Rediscovering Ramanujan

Edgar Valdebenito

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ABSTRACT: In this note, we revisit Ramanujan-type series for $1/\pi$.

Notations:

$\mathbb{N} = \{1, 2, 3, ...\}$; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{Z} = \{ ..., -3, -2, -1, 0, 1, 2, 3, ...\}$; $\mathbb{Z}^- = \mathbb{Z} - \mathbb{N}_0$; $\mathbb{Z}_0^- = \mathbb{Z} - \mathbb{N}$

$\mathbb{R}$ **real numbers ;** $\mathbb{C}$ **complex numbers**

The hypergeometric function is defined by:

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!} $$

where the Pochhammer symbol $(a)_n$ $(a, n \in \mathbb{C})$ is defined , in terms of Gamma function Γ , by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (a + n \in \mathbb{C} - \mathbb{Z}_0^- ,\ n \in \mathbb{C} - \{0\} ;\ a \in \mathbb{C} - \mathbb{Z}_0^-,\ n = 0) $$

$$ (a)_n = a(a+1)(a+2) \ldots (a+n-1) \ ,\ n \in \mathbb{N} ;\ (a)_0 = 1 ;\ (0)_0 = 1 $$

Here it is supposed that the variable x , the numerator parameters $a_1, ..., a_p$, and the denominator parameters $b_1, ..., b_q$ take on complex values, provided that

$$ b_k \in \mathbb{C} - \mathbb{Z}_0^- ;\ k = 1, ..., q . $$

Then, if a numerator parameter is in $\mathbb{Z}_0^-$, the series ${}_pF_q$ is found to terminate and becomes a polynomial in x .

With none of the numerator and denominator parameters being zero or a negative integer, the series ${}_pF_q$:

(i) diverges for all $x \in \mathbb{C} - \{0\}$, if $p > q + 1$;

(ii) converges for all $x \in \mathbb{C}$, if $p \leq q$;

(iii) converges for $|x| < 1$ and diverges for $|x| > 1$ if $p = q + 1$;

(iv) converges absolutely for $|x| = 1$, if $p = q + 1$ and $\Re(w) > 0$;

(v) converges conditionally for $|x| = 1$ $(x \neq 1)$, if $p = q + 1$ and $-1 < \Re(w) \leq 0$;

(vi) diverges for $|x| = 1$, if $p = q + 1$ and $\Re(w) \leq -1$.

where

$$w = \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k$$

Binomial Coefficient:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{(-1)^k\,(-n)_k}{k!}\ ,\ 0 \leq k \leq n$$

Floor function:

$$\lfloor x \rfloor = \max\,\{a \in \mathbb{Z}\ :\ a \leq x\}$$

Ceiling function:

$$\lceil x \rceil = \min\,\{a \in \mathbb{Z}\ :\ a \geq x\}$$

Integer Part function:

$$[x] = \begin{cases} \lfloor x \rfloor & x \geq 0 \\ \lceil x \rceil & x < 0 \end{cases}$$

1. Introduction. Ramanujan-Like Series

Equation (1):

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(1+6\,n)}{8^n} \left(\frac{(1/2)_n}{1_n}\right)^3 = \sum_{n=0}^{\infty} \frac{(-1)^n\,(1+6\,n)}{8^n} \left(\binom{2\,n}{n} 2^{-2\,n}\right)^3$$

where

$$\frac{(1/2)_n}{1_n} = \binom{2n}{n} 2^{-2n}$$

2. Elementary Transformation (of equation (1))

$$\frac{8}{3\pi} = \sum_{n=0}^{\infty} (-1)^n\, 2^{-5n} \sum_{k=0}^{[n/2]} (-2)^k (1 + 6k) \binom{2k}{k}^3 \binom{2n-4k}{n-2k}$$

$$\frac{8}{3\pi} = \sum_{n=0}^{\infty} (-1)^n\, 2^{-9n} \sum_{k=0}^{[n/2]} (-1)^k\, 2^{13k} (1 + 6n - 12k) \binom{2n-4k}{n-2k}^3 \binom{2k}{k}$$

