FIXED POINT PROPERTIES OF PRECOMPLETELY AND POSITIVELY NUMBERED SETS

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Abstract. In this paper, we prove a joint generalization of Arslanov's completeness criterion and Visser's ADN theorem for precomplete numberings. Then we consider the properties of completeness and precompleteness of numberings in the context of the positivity property. We show that the completions of positive numberings are not their minimal covers and that the Turing completeness of any set A is equivalent to the existence of a positive precomplete A-computable numbering of any infinite family with positive A-computable numbering. In addition, we prove that each Σ^0_n-computable numbering ($n \geqslant 2$) of a Σ^0_n-computable non-principal family has a Σ^0_n-computable minimal cover ν such that for every computable function f there exists an integer n with $\nu(f(n)) = \nu(n)$.

Keywords: numbering, fixed point, complete numbering, precomplete numbering, positive numbering.

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§1. Introduction. Kleene's fixed point theorems [25] state that for every computable function f there is an integer n such that $W_{f(n)} = W_n$, and, moreover, for every binary computable function g there is a computable function h such that $W_{g(x,h(x))} = W_{h(x)}$ for each x. These two theorems are also called the Recursion Theorem and the Recursion Theorem with Parameters, respectively, and they play a significant role in many branches of computability theory and related areas such as λ-calculus and combinatory algebra. In Ershov's monograph [14], the recursion theorems are even compared in significance and the place occupied with main existence theorems in the theory of differential equations.

The proofs of Kleene's recursion theorems are based on the fact that for every partially computable function ψ there exists a computable function f such that $W_{f(x)} = W_{\psi(x)}$ if $\psi(x)$ converges, and $W_{f(x)} = \emptyset$ otherwise. This property was generalized by Mal'tsev [26–28] to arbitrary numberings, i.e. surjective mappings of $\mathbb{N}$ onto arbitrary sets, and was called *completeness*. Later Ershov weakened this definition by introducing the notion of *precomplete* numberings [13,14]. They can be characterized by the following theorem.

Theorem 1.1 (Ershov's recursion theorem [14]). *A numbering ν is precomplete if and only if for every binary partial computable function ψ there exists a computable function h such that $\nu(\psi(x, h(x))) = \nu(h(x))$ for each x with $\langle x, h(x)\rangle \in \operatorname{dom}\psi$.*

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The complete and precomplete numbering are the main object of study in this paper. This research topic has become especially relevant in the last two decades and many works have already been published on it. These include papers by Arslanov [2], Barendregt and Terwijn [11], Jain and Nessel [23], Selivanov [32], Terwijn [35], etc. (see also [17–19]).

Many of the well-known variations of the fixed point theorem in the Gödel numbering $x \mapsto W_x$ hold in arbitrary precomplete numberings. In particular, these include Arslanov's completeness criterion [1,3] (see Selivanov [31]) and the uniform version of Visser's ADN theorem [36] (see Barendregt and Terwijn [11]) which has applications in λ-calculus, c.e. equivalences, the theory of numberings, etc. (cf., e.g., [10, 12, 29, 36]). It was proved by Terwijn [35] that, in the Gödel numbering, the completeness criterion and the ADN theorem can be jointly generalized.

Theorem 1.2 (Terwijn [35]). *Suppose that A is a c.e. set with $A <_T \emptyset'$, and suppose that δ is a partially A-computable function such that $W_{\delta(n)} \neq W_n$ for each $n \in \operatorname{dom} \delta$. Then for every partially computable function ψ there exists a computable function f such that*

$$\psi(n) \downarrow \Rightarrow W_{f(n)} = W_{\psi(n)},$$
$$\psi(n) \uparrow \Rightarrow \delta(f(n)) \uparrow$$

for each integer n.

The research in this paper is motivated by the following question:

Question 1.3 (Barendregt and Terwijn [11]). Does the joint generalization Theorem 1.2 hold for arbitrary precomplete numberings?

A positive answer to this question is given in § 3 of this paper.

Usually, the properties of completeness and precompleteness of numberings are studied (cf., e.g., [6–8, 20]) in the context of their *principality* (i.e., their computability and the reducibility to them of all other computable numberings of a given family). On the other hand, Ershov proved [14] that precomplete numberings can be minimal and, moreover, *positive*. In the rest of the paper, we study the properties of completeness and precompleteness in the context of positivity. In particular, we obtain a partial answer to one of the open questions of Badaev, Goncharov, and Sorbi [7].

