6}$ *is an irrational number.*

Proof. An argument by contradiction. Suppose that $\sqrt{6} \in \mathbb{Q}$, and as $\sqrt{6} > 0$ then $\exists\, p\,, q \in \mathbb{N}^*$ such that $\sqrt{6} = \frac{p}{q}$ and $p \wedge q = 1$, then $\left(\sqrt{6}\right)^2 = \left(\frac{p}{q}\right)^2$, then $6 = \frac{p^2}{q^2}$ and $6q^2 = p^2 \Rightarrow p^2$ is even and $p \in \mathbb{N}^* \Rightarrow p$ is even or $p = 2k\colon k \in \mathbb{N}^*$ $\Rightarrow 6q^2 = (2k)^2 = 4k^2 \Rightarrow 3q^2 = 2k^2$ and $3 \wedge 2 = 1 \Rightarrow 2$ divides q^2 and 2 is prime $\Rightarrow$ 2 divides q and $q \in \mathbb{N}^* \Rightarrow q$ is even or $q = 2k'\colon k' \in \mathbb{N}^*$, hence $p \wedge q \geq 2$, and we get a contradiction because $p \wedge q = 1$.

Main Theorem 1. $\sqrt{3} - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\sqrt{3} + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.
In other words, $\sqrt{3} - \sqrt{2}$ *and* $\sqrt{3} + \sqrt{2}$ *both are irrational numbers .*

Proof. An argument by contradiction. First, suppose that $\sqrt{3} - \sqrt{2} \in \mathbb{Q}$, then $\exists\, r \in \mathbb{Q}$ such that $\sqrt{3} - \sqrt{2} = r$ implies that $\left(\sqrt{3} - \sqrt{2}\right)^2 = r^2 \in \mathbb{Q}$ $\Rightarrow 5 - 2\sqrt{6} = r^2 \in \mathbb{Q} \Rightarrow \sqrt{6} = \frac{5 - r^2}{2} \in \mathbb{Q}$, and we get a contradiction . Second, suppose that $\sqrt{3} + \sqrt{2} \in \mathbb{Q}$, then $\exists\, r \in \mathbb{Q}$ such that $\sqrt{3} + \sqrt{2} = r$ implies that $(\sqrt{3} + \sqrt{2})^2 = r^2 \in \mathbb{Q} \Rightarrow 5 + 2\sqrt{6} = r^2 \in \mathbb{Q}$ $\Rightarrow \sqrt{6} = \frac{r^2 - 5}{2} \in \mathbb{Q}$, and we get a contradiction.

Theorem 2. $\forall\, n \in \mathbb{N}^*$ *we have* $\sin(n) \neq 0$. Several proofs are possible.

Proof. Indeed, $\forall\, n \in \mathbb{N}^*$ we have $\cos(n) \in \mathbb{R} \setminus \mathbb{Q}$ see [**1, Theorem 2.5**] , then $|\cos(n)| \neq 1$ and $\cos^2(n) + \sin^2(n) = 1 \Rightarrow \sin(n) \neq 0$.

Main Theorem 2. $e \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\pi \in \mathbb{R} \setminus \mathbb{Q}$.
In other words, e and π both are irrational numbers.

Proof. An argument by contradiction . First, Suppose that $e \in \mathbb{Q}$, and as $2 < e < 3$ then $\exists\, p\,, q \in \mathbb{N}^*$ such that $e = \frac{p}{q}$ and $q > 1$ and $p \wedge q = 1$, then $q!\,e = q!\frac{p}{q} = (q-1)!p \Rightarrow q!\,e \in \mathbb{N}^*$.
We also have $q!\sum_{m=0}^{q}\frac{1}{m!} = \sum_{m=0}^{q}\frac{q!}{m!} = q! + q! + \frac{q!}{2!} + \cdots + 1 \Rightarrow q!\sum_{m=0}^{q}\frac{1}{m!} \in \mathbb{N}^*$, and $e = \sum_{m=0}^{+\infty}\frac{1}{m!} > \sum_{m=0}^{q}\frac{1}{m!} \Rightarrow q!\,e > q!\sum_{m=0}^{q}\frac{1}{m!}$ and $q!\left(e - \sum_{m=0}^{q}\frac{1}{m!}\right) \in \mathbb{N}^*$.