3. Integer Sequences

$$f(n) = \sum_{k=0}^{[n/2]} (-2)^k (1 + 6k) \binom{2k}{k}^3 \binom{2n-4k}{n-2k}\ ,\ n = 0, 1, 2, 3, \ldots$$

$$f(n) = \{1, 2, -106, -204, 10\,630, 20\,476, -1\,155\,524, \ldots\}$$

$$g(n) = \sum_{k=0}^{[n/2]} (-1)^k\, 2^{13k} (1 + 6n - 12k) \binom{2n-4k}{n-2k}^3 \binom{2k}{k}\ ,\ n = 0, 1, 2, 3, \ldots$$

$$g(n) = \{1, 56, -13\,576, -765\,504, 365\,221\,912, 20\,554\,303\,552, \ldots\}$$

4. Hypergeometric Relations for $f(n)$

$$f(2n) = \binom{4n}{2n} {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} - n, -n; 1, 1, \frac{1}{4} - n, \frac{3}{4} - n; -8\right) - 96 \binom{4n-4}{2n-2} {}_5F_4\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1 - n, \frac{3}{2} - n; 2, 2, \frac{5}{4} - n, \frac{7}{4} - n; -8\right)$$

$$f(2n+1) = \binom{4n+2}{2n+1} {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} - n, -n; 1, 1, -\frac{1}{4} - n, \frac{1}{4} - n; -8\right) - 96 \binom{4n-2}{2n-1} {}_5F_4\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2} - n, 1 - n; 2, 2, \frac{3}{4} - n, \frac{5}{4} - n; -8\right)$$

5. Hypergeometric Relations for $g(n)$

$$g(2n) = (1+12n)\binom{4n}{2n}^3 {}_7F_6\left(\frac{1}{2}, \frac{1}{2}-n, \frac{1}{2}-n, -\frac{1}{2}-n, -n, -n, -n; \frac{1}{4}-n, \frac{1}{4}-n, \frac{1}{4}-n, \frac{3}{4}-n, \frac{3}{4}-n, \frac{3}{4}-n; -8\right) + 196\,608\binom{4n-4}{2n-2}^3 {}_7F_6\left(\frac{3}{2}, 1-n, 1-n, 1-n, \frac{3}{2}-n, \frac{3}{2}-n, \frac{3}{2}-n; \frac{5}{4}-n, \frac{5}{4}-n, \frac{5}{4}-n, \frac{7}{4}-n, \frac{7}{4}-n, \frac{7}{4}-n; -8\right)$$

$$g(2n+1) = (7+12n)\binom{4n+2}{2n+1}^3 {}_7F_6\left(\frac{1}{2}, -\frac{1}{2}-n, -\frac{1}{2}-n, -\frac{1}{2}-n, -n, -n, -n; -\frac{1}{4}-n, -\frac{1}{4}-n, -\frac{1}{4}-n, \frac{1}{4}-n, \frac{1}{4}-n, \frac{1}{4}-n; -8\right) + 196\,608\binom{4n-2}{2n-1}^3 {}_7F_6\left(\frac{3}{2}, \frac{1}{2}-n, \frac{1}{2}-n, \frac{1}{2}-n, 1-n, 1-n, 1-n; \frac{3}{4}-n, \frac{3}{4}-n, \frac{3}{4}-n, \frac{5}{4}-n, \frac{5}{4}-n, \frac{5}{4}-n; -8\right)$$

6. Other Formulas

Entry 1.

$$\frac{3}{\pi} = \sum_{n=0}^{\infty}(-1)^n 2^{-5n} \sum_{k=0}^{[n/2]} \frac{(-2)^k (1+6k)}{1-2n+4k}\binom{2n-4k}{n-2k}\binom{2k}{k}^3$$

Entry 2.