Our notation from computability theory is mostly standard. In the following, φ_e denotes the partially computable function with the Gödel number e. We write $\varphi_e(x)\downarrow$ if this computation converges, and $\varphi_e(x)\uparrow$ otherwise. For a partial function ψ we denote its domain and range by $\operatorname{dom}\psi$ and $\operatorname{ran}\psi$ respectively. For every e, the domain of the partially computable function φ_e will be denoted by W_e. We let $c(x, y)$ denote the computable pairing function $2^x(2y+1)-1$. Instead of $c(x, c(y, z))$ we will simply write $c(x, y, z)$. If η is an equivalence relation, then the notation $[x]_\eta$ is used to denote the η-equivalence class of the element x. For unexplained notions we refer to Soare [33, 34].

§2. Preliminaries on numberings. For the main concepts and notions of the theory of numberings we refer to the book by Ershov [14] and his paper [15].

Definition 2.1. A numbering ν of a set S is said to be *complete with respect to a special element* $a \in S$ if for every partially computable function ψ there exists a computable function f such that, for each x, $\nu(f(x)) = \nu(\psi(x))$ if $\psi(x)$ converges, and $\nu(f(x)) = a$ otherwise.

We say that a numbering ν is *complete* if it is complete with respect to some special element.

Definition 2.2. A numbering ν of a set S is said to be *precomplete* if for every partially computable function ψ there exists a computable function f such that $\nu(f(x)) = \nu(\psi(x))$ whenever $\psi(x)$ converges.

Since the partial function $\langle e, x\rangle \mapsto \varphi_e(x)$ is partially computable, any numbering ν is precomplete if and only if there exists a binary computable function f such that $\nu(f(e,x)) = \nu(\varphi_e(x))$ whenever $\varphi_e(x)$ converges. It is immediate to see that every complete numbering is precomplete.

In the theory of numberings there is a well known and powerful construction, due to Ershov [13], which allows, given any numbering ν of a set S, to find a complete numbering of S with respect to any special element $a \in S$. This construction is defined as follows.

Definition 2.3. Given numbering ν of a set S and $a \in S$, define a numbering ν_a, which is called the *completion* of ν with respect to a, by

$$\nu_a(c(e,x)) = \begin{cases} \nu(\varphi_e(x)), & \text{if}\, \varphi_e(x) \downarrow, \\ a, & \text{if}\, \varphi_e(x) \uparrow . \end{cases}$$

A numbering ν is said to be *positive* (*decidable, single-valued*) if its *numeration equivalence*

$$\eta_\nu = \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : \nu(x) = \nu(y)\}$$

is c.e. (computable, coincides with the equality relation, respectively). Given numberings μ and ν, we say that μ is *reducible* to ν denoted $\mu \leqslant \nu$ if there exists a computable function f such that $\mu(x) = \nu(f(x))$ for each x. We note that if $\mu \leqslant \nu$, then $\mu(\mathbb{N}) \subseteq \nu(\mathbb{N})$. We write $\mu < \nu$ if $\mu \leqslant \nu$ and $\nu \nleqslant \mu$. Numberings ν and μ are called *equivalent* if $\mu \leqslant \nu$ and $\nu \leqslant \mu$. For numberings ν_0 and ν_1, their *direct sum* is defined by $(\nu_0 \oplus \nu_1)(2x + i) = \nu_i(x)$, $i = 0, 1$, $x \in \mathbb{N}$. A numbering ν of a set S is said to be a *minimal cover* of a numbering μ of S is $\mu < \nu$ and there is no numbering α such that $\mu < \alpha < \nu$.

In the second half of the 1990s, Goncharov and Sorbi in their paper [22] proposed a general approach to defining the computability of families of constructive objects that can be formally described in some language equipped with some Gödel numbering. The greatest progress in this area has been made in the study of generalized computable families and their numberings in the arithmetic [4,30] and hyperarithmetic [5, 21] hierarchies. By the mid-2010s, generalized computable numberings began to be intensively studied from the standpoint of the uniform enumerability of families with respect to an arbitrary oracle A (see, e.g., [6, 16]). It is this generalization that we will consider in § 5 of this paper. For a set A and a family of A-c.e. sets $\mathcal{R}$, a numbering ν of $\mathcal{R}$ is said to be *A-computable* if the set

$$G_\nu = \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : y \in \nu(x)\}$$

is A-c.e. Families with A-computable numberings are also called *A-computable.* If we skip the oracle A in these definitions, we arrive at the classical notions of *computable numberings* and *computable families.* According to Goncharov and Sorbi's paper [22], $\emptyset^{(n-1)}$-computable numberings and families ($n \geqslant 2$) are called *Σ^0_n-computable.*

We say that an A-computable numbering ν of a family $\mathcal{R}$ is *principal*, if $\mu \leqslant \nu$ for each A-computable numbering μ of $\mathcal{R}$. Families with A-computable principal numberings are also called *principal.*

§3. A joint generalization of Arslanov's completeness criterion and Visser's ADN theorem. First, we present the previously mentioned generalization of the completeness criterion and the ADN theorem.

