A proof of a result of James Stirling

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Abstract

A recent paper [1] contains a proof of a result of James Stirling [2],

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

What is following is another proof of this equality.

Since,

$$\frac{1}{\binom{2n}{n}} = \frac{n!^2}{(2n)!} = (2n+1)\mathrm{B}(n+1, n+1) = (2n+1) \int_0^1 x^n (1-x)^n dx$$

and,

$$|u| \leq 1, \mathrm{Li}_2(u) = -\int_0^u \frac{\ln(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{u^n}{n^2}$$

then,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} &= \sum_{n=1}^{\infty} \left(\left(\frac{2}{n} + \frac{1}{n^2} \right) \int_0^1 x^n (1-x)^n dx \right) \\
&= 2 \int_0^1 \left(\sum_{n=1}^{\infty} \left(\frac{x^n (1-x)^n}{n} \right) \right) dx + \\
&\int_0^1 \left(\sum_{n=1}^{\infty} \left(\frac{x^n (1-x)^n}{n^2} \right) \right) \\
&= \underbrace{\int_0^1 \mathrm{Li}_2\Big(x(1-x)\Big) dx}_{=A} - 2 \underbrace{\int_0^1 \ln\Big(1 - x(1-x)\Big) dx}_{=B}
\end{aligned}$$

$$
\begin{aligned}
A &\overset{\text{IBP}}{=} \underbrace{\Big[x\mathrm{Li}_2\Big(x(1-x)\Big)\Big]_0^1}_{=0} + \int_0^1 \frac{(1-2x)\ln\Big(1-x(1-x)\Big)}{1-x}dx \\
&= 2B - \underbrace{\int_0^1 \frac{\ln\Big(1-x(1-x)\Big)}{1-x}dx}_{u=1-x} \\
&= 2B - \int_0^1 \frac{\ln\Big(1-u(1-u)\Big)}{u}du \\
&= 2B - \underbrace{\int_0^1 \frac{\ln(1+u^3)}{u}du}_{z=u^3} + \int_0^1 \frac{\ln(1+u)}{u}du \\
&= 2B + \frac{2}{3}\int_0^1 \frac{\ln(1+u)}{u}du \\
&= 2B + \frac{2}{3}\underbrace{\int_0^1 \frac{\ln(1-u^2)}{u}du}_{z=u^2} - \frac{2}{3}\int_0^1 \frac{\ln(1-u)}{u}du \\
&= 2B - \frac{1}{3}\int_0^1 \frac{\ln(1-u)}{u}du \\
&= 2B + \frac{1}{3}\mathrm{Li}_2(1) = 2B + \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^2}
\end{aligned}
$$

Therefore,

$$
\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^2}}
$$

NB: B is the Beta Euler function.

References

[1] A. Lasjaunias and J-P. Tran, A note on the equality $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$
https://arxiv.org/pdf/2312.02245v1.pdf

[2] James Stirling, Methodus differentialis, sive Tractatus de summatione et interpolatione serierum infinitarum (1730, proposition 11, example 1)
https://gallica.bnf.fr/ark:/12148/bpt6k62011b/f2.item