Seven Internal Contradictions of Set Theory

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Abstract: Seven internal contradictions of set theory are discussed.

1. Scrooge McDuck's bankrupt

Scrooge Mc Duck earns 1000 $ daily and spends only 1 $ per day. As a cartoon-figure he will live forever and his wealth will increase without bound. But according to set theory he will go bankrupt if he spends the dollars in the same order as he receives them. Only if he always spends them in another order, for instance every day the second dollar received, he will get rich. These different results prove set theory to be useless for all practical purposes.

The above story is only the story of Tristram Shandy in simplified terms, which has been narrated by Fraenkel, one of the fathers of ZF set theory.

"Well known is the story of Tristram Shandy who undertakes to write his biography, in fact so pedantically, that the description of each day takes him a full year. Of course he will never get ready if continuing that way. But if he lived infinitely long (for instance a 'countable infinity' of years [...]), then his biography would get 'ready', because, expressed more precisely, *every* day of his life, how late ever, finally would get its description because the year scheduled for this work would some time appear in his life." [A. Fraenkel: "Einleitung in die Mengenlehre", 3rd ed., Springer, Berlin (1928) p. 24] "If he is mortal he can never terminate; but did he live forever then no part of his biography would remain unwritten, for to each day of his life a year devoted to that day's description would correspond." [A.A. Fraenkel, A. Levy: "Abstract set theory", 4th ed., North Holland, Amsterdam (1976) p. 30]

2. Failed enumeration of the fractions

All natural numbers are said to be enough to index all positive fractions. This can be disproved when the natural numbers are taken from the first column of the matrix of all positive fractions

1/1, 1/2, 1/3, 1/4, ...
2/1, 2/2, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...
4/1, 4/2, 4/3, 4/4, ...
... .

To cover the whole matrix by the integer fractions amounts to the idea that the letters X in

XOOO...
XOOO...
XOOO...
XOOO...
...

can be redistributed to cover all positions by exchanging them with the letters O. (X and O must be exchanged because where an index has left, there is no index remaining.) But where should the O remain if not within the matrix at positions not covered by X?

3. Violation of translation invariance

Translation invariance is fundamental to every science. With $n, m \in \mathbb{N}$ and $q \in \{\mathbb{Q} \cap (0, 1]\}$ there is precisely the same number of rational points $n + q$ in $(n, n+1]$ as $m + q$ in $(m, m+1]$. However, half of all positive rational numbers of Cantor's enumeration

1/1, 1/2, 2/1, 1/3, 2/2, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, 2/4, 3/3, 4/2, 5/1, ...

are of the form $0 + q$ and lie in the first unit interval between 0 and 1. There are less rational points in (1, 2] but more than in (2, 3] and so on. ["What fraction of fractions does Cantor's famous sequence enumerate?", MathOverflow, Question 362791]

4. Violation of inclusion monotony

Every endsegment $E(n) = \{n, n+1, n+2, ...\}$ of natural numbers has an infinite intersection with all other infinite endsegments

$$\forall k \in \mathbb{N}_{\text{def}}: \cap\{E(1), E(2), ..., E(k)\} = E(k) \wedge |E(k)| = \aleph_0.$$

Set theory however comes to the conclusion that there are only infinite endsegments and that their intersection is empty. This violates the inclusion monotony of the endsegments according to which, as long as only non-empty endsegments are concerned, their intersection is non-empty. All infinite endsegments have an infinite set of natural numbers, remaining from $E(1) = \mathbb{N}$, in common.

5. Completed infinity implies a smallest unit fraction

The completed infinite, *das vollendete Unendliche* or *Vollendetunendliche* as Cantor called it [letter to Lipschitz (19 Nov. 1883) and E. Zermelo: "Georg Cantor – Gesammelte Abhandlungen mathematischen und philosophischen Inhalts", Springer, Berlin (1932) p. 391], is a prerequisite of set theory. All natural numbers are required to exist for counting and with them also all unit fractions. But *all* unit fractions $1/n$ have finite distances d_n from each other

$$\forall n \in \mathbb{N}: 1/n - 1/(n+1) = d_n > 0.$$

Therefore the function **N**umber of **U**nit **F**ractions between 0 and x, NUF(x), cannot be infinite for all $x > 0$. The modern claim

$$\forall x \in (0, 1]: \text{NUF}(x) = \aleph_0$$

is wrong. The function can increase only one by one because between two steps the increase pauses during the gap of more than one point between the unit fractions.

If every positive point has $\aleph_0$ unit fractions at its left-hand side, then there is no positive point with less than $\aleph_0$ unit fractions at its left-hand side, then all positive points have $\aleph_0$ unit fractions at their left-hand side, then the interval (0, 1] has $\aleph_0$ unit fractions at its left-hand side, then $\aleph_0$ unit fractions are negative. Contradiction.

6. There are more paths than nodes in the complete infinite Binary Tree

$$\begin{array}{ccccccc}
 & & & a_0. & & & \\
 & & / & & \backslash & & \\
 & a_1 & & & & a_2 & \\
 / & & \backslash & & / & & \backslash \\
a_3 & & a_4 & & a_5 & & a_6 \\
/\backslash & & /\backslash & & /\backslash & & /\backslash \\
a_7 \ldots . & & & & & &
\end{array}$$

Since each of n paths in the complete infinite Binary Tree contains at least one node a_k differing from all other paths, there are not less nodes than paths possible. Everything else would amount to having more houses than bricks.

7. The diagonal does not define a number

An irrational number is the limit of a sequence of rational numbers defined by a finite expression. The classic example is π as defined by the algorithm of Aristotle or the formulas by Wallis or Gregory-Leibniz, or Euler, and many more. An endless digit sequence without a finite definition of the digits however cannot define any real number [W. Mückenheim: "Sequence and Limits", Advances in Pure Mathematics 5,2 (2015)]. After every known digit almost all digits will follow. Therefore the digit sequence of Cantor's famous diagonal argument does not represent a real number.

Details about these and other clashes of set theory and further references are given in W. Mückenheim: "Transfinity - A Source Book" (2023).