Theorem 3.1 (Arslanov's completeness criterion, Selivanov [31]). *Let ν be a precomplete numbering. Suppose that A is a c.e. set with $A <_T \emptyset'$. Then for every function $f \leqslant_T A$ there exists an integer n such that $\nu(f(n)) = \nu(n)$.*

The following theorem also shows that for any numbering its precompleteness is equivalent to the fact that it satisfies the ADN theorem.

Theorem 3.2 (Visser's ADN theorem [36]). *Let ν be a precomplete numbering. Suppose that δ is a partially computable function such that $\nu(\delta(n)) \neq \nu(n)$ for each $n \in \operatorname{dom}\delta$. Then for every partially computable function ψ there exists a computable function f such that*

$$\psi(n)\downarrow \Rightarrow \nu(f(n)) = \nu(\psi(n)),$$
$$\psi(n)\uparrow \Rightarrow \delta(f(n))\uparrow$$

for each integer n.

The following theorem provides the joint generalization of the two previous theorems.

Theorem 3.3. *Let ν be a precomplete numbering. Suppose that A is a c.e. set with $A <_T \emptyset'$, and suppose that δ is a partially A-computable function such that $\nu(\delta(n)) \neq \nu(n)$ for each $n \in \operatorname{dom}\delta$. Then for every partially computable function ψ there exists a computable function f such that*

$$\psi(n)\downarrow \Rightarrow \nu(f(n)) = \nu(\psi(n)), \tag{1}$$

$$\psi(n)\uparrow \Rightarrow \delta(f(n))\uparrow \tag{2}$$

for each integer n.

Proof. Let ν be a precomplete numbering. Fix a binary computable function h for which $\nu(h(e,y)) = \nu(\varphi_e(y))$ whenever $\varphi_e(y)\downarrow$. Since every precomplete numbering μ is *cylindric*, i.e., there exists a binary computable function g such that $g(x,u) \neq g(x,v)$ but $\mu(g(x,u)) = \mu(g(x,v))$ for all x and for all different u, v, we can assume that the function h is strictly increasing with respect to the second variable.

Suppose A is a c.e. set, and suppose that δ is a partially A-computable function for which $\nu(\delta(y)) \neq \nu(y)$ for each $y \in \operatorname{dom}\delta$ and the statement of the theorem does not hold, i.e., there exists a partially computable function ψ such that there is no computable function f satisfying, for each n, both (1) and (2). Without loss of generality, ψ is not total. To prove that $\emptyset' \leqslant_T A$, we define a Turing functional Θ such that $\emptyset'(x) = \Theta^A(x)$ for each x.

Fix an arbitrary integer x. To define the value $\Theta^A(x)$ of Θ at the pair $\langle A, x\rangle$, we construct some partially computable function ξ. Using the Recursion Theorem we initially fix an index p such that $\xi = \varphi_p$. Let f be a computable function defined by $f(n) = h(p, n)$.

Construction

Stage 0. Let $\xi_0(n)\uparrow$ for each n. Fix an index z such that $\delta = \Phi_z^A$. For all integers s and n, we let $\delta_s(n) = \Phi_{z,s}^{A_s}(n)$. We assume that $\Phi_{z,s}^{A_s}(n)\uparrow$ whenever $s \leqslant n$.

Stage $s+1$. First, we will assume that $x \notin \emptyset'_s$. Let

$$n_0 < n_1 < \cdots < n_m$$

be all the integers $n \leqslant s$ with

(3) $$\delta_{s+1}(f(n))\downarrow \,\&\, \xi_s(n)\uparrow .$$

If such sequence exists, then, for every $i \leqslant m$, we choose the least integer t_i such that

(4) $$A_{t_i} \upharpoonright \mathrm{use}(A_{t_i}; z, f(n_i), t_i) = A_{s+1} \upharpoonright \mathrm{use}(A_{t_i}; z, f(n_i), t_i).$$

Then, we choose the least $k \leqslant m$ for which

$$t_k = \min\{t_i : i \leqslant m\}$$

and define

(5) $$\Theta_{s+1}^{A_{s+1}}(x) = 0,$$

(6) $$\theta^{A_{s+1}}(x, s+1) = \mathrm{use}(A_{t_k}; z, f(n_k), t_k),$$

where θ is the use-function of the functional Θ. For every $n \leqslant s$ for which $\psi_s(n)\downarrow$ and $\xi_s(n)\uparrow$ (except for $n = n_k$ if it exists), we define