$|x| := \max\{-x\,, x : x \in \mathbb{R}\}$ *the absolute value of x.*

$]0,1[\; := \{x \in \mathbb{R} : 0 < x < 1\}$ *the open interval with endpoints 0 and 1.*

Now, $q!\left(e-\sum_{m=0}^{q}\frac{1}{m!}\right)= q!\left(\sum_{m=0}^{+\infty}\frac{1}{m!}-\sum_{m=0}^{q}\frac{1}{m!}\right)=q!\sum_{m=q+1}^{+\infty}\frac{1}{m!}=\sum_{m=q+1}^{+\infty}\frac{q!}{m!}$,
and $0<\sum_{m=q+1}^{+\infty}\frac{q!}{m!}= \frac{1}{(q+1)}+\frac{1}{(q+1)(q+2)}+\frac{1}{(q+1)(q+2)(q+3)}+\cdots$

$$<\frac{1}{(q+1)}+\frac{1}{(q+1)(q+1)}+\frac{1}{(q+1)(q+1)(q+1)}+\cdots=\sum_{i=1}^{+\infty}\frac{1}{(q+1)^i}=\frac{1}{q}<1$$, we get

a contradiction because we have found an integer on $]0,1[$.

Second, suppose that $\pi\in\mathbb{Q}$, and as $3<\pi<4$ then $\exists\, p,q\in\mathbb{N}^*$ such that $\pi=\frac{p}{q}$ and $p\wedge q=1\Rightarrow p=q\pi$ and $\sin(p)=\sin(q\pi)=0$, we get a contradiction according to [**Theorem 2**].

Properties. The sine function satisfies the following properties :

The sine function (or $\sin(\theta)$) is defined , continuous , odd and $2\pi-$periodic on $\mathbb{R}$.

$\forall\,\theta\in\mathbb{R}$ we have $\sin(2k\pi+\theta)=\sin(\theta)$ and $\sin(2k\pi-\theta)=-\sin(\theta) : k\in\mathbb{Z}$.

$\forall\,\theta\in\mathbb{R}$ we have $\sin(\theta)=0\Leftrightarrow\theta\in\{k\pi : k\in\mathbb{Z}\}$.

Let $\{\theta_n : n\in\mathbb{N}^*\}\subset\mathbb{R}$ we have $\lim_{n\to+\infty}\sin(\theta_n)=0\Leftrightarrow\lim_{n\to+\infty}\theta_n\in\{k\pi : k\in\mathbb{Z}\}$.

Lemma. *We have* $\lim_{n\to+\infty}\sum_{m=n+1}^{+\infty}\frac{n!}{m!}=0$.

Proof. $\forall\, n\in\mathbb{N}^*$, $\sum_{m=n+1}^{+\infty}\frac{n!}{m!}=\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots$

$$<\frac{1}{n+1}+\frac{1}{(n+1)(n+1)}+\frac{1}{(n+1)(n+1)(n+1)}+\cdots$$

$$=\sum_{i=1}^{+\infty}\frac{1}{(n+1)^i}=\frac{1}{n}\quad,$$

then $0<\sum_{m=n+1}^{+\infty}\frac{n!}{m!}<\frac{1}{n}$ and $\lim_{n\to+\infty}\frac{1}{n}=0\Rightarrow\lim_{n\to+\infty}\sum_{m=n+1}^{+\infty}\frac{n!}{m!}=0$.

Two other proofs that e is an irrational number are available at [**2**, **Théorème 15**. **2**] by Dimitris Koukoulopoulos (This proof was found by Fourier in 1815) and at [**3**] by Jonathan Sondow , and tow other proofs that π is an irrational number are available at [**4**] by Ivan Niven and at [**5**] by Miklós Laczkovich (This proof was found by Lambert in 1761).

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Theorem 3. *We have*
$$\begin{cases} \lim\limits_{n\to+\infty} \sin\left(n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}\right) = 0 \\ \lim\limits_{n\to+\infty} \sin\left(n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}\right) = 0 \\ \lim\limits_{n\to+\infty} \sin\left(n!\,\pi e - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right) = 0 \\ \lim\limits_{n\to+\infty} \sin\left(n!\,pe - p.\sum_{m=0}^{n} \frac{n!}{m!}\right) = 0 \end{cases}.$$

Proof. First,

$$\lim_{n\to+\infty} \sin\left(n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n\to+\infty} \sin\left(n!\,\pi - n!\,e + \sum_{m=0}^{n} \frac{n!}{m!}\right)$$

$$= \lim_{n\to+\infty} \sin\left(n!\,\pi - \sum_{m=0}^{+\infty} \frac{n!}{m!} + \sum_{m=0}^{n} \frac{n!}{m!}\right)$$

$$= \lim_{n\to+\infty} \sin\left(n!\,\pi - \sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right)$$

$$= \lim_{n\to+\infty} -\sin\left(\sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right) = -\sin(0) = 0\,.$$

