Convergence Condition for the Newton-Raphson Method. Application in Real Polynomial Functions.

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ABSTRACT

The Newton-Raphson method applies to the numerical calculation of the roots of Real functions, through successive approximations towards the Root of the function. The Newton-Raphson method has the drawback that it does not always converge. This work establishes the convergence condition of the Newton-Raphson method for Real functions in general; once the convergence condition is met, the method will always converge towards the Root of the function. In this work, the development of the application of the convergence condition is established to specifically solve Real polynomial functions.

KEYWORDS: Newton-Raphson method, roots of Real polynomial functions, numerical calculation.

1. Convergence condition in Real Functions

For Real functions $f(x)$; $f : \mathbb{R} \rightarrow \mathbb{R}$, if the method of the tangents (Newton-Raphson) is started at an inflexion point, it will always converge towards a $(x, f(x))$=$(x, 0)$ point.

2. Application of the convergence condition in real polynomial functions

Here we propose the calculation of the roots of a Real polynomial function of order H, $f(x) = \sum_{i=0}^{H} a_i x^i$; $f : \mathbb{R} \rightarrow \mathbb{R}$, in two phases:

First Phase: Numerical calculation of $H - 2$ roots of the function with a successive approximation method that starts for the calculation of each root with the value of each of the roots of the second derivative of the function. *This start condition causes safe convergence towards the value of the root of the function.*

Second Phase: Direct calculation of the two remaining roots through a second degree equation obtained once $H - 2$ roots of the function are known.

2.1. Successive approximation method

In this method, the first approximation to the value of a root of the function will be the value of a root of the second derivative of the function; the second approximation to the value of the root of the function will be the value of x of the intersection point of the abscissa axis with the line tangent to the function

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at the point whose abscissa corresponds to the value of the root of the second derivative of the function. For this new value of x, the tangent line to the function is specified, and from the point of intersection of that tangent line with the abscissa axis, another value of x is determined that will be even closer to the value of the root of the function. Each time this procedure is repeated, a value of x closer to the root of the function will be achieved until a value as close as desired to the value of the root of the function is obtained. *With this method there will always be convergence towards the value of the root of the function.*

A variant of this method will be to test the function until the value of the root is reached or to get a value as close as one wants to the root, from any approximation x obtained in the way described above that is different from the value of the root of the second derivative. The testing is done with values greater than x if x is greater than the value of the root of the second derivative of the function and with values less than x if x is less than the value of the root of the second derivative of the function. It is possible that the value of the root of the second derivative of the function coincides with the value of the root of the function.

2.2. Roots of the second derivative

To obtain the values of the roots of the second derivative of the Real polynomial function to be solved, the successive derivatives of such function are previously determined until the last derivative is a linear function. From this group of successive derivatives, the roots of the intermediate derivatives are calculated, starting from the last derivative if the order of the function to solve is odd and from the penultimate derivative if the order of the function to solve is even, until the values of the roots of the second derivative of the function to solve are calculated. The last derivative equal to zero is a linear equation and the penultimate derivative equal to zero is an equation of the second degree, both are equations of direct resolution. To calculate the roots of each intermediate derivative, we proceed with the approximation method described above.

The successive derivatives of a Real polynomial function are Real polynomial functions. The number of roots of the second derivative of a Real polynomial function is equal to $H-2$, where H is the order of the function. Thus, knowing $H-2$ roots of the function, such function can be reduced to quadratic function that allows direct calculation of the remaining two roots. These two roots will be the smallest and the largest of the roots of the function.

2.3. Definitions

Let $P_0(x)$ be a Real polynomial function; domain $\in \mathbb{R}$, co-domain $\in \mathbb{R}$.

Let $x(k,n)$ be the ordinal root k of the function $P_n(x)$

Let $x(l,k,n)$ be the ordinal approximation l to the root $x(k,n)$

Starting from $n=1$, $P_n(x)$ is the nth derivative function of the Real polynomial function $P_0(x)$

Ec_n: $P_n(x)=0$

$$x(l+1,k,n) = \frac{-P_n(x(l,k,n)) + P_{n+1}(x(l,k,n)) * x(l,k,n)}{P_{n+1}(x(l,k,n))}$$

2.4. Example

Let
$P_0(x) = x^5 - 19x^4 + 133x^3 - 421x^2 + 586x - 280$

P_0'$(x) = P_1(x) = 5x^4 - 76x^3 + 399x^2 - 842x + 586$
P_0"$(x) = P_2(x) = 20x^3 - 228x^2 + 798x - 842$
P_0"'$(x) = P_3(x) = 60x^2 - 456x + 798$
P_0""$(x) = P_4(x) = 120x - 456$
$P_1(x); P_2(x); P_3(x); P_4(x)$ are the successive derivatives of the function $P_0(x)$