(7) $$\xi_{s+1}(n) = \psi_s(n).$$

Now suppose that $x \in \emptyset'_s$. If $\Theta_s^{A_s}(x)\uparrow$, then we define $\Theta^A(x) = 1$ and interrupt the construction. If $\Theta_s^{A_s}(x)\downarrow$, then, according to the above, we can take an n such that $\delta_s(f(n))\downarrow$, $\xi_s(n)\uparrow$, and

$$\theta^{A_s}(x, s) = \mathrm{use}(A_s; z, f(n), s).$$

Define

$$\xi_{s+1}(n) = \delta_s(f(n)).$$

Then there exists a $t > s$ such that

$$A_s \upharpoonright \mathrm{use}(A_s; z, f(n), s) \neq A_t \upharpoonright \mathrm{use}(A_s; z, f(n), s).$$

Indeed, assume that the required t does not exist. Then

$$\nu(\delta(f(n))) = \nu(\delta_s(f(n))) = \nu(\xi(n)) = \nu(\varphi_p(n)) = \nu(h(p, n)) = \nu(f(n)).$$

This is a contradiction, because $\nu(\delta(y)) \neq \nu(y)$ for each $y \in \operatorname{dom}\delta$. Thus, as in the previous case, we can define $\Theta^A(x) = 1$. Then we also itterrupt the construction.

End of construction

It follows directly from the construction that if $x \in \emptyset'$, then $\Theta^A(x) = 1$. Suppose $x \notin \emptyset'$. To prove that $\Theta^A(x)\downarrow= 0$, taking into account checks (3), (4) and assignments (5), (6), it suffices to show that there exists an n such that $\delta(f(n))\downarrow$ and $\psi(n)\uparrow$. Toward a contradiction, assume that such n does not exist. By assignments (7), we have

$$\psi(n)\downarrow \Rightarrow \xi(n)\downarrow \,\&\, \nu(\psi(n)) = \nu(\xi(n)) = \nu(\varphi_p(n)) = \nu(h(p,n)) = \nu(f(n))$$

for each n. Since, for every n,

$$\psi(n)\uparrow \Rightarrow \delta(f(n))\uparrow,$$

the function f satisfies both (1) and (2). This contradiction completes the proof of the theorem. ⊣

§4. Completions and minimal covers. The paper by Badaev, Goncharov, and Sorbi [7] raised the question of the existence of Σ^0_n-computable numberings ($n \geqslant 1$) for which their completions are their own minimal covers. It was proved in [8] that for every $n \geqslant 1$ there exist a family $\mathcal{R}$ and its Σ^0_n-computable numbering μ such that for some $a \in \mathcal{R}$ there is a numbering ν with $\mu < \nu < \mu_a$ (moreover, the upper semilattice of the equivalence classes of Σ^0_{n+1}-computable numberings of $\mathcal{R}$ in the restriction to the segment $[\mu, \mu_a]$ is isomorphic to the semilattice of c.e. m-degrees). Let us show that if the numbering μ of a non-singleton set S is positive, then for any $a \in S$ there exists a numbering ν such that $\mu < \nu < \mu^{\mathbf{0}}_a$.

THEOREM 4.1. *Suppose that μ is a numbering of a non-singleton set S, and suppose that a is an element of S such that $\mu^{-1}(a)$ is c.e. Then there exists a numbering ν with $\mu < \nu < \mu_a$.*

PROOF. In order to prove the theorem, it is sufficient to define a numbering $\alpha \leqslant \mu_a$ such that $\alpha \nleqslant \mu$ and $\mu_a \nleqslant \alpha$. Then we will set $\nu = \mu \oplus \alpha$. It is not hard to show that the condition $\mu < \mu \oplus \alpha \leqslant \mu_a$ will then hold. By Mal'tsev's result [27], for every complete numbering of a non-singleton set, the set of numbers of its special element is productive. Since $\mu^{-1}(a)$ is c.e. and μ_a is complete with respect to a, we have $\mu < \mu_a$. According to [14], complete numberings are not splittable into non-trivial direct sums of numberings. Hence, $\nu = \mu \oplus \alpha < \mu_a$.

To define the numbering α, we construct a partially computable function ψ. Using the Double Recursion Theorem we initially fix an index n such that

$$\psi = \varphi_n.$$

The numbering α will be defined by the equality

$$\alpha(x) = \mu_a(c(n, x))$$

for each x. To meet the condition $\mu_a \nleqslant \alpha$, we also construct a partially computable function ξ. Using the Double Recursion Theorem we initially fix an index k such that

$$\xi = \varphi_k.$$

Fix a number m for which $\mu(m) \neq a$. We are going to define the partial function ψ in such a way that $\operatorname{ran}\psi = \{m\}$.