Second,

$$\lim_{n\to+\infty} \sin\left(n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n\to+\infty} \sin\left(n!\,\pi + n!\,e - \sum_{m=0}^{n} \frac{n!}{m!}\right)$$

$$= \lim_{n\to+\infty} \sin\left(n!\,\pi + \sum_{m=0}^{+\infty} \frac{n!}{m!} - \sum_{m=0}^{n} \frac{n!}{m!}\right)$$

$$= \lim_{n\to+\infty} \sin\left(n!\,\pi + \sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right)$$

$$= \lim_{n\to+\infty} \sin\left(\sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right) = \sin(0) = 0.$$

Third,

$$\lim_{n\to+\infty} \sin\left(n!\,\pi e - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n\to+\infty} \sin\left(\pi.\sum_{m=0}^{+\infty} \frac{n!}{m!} - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right)$$
$$= \lim_{n\to+\infty} \sin\left(\pi.\sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right) = \sin(0) = 0.$$

Fourth, let $p \in \mathbb{N}^*$ we have

$$\lim_{n\to+\infty} \sin\left(n!\,pe - p.\sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n\to+\infty} \sin\left(p.\sum_{m=0}^{+\infty} \frac{n!}{m!} - p.\sum_{m=0}^{n} \frac{n!}{m!}\right)$$
$$= \lim_{n\to+\infty} \sin\left(p.\sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right) = \sin(0) = 0.$$

Main Theorem 3. *$\pi - e \in \mathbb{R} \setminus \mathbb{Q}$ and $\pi + e \in \mathbb{R} \setminus \mathbb{Q}$ and $\pi e \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{\pi}{e} \in \mathbb{R} \setminus \mathbb{Q}$. In other words, $\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$ all are irrational numbers.*

Before starting the proof, we recall that
$\forall n \in \mathbb{N}^*$ we have $n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} > 0$ and $n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} > 0$,
and according to [**Main Theorem 2**] we have $\{k\pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$.

Proof. An argument by contradiction . First, suppose that $\pi - e \in \mathbb{Q}$, and as $\pi - e > 0$, then $\exists\, p , q \in \mathbb{N}^*$ such that $\pi - e = \frac{p}{q}$ and $p \wedge q = 1$,
then $\lim_{n \to +\infty} \sin\left(n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n \to +\infty} \sin\left(n!\,\frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!}\right)$.
We put $a_n = n!\,\frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $\lim_{n \to +\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\}$,
this implies that $\lim_{n \to +\infty} \sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

Second, suppose that $\pi + e \in \mathbb{Q}$, and as $\pi + e > 0$, then $\exists\, p , q \in \mathbb{N}^*$ such that $\pi + e = \frac{p}{q}$ and $p \wedge q = 1$,
then $\lim_{n \to +\infty} \sin\left(n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n \to +\infty} \sin\left(n!\,\frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!}\right)$.
We put $a_n = n!\,\frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $\lim_{n \to +\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\}$,
this implies that $\lim_{n \to +\infty} \sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

Third, suppose that $\pi e \in \mathbb{Q}$, and as $\pi e > 0$, then $\exists\, p , q \in \mathbb{N}^*$ such that $\pi e = \frac{p}{q}$ and $p \wedge q = 1$,

$$\lim_{n \to +\infty} \sin\left(n!\,\pi e - \pi . \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n \to +\infty} \sin\left(n!\,\frac{p}{q} - \pi . \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n \to +\infty} (-1)^{n+1} . \sin\left(n!\,\frac{p}{q}\right) .$$

(then)

We put $a_n = n!\,\frac{p}{q} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $\lim_{n \to +\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\}$,
this implies that $\lim_{n \to +\infty} \sin(a_n) \neq 0$ and $\lim_{n \to +\infty} (-1)^{n+1} . \sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

$$\pi - e\,,\ \pi + e\,,\ \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers}$$

Fourth, suppose that $\frac{\pi}{e} \in \mathbb{Q}$, and as $\frac{\pi}{e} > 0$, then $\exists\, p\,, q \in \mathbb{N}^*$ such that $\frac{\pi}{e} = \frac{p}{q}$ and $p \wedge q = 1$ implies that $pe = q\pi$,

then $\lim\limits_{n\to+\infty} \sin\left(n!\,pe - p.\sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim\limits_{n\to+\infty} \sin\left(n!\,q\pi - p.\sum_{m=0}^{n} \frac{n!}{m!}\right)$

$$= \lim_{n\to+\infty} -\sin\left(p.\sum_{m=0}^{n} \frac{n!}{m!}\right) .$$

We put $a_n = p.\sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \in \mathbb{N}^*\} \subset \mathbb{N}^*$, then $\lim\limits_{n\to+\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\}$,

this implies that $\lim\limits_{n\to+\infty} \sin(a_n) \neq 0$ and $\lim\limits_{n\to+\infty} -\sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

Finally, we conclude that $\pi - e\,, \pi + e\,, \pi e$ and $\frac{\pi}{e}$ all are irrational numbers.

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