$H = 5$: the order of $P_0(x)$ is odd $\rightarrow$ Ec_4: $P_4(x) = 0$; Ec_4: $120x$–$456 = 0 \rightarrow$ $x(1,4) = 456/120 = 3.80$

$x(1,4)$ corresponds to the value of the second derivative of the function $P_2(x)$, so $x(1,4)$ will be the first approximation to a root of the function $P_2(x)$. Then:
$x(1,1,2) = x(1,4) \rightarrow x(1,1,2) = 3.80$

The approximations to the roots of the function are defined by the following formula:

$$x(l+1,k,n) = \frac{-P_n(x(l,k,n)) + P_{n+1}(x(l,k,n)) * x(l,k,n)}{P_{n+1}(x(l,k,n))}$$

Thus,
$x(1,1,2) = 3.80; P_2(x(1,1,2)) = -4.48$
$x(2,1,2) = 3.734502924; P_2(x(2,1,2)) = -0.0005619475$
$x(3,1,2) = 3.73442045758701; P_2(x(3,1,2)) = -2.67369E - 08$
$x(4,1,2) = 3.73442045719464; P_2(x(4,1,2)) = 0$
Then $x(1,2) = x(4,1,2) \rightarrow x(1,2) = 3.73442045719464$

$$\frac{P_2(x)}{(x - x(1,2))} = \frac{P_2(x)}{(x - 3.73442045719464)} = 20x^2 - 153.3115909x + 225.4700588$$

$20x^2 - 153.3115909x + 225.4700588 = 0 \rightarrow x(2,2) = 1.984337851; x(3,2) = 5.681241692$

$x(1,2); x(2,2); x(3,2)$ correspond to the values of each root of the second derivative of the function $P_0(x)$, so $x(1,2); x(2,2); x(3,2)$ will each be the first approximation to one of the roots of the function $P_0(x)$. Then:

$x(1,1,0) = x(1,2); x(1,2,0) = x(2,2); x(1,3,0) = x(3,2)$
$x(1,1,0) = 3.73442045719464; x(1,2,0) = 1.984337851; x(1,3,0) = 5.681241692$

Thus,
$x(1,1,0) = 3.73442045719464; P_0(x(1,1,0)) = -5.205518732$
$x(2,1,0) = 3.989557854; P_0(x(2,1,0)) = -0.188927438$
$x(3,1,0) = 3.999947413; P_0(x(3,1,0)) = -0.000946592$
$x(4,1,0) = 3.99999999861755; P_0(x(4,1,0)) = -2.48833E-08$
$x(5,1,0) = 3.99999999999995; P_0(x(5,1,0)) = -9.09495E-13$
$x(6,1,0) = 4; P_0(x(6,1,0)) = 0$
Then $x(1,0) = x(6,1,0) \rightarrow x(1,0) = 4$

$x(1,2,0) = 1.984337851; P_0(x(1,2,0)) = 0.470028549$
$x(2,2,0) = 1.999997258; P_0(x(2,2,0)) = 8.226E-05$
$x(3,2,0) = 2; P_0(x(3,2,0)) = 0$
Then $x(2,0) = x(3,2,0) \rightarrow x(2,0) = 2$

$x(1,3,0) = 5.681241692;\ P_0(x(1,3,0)) = -26.02866987$
$x(2,3,0) = 5.122483594;\ P_0(x(2,3,0)) = -3.322773707$
$x(3,3,0) = 5.012417523;\ P_0(x(3,3,0)) = -0.302023732$
$x(4,3,0) = 5.000162194;\ P_0(x(4,3,0)) = -0.00389334$
$x(5,3,0) = 5.000000028;\ P_0(x(5,3,0)) = -6.72E-07$
$x(6,3,0) = 5;\ P_0(x(6,3,0)) = 0$
Then $x(3,0) = x(6,3,0) \rightarrow x(3,0) = 5$

$$\frac{P_0(x)}{(x - x(1,0)) * (x - x(2,0)) * (x - x(3,0))} = \frac{P_0(x)}{(x-4) * (x-2) * (x-5)} = x^2 - 8x + 7$$

$x^2 - 8x + 7 = 0 \rightarrow x(4,0) = 1; x(5,0) = 7$

Then the roots of the function $P_0(x) = x^5 - 19x^4 + 133x^3 - 421x^2 + 586x - 280$ are $x(1,0) = 4$; $x(2,0) = 2$; $x(3,0) = 5$; $x(4,0) = 1$; $x(5,0) = 7$

Conclusions:

This Article proposes the novelty of the convergence condition of the Newton-Raphson Method in general for Real functions. It also presents the development of the application of the convergence condition in Real polynomial functions, which settles everything related to the numerical calculation of Real polynomial functions. According to the Abel-Ruffini theorem, the resolution of polynomial functions of order higher than 4 is only possible through numerical calculation.

References:

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