Construction

Stage 0. Let $\psi_0(x)\uparrow$ and $\xi_0(x)\uparrow$ for each x. To meet the conditions $\alpha \nleqslant \mu$ and $\mu_a \nleqslant \alpha$, in the construction, we will define standard for finite injury priority constructions binary computable restraint functions r and p. For every e, we set

$$r(e,0) = p(e,0) = c(e,0).$$

At each subsequent stage $s+1$, we will assume that $p(e,s+1) = r(e,s)$ and $p(e,s+1) = p(e,s)$ for each e unless explicitly stated otherwise. Let $\{A_s\}_{s\in\mathbb{N}}$ be a computable enumeration of the c.e. set $A = \mu^{-1}(a)$.

Stage $s+1 = 2c(e,u)+1$. In these stages, we meet the condition $\alpha \nleqslant \mu$. If one of the following conditions is true:

- $\psi_s(r(e,s))\downarrow$;
- $\varphi_{e,s}(r(e,s))\uparrow$;
- $\varphi_{e,s}(r(e,s))\downarrow\notin A_s$;

then we go to the next stage. Otherwise, we define

$$\psi_{s+1}(r(e,s)) = m,$$
$$r(i,s+1) = p(i-1,s+1) = c(i,s+1)$$

for each $i > e$.

Stage $s+1 = 2c(e,u)+2$. In these stages, we meet the condition $\mu_a \nleqslant \alpha$. If one of the following conditions is true:

- there exist $x < y \leqslant s$ such that
$$\varphi_{e,s}(c(k,e,x))\downarrow = \varphi_{e,s}(c(k,e,y))\downarrow,$$
$$\xi_s(c(e,x))\downarrow \;\&\; \xi_s(c(e,y))\uparrow;$$
- there exists an $x \leqslant s$ for which
$$\varphi_{e,s}(c(k,e,x))\downarrow < p(e,s),$$
$$\xi(c(e,x))\uparrow \;\&\; \psi_s(\varphi_{e,s}(c(k,e,x))) = m,$$

then we go to the next stage. Otherwise, we consider three cases.

i) There exist $x < y \leqslant s$ such that

$$\varphi_{e,s}(c(k,e,x))\downarrow = \varphi_{e,s}(c(k,e,y))\downarrow,$$
$$\xi_s(c(e,x))\uparrow \;\&\; \xi_s(c(e,y))\uparrow.$$

In this case, we define $\xi_{s+1}(c(e,x)) = m$.

ii) Case i) is not satisfied, but there exists an $x \leqslant s$ for which

$$\varphi_{e,s}(c(k,e,x))\downarrow \geqslant p(e,s) \;\&\; \psi_s(\varphi_e(c(k,e,x)))\uparrow,$$
$$\xi_s(c(e,x))\uparrow.$$

In this case, we define

$$\psi_{s+1}(\varphi_e(c(k,e,x))) = m,$$
$$r(i+1,s+1) = p(i,s+1) = c(i+1,s+1)$$

for each $i \geqslant e$ (recall that we are doing $\operatorname{ran}\psi = \{m\}$). Since $w < s$ whenever $\varphi_{j,s}(z)\downarrow= w$ $(j,z,w \in \mathbb{N})$ and $p(e,s) > r(e,s)$, we have

$$r(e+1,s+1) > \varphi_{e,s}(c(k,e,x)) \geqslant p(e,s) > r(e,s).$$

iii) If cases i) and ii) are not satisfied, then we go to the next stage.

End of construction

A standard priority argument shows that for every e there exist finite limits $\lim_s r(e,s)$ and $\lim_s p(e,s)$. It follows directly from the construction, that $\operatorname{ran}\psi = \{m\}$.

LEMMA 1. *$\alpha \not\leqslant \mu$.*

PROOF. Choose an arbitrary e such that the partial function φ_e is total. We are going to show that $\alpha \neq \mu \circ \varphi_e$. Fix an s such that $r(e,s) = r(e,t)$ for each $t \geqslant s$.

If $\psi(r(e,s))\downarrow$, then by the construction, $\varphi_e(r(e,s))\downarrow\in A$. Therefore,

$$\mu(\varphi_e(r(e,s))) = a \neq \mu(m) = \mu(\psi(r(e,s))) =$$
$$= \mu(\varphi_n(r(e,s))) = \mu_a(c(n,r(e,s))) = \alpha(r(e,s)).$$

Hence, $\alpha \neq \mu \circ \varphi_e$.

If $\psi(r(e,s))\uparrow$, then $\varphi_e(r(e,s))\downarrow\notin A$ and

$$\mu(\varphi_e(r(e,s))) \neq a = \mu_a(c(n,r(e,s))) = \alpha(r(e,s)).$$

Thus again, $\alpha \neq \mu\circ\varphi_e$. Since the choice of e is arbitrary, the lemma is proved. ⊣

LEMMA 2. *$\mu_a \not\leqslant \alpha$.*

PROOF. Choose an arbitrary e such that φ_e is total. Let us prove that $\mu_a \neq \alpha \circ \varphi_e$. Fix an s such that $p(e,s) = p(e,t)$ for each $t \geqslant s$.

If there exist $x < y$ such that

$$\varphi_e(c(k,e,x))\downarrow= \varphi_e(c(k,e,y))\downarrow,$$
$$\xi(c(e,x)) = m \,\&\, \xi(c(e,y))\uparrow,$$

then

$$\mu_a(c(k,e,x)) = \mu(\xi(c(e,x))) = \mu(m) \neq a = \mu_a(c(k,e,y)).$$

Hence, $\mu_a \neq \alpha \circ \varphi_e$.

If such $x < y$ do not exist, then there exists an x for which

$$\varphi_e(c(k,e,x))\downarrow> p(e,s) \,\&\, \xi(c(e,x))\uparrow.$$

By the construction, we have $\psi(\varphi_e(c(k,e,x))) = m$. Hence,

$$\alpha(\varphi_e(c(k,e,x))) = \mu_a(c(n,\varphi_e(c(k,e,x)))) =$$
$$= \mu(\psi(\varphi_e((c(k,e,x))))) = \mu(m) \neq a = \mu_a(c(k,e,x)).$$

It follows that in this case again $\mu_a \neq \alpha \circ \varphi_e$. Since the choice of e is arbitrary, the lemma is proved. ⊣

This contradiction completes the proof of the theorem. $\dashv$

Corollary 4.2. *Suppose that μ is a positive numbering of a non-singleton set S. Then for every $a \in S$ there exists a numbering ν such that $\mu < \nu < \mu_a$.*

Fix an integer $n \geqslant 2$. The question of the existence of arbitrary minimal covers of Σ^0_n-computable numberings was studied in Badaev and Podzorov's paper [9]. Some important sufficient conditions for the existence of minimal covers are given there, but in the general case the question remains open. The following theorem shows that for Σ^0_n-computable numberings of non-principal families minimal covers always exist and, moreover, they can be chosen to satisfy the Recursion Theorem.

Theorem 4.3. *Let $\emptyset' \leqslant_T A$ and let $\mathcal{R}$ be an A-computable non-principal family. Then every A-computable numbering μ of $\mathcal{R}$ has an A-computable minimal cover ν such that for each computable function f there exists an integer n with $\nu(f(n)) = \nu(n)$.*

Proof. First, we show that for every low$_2$ set $Y \leqslant_T \emptyset'$ and every A-computable numbering μ of an A-computable non-principal family there exists its A-computable numbering α such that

$$\alpha \neq \mu \circ g$$

for each function $g \leqslant_T Y$. Since

$$Y <_T \emptyset' <_T \emptyset'' \equiv_T Y'',$$

$\emptyset'$ is high over Y. Therefore, there exists an $\emptyset'$-computable sequence $\{g_n\}_{n\in\mathbb{N}}$ of all Y-computable functions (cf., e.g., [24, 33]). If for every A-computable numbering α of $\mathcal{R}$ there exists a function $g \leqslant_T Y$ such that $\alpha = \mu \circ g$, then the A-computable numbering

$$c(n, x) \mapsto \mu(g_n(x))$$

of $\mathcal{R}$ is principal. This is a contradiction.

It was proved in [17] that for every non-computable c.e. set Y there exists a c.e. equivalence relation $\eta \leqslant_T Y$ on $\mathbb{N}$ such that all the η-equivalence classes are finite and, for each e, if $\operatorname{ran}\varphi_e$ is infinite, then

$$\mathbb{N}_{/\eta} =^* \{[\varphi_e(x)]_\eta : x \in \mathbb{N},\ \varphi_e(x)\downarrow\}, \tag{8}$$

where for any sets X and Z the notation $X =^* Z$ means that their symmetric difference is finite.

Let $Y \leqslant_T \emptyset'$ be a non-computable low$_2$ c.e. set and let

$$a_0 < a_1 < a_2 < \dots$$

be a Y-computable sequence such that $\langle a_i, a_j\rangle \notin \eta$ for all different i, j and

$$\mathbb{N}_{/\eta} = \{[a_i]_\eta : i \in \mathbb{N}\}.$$

To define a numbering ν that satisfies the conclusion of the theorem, we construct some numbering β and then set

$$\nu = \beta \oplus \mu.$$

For all x and i we define

$$\beta_0(x) = \alpha(i)$$

whenever $x \in [a_{2i+1}]_\eta$. Since $\alpha \neq \mu \circ g$ for each function $g \leqslant_T Y$, we will have $\beta \nleqslant \mu$. Hence,

$$\mu < \nu.$$

To define a partial mapping β_{s+1}, we choose the least i such that $\beta_s(x)$ is undefined for each $x \in [a_{2i}]_\eta$. We consider several cases.

i) If $\varphi_s(2a_{2i})\uparrow$, then we define

$$\beta_{s+1}(x) = \alpha(0)$$

for each $x \in [a_{2i}]_\eta$.

ii) If $\varphi_s(2a_{2i})\downarrow$ is odd, then we define

$$\beta_{s+1}(x) = \mu\left(\frac{\varphi_s(2a_{2i}) - 1}{2}\right)$$

for each $x \in [a_{2i}]_\eta$. Thus, we have

$$\nu(2a_{2i}) = \beta(a_{2i}) = \mu\left(\frac{\varphi_s(2a_{2i}) - 1}{2}\right) = \nu(\varphi_s(2a_{2i})).$$

iii) If $\varphi_s(2a_{2i})\downarrow$ is even and $\beta_s\left(\frac{\varphi_s(2a_{2i})}{2}\right)$ is defined, then we define

$$\beta_{s+1}(x) = \beta_s\left(\frac{\varphi_s(2a_{2i})}{2}\right)$$

for each $x \in [a_{2i}]_\eta$. We have

$$\nu(2a_{2i}) = \beta(a_{2i}) = \beta\left(\frac{\varphi_s(2a_{2i})}{2}\right) = \nu(\varphi_s(2a_{2i})). \tag{9}$$

iv) If $\varphi_s(2a_{2i})\downarrow$ is even and $\beta_s\left(\frac{\varphi_s(2a_{2i})}{2}\right)$ is undefined, then we fix a k such that $\left\langle a_{2k}, \frac{\varphi_s(2a_{2i})}{2}\right\rangle \in \eta$ and define

$$\beta_{s+1}(x) = \alpha(0)$$

for each $x \in \bigcup_{j=i}^{k} [a_{2j}]_\eta$. In this case, we also have that (9) hold.

Thus, for each computable function f there exists an integer n such that $\nu(f(n)) = \nu(n)$.

It remains to show that ν is a minimal cover of μ. Let γ be a numbering of $\mathcal{R}$ such that

$$\mu \leqslant \gamma \leqslant \nu.$$

Fix an index e such that φ_e is total and $\gamma = \nu \circ \varphi_e$. Since $\nu = \beta \oplus \mu$, we have $\gamma \leqslant \mu$ if $\operatorname{ran} \varphi_e$ contains only a finite number of even integers. Otherwise, since η is c.e. and $\beta(x) = \beta(y)$ for all x, y with $\langle x, y\rangle \in \eta$, it follows from (8) that $\beta \leqslant \gamma$.

$\dashv$

§5. Positive precomplete numberings. If a set A is Turing complete, then every A-computable family has a precomplete (and even complete) A-computable numbering (cf., e.g., [6]). Let us show that if an infinite A-computable family has a positive A-computable numbering, then it can also be chosen to be precomplete. Note that if a non-singleton set is finite, then any of its positive numberings is decidable (cf., e.g., [14]) and, therefore, is not precomplete (moreover, in decidable numberings, not all computable functions have fixed points).

Theorem 5.1. *For a set A the following statements are equivalent:*

1) $\emptyset' \leqslant_T A$;
2) *every infinite family possessing an A-computable numbering ν with $\eta_\nu \leqslant_T \emptyset'$ has a positive precomplete A-computable numbering;*
3) *every infinite family possessing a positive A-computable numbering has a positive precomplete A-computable numbering;*
4) *every family possessing a single-valued A-computable numbering has a positive precomplete A-computable numbering;*

Proof. Let us first prove the implication $(1 \Rightarrow 2)$. Suppose that $\emptyset' \leqslant_T A$ and suppose that $\mathcal{R}$ is an infinite family with positive A-computable numbering ν. Let η be the c.e. equivalence relation on $\mathbb{N}$ generated by the binary relation

$$\{\langle c(e,x), \varphi_e(x)\rangle \in \mathbb{N}\times\mathbb{N} : \varphi_e(x)\downarrow\}.$$

If φ_e is nowhere defined, then $\langle c(e,x), c(e,y)\rangle \notin \eta$ holds for all different x and y. Hence, the quotient set $\mathbb{N}_{/\eta}$ is infinite. Fix $\emptyset'$-computable sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\forall m \forall n\, [m \neq n \Rightarrow \langle y_m, y_n\rangle \notin \eta_\nu \,\&\, \langle z_m, z_n\rangle \notin \eta],$$

$$\mathbb{N}_{/\eta_\nu} = \{[y_n]_{\eta_\nu} : n \in \mathbb{N}\},\ \mathbb{N}_{/\theta} = \{[z_n]_\eta : n \in \mathbb{N}\}.$$

Now we define an A-computable numbering μ of the family $\mathcal{R}$ by letting

$$\mu(z) = \nu(y_n)$$

for all $n \in \mathbb{N}$ and $z \in [z_n]_\eta$. Since

$$\mu(x) = \mu(y) \Leftrightarrow \langle x, y\rangle \in \eta$$

for all x, y, the numbering μ is positive. For every partially computable function φ_e we have

$$\forall x\, [\varphi_e(x)\downarrow \Rightarrow \mu(c(e,x)) = \mu(\varphi_e(x))].$$

Hence, μ is precomplete.

Since $\eta_\nu \leqslant_T \emptyset'$ for each positive numbering ν, we have the implication $(2 \Rightarrow 3)$. The implication $(3 \Rightarrow 4)$ is obvious.

We now prove the implication $(4 \Rightarrow 1)$. Let μ be a positive precomplete A-computable numbering of the family

$$\mathcal{R} = \{\{x\} : x \in \mathbb{N}\}.$$

Let $\{\eta_\mu^s\}_{s\in\mathbb{N}}$ be a computable enumeration of the relation η_μ. Without loss of generality, we assume that $\mu(0) \neq \mu(1)$. Fix an x_0 such that $\mu(0) = \{x_0\}$.

Since μ is precomplete, we can choose a binary computable function f such that

$$\mu(f(e,x)) = \mu(\varphi_e(x))$$

whenever $\varphi_e(x)\downarrow$. To prove that $\emptyset' \leqslant_T A$, we need a partially computable function ψ defined as follows:

- using the Recursion Theorem we initially fix an index n such that $\psi = \varphi_n$;
- for every x, we let

$$\psi(x) = \begin{cases} 0, & \text{if } \exists s\, [x \in \emptyset'_{s+1} \setminus \emptyset'_s \,\&\, \langle f(n,x), 0\rangle \notin \eta^s_\mu], \\ 1, & \text{if } \exists s\, [x \notin \emptyset'_s \,\&\, \langle f(n,x), 0\rangle \in \eta^s_\mu], \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Since $\mu(0) \neq \mu(1)$ and

$$\forall x\, [\psi(x) = \varphi_n(x) = 1 \Rightarrow \mu(1) = \mu(\varphi_n(x)) = \mu(f(n,x)) = \mu(0)],$$

we have $\psi(x) \neq 1$ for each x. It follows that

$$\forall s\, [x \notin \emptyset'_s \Rightarrow \langle f(n,x), 0\rangle \notin \eta^s_\mu] \tag{10}$$

for each x. Let us show that

$$\mu(f(n,x)) \neq \mu(0)$$

whenever $\psi(x)\uparrow$. Toward a contradiction, assume that there exists an x such that $\psi(x)\uparrow$ and

$$\langle f(n,x), 0\rangle \in \eta_\mu.$$

If $x \notin \emptyset'$, then $\psi(x)\downarrow = 1$. This is a contradiction. Assume that $x \in \emptyset'$. Fix an s such that $x \in \emptyset'_{s+1} \setminus \emptyset'_s$. Then $\psi(x)\downarrow = 0$ if

$$\langle f(n,x), 0\rangle \notin \eta^s_\mu,$$

and $\psi(x)\downarrow = 1$ otherwise. We again obtain a contradiction with $\psi(x)\uparrow$.

Now, for each x, taking into account (10), we have

$$x \in \emptyset' \Leftrightarrow \exists s\, [x \in \emptyset'_{s+1} \setminus \emptyset'_s \,\&\, \langle f(n,x), 0\rangle \notin \eta^s_\mu] \Leftrightarrow$$

$$\Leftrightarrow \mu(f(n,x)) = \mu(\psi(x)) = \mu(0) \Leftrightarrow x_0 \in \mu(f(n,x)).$$

Therefore, $\emptyset' \leqslant_T A$. $\dashv$

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