$$\frac{32}{\pi\sqrt{129}} = \sum_{n=0}^{\infty}(-1)^n 2^{-9n} \sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}^3 (1+6k)$$

$$\frac{32}{\pi\sqrt{129}} = \sum_{n=0}^{\infty}(-1)^n 2^{-9n}\binom{2n}{n} {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6}, -n; \frac{1}{6}, 1, 1, \frac{1}{2}-n; 16\right)$$

$$\frac{32}{\pi\sqrt{127}} = \sum_{n=0}^{\infty}2^{-9n} \sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{2k}{k}^3 (1+6k)$$

$$\frac{32}{\pi\sqrt{127}} = 1 + \sum_{n=1}^{\infty} 2^{-9n} \left(\binom{2n}{n} {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -n; 1, 1, \frac{1}{2} - n; -16\right) - 48 \binom{2n-2}{n-1} {}_4F_3\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1-n; 2, 2, \frac{3}{2} - n; -16\right)\right)$$

Entry 3.

$$\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} 2^{-12\,n!} \sum_{k=0}^{n \cdot n! - 1} \binom{2k + 2n!}{k + n!}^3 \frac{(42k + 42n! + 5)}{2^{12k}}$$

$$\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} 2^{-12\,n!} \binom{2n!}{n!}^3 \sum_{k=0}^{n \cdot n! - 1} \left(\frac{(2n! + 1)_{2k}}{(n! + 1)_k^2}\right)^3 \frac{(42k + 42n! + 5)}{2^{12k}}$$

Entry 4.

$$\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} 2^{-12n^2} \sum_{k=0}^{2n} \binom{2k + 2n^2}{k + n^2}^3 \frac{\left(42k + 42n^2 + 5\right)}{2^{12k}}$$

$$\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} 2^{-12n^2} \binom{2n^2}{n^2}^3 \sum_{k=0}^{2n} \left(\frac{\left(2n^2 + 1\right)_{2k}}{\left(n^2 + 1\right)_k^2}\right)^3 \frac{\left(42k + 42n^2 + 5\right)}{2^{12k}}$$

Entry 5.

$$\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} 2^{-6n(n-1)} \sum_{k=1}^{n} \binom{2k + n^2 - n}{k + \frac{n(n-1)}{2}}^3 \frac{(42k + 21n(n-1) + 5)}{2^{12k}}$$

Entry 6.

$$\frac{16}{\pi} = 5 + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2^{k+1}(2n - 2k + 1)}{2^k(2n - 2k + 1)}^3 \frac{\left(2^{k+1}(42n - 42k + 21)\right) + 5}{2^{2^{k+2}(6n - 6k + 3)}}$$

$$\frac{1}{64\pi} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{2^{k+1}(2n+1) - 2}{2^k(2n+1) - 1}^3 \frac{3 \cdot 2^{k+1}(2n+1) - 5}{2^{2^{k+3}(2n+1)}}$$

Entry 7.

$$\frac{1}{32\pi} = \sum_{n=1}^{\infty} \binom{2n}{n}^3 \left(\frac{n}{2n-1}\right)^3 \frac{(42n - 37)}{2^{12n}}$$

$$\frac{1}{8\pi} = \sum_{n=1}^{\infty} \binom{2n}{n}^3 \left(\frac{n}{2n-1}\right)^3 \frac{(6n-5)}{2^{8n}}$$

Entry 8.

$$\frac{8}{\pi} = 3\,{}_4F_3\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 2\,; 1, 1, 1; -\frac{1}{4}\right) - \frac{51}{128}\,{}_3F_2\left(\frac{5}{4}, \frac{3}{2}, \frac{7}{4}\,; 2, 2; -\frac{1}{4}\right)$$

$$\frac{4}{\pi} = 6\,{}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2\,; 1, 1, 1; \frac{1}{4}\right) - 5\,{}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\,; 1, 1; \frac{1}{4}\right)$$

$$\frac{16}{\pi} = 42\,{}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2\,; 1, 1, 1; \frac{1}{64}\right) - 37\,{}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\,; 1, 1; \frac{1}{64}\right)$$

Entry 9.

$$\frac{4}{\pi} = \sum_{n=1}^{\infty} \frac{n}{2^{8n}} \binom{2n}{n}^3 \left(5 + 32\left(\frac{n}{2n-1}\right)^3\right)$$

$$\frac{16}{\pi} = \sum_{n=1}^{\infty} \frac{n}{2^{12n}} \binom{2n}{n}^3 \left(37 + 2560\left(\frac{n}{2n-1}\right)^3\right)$$

$$\frac{2}{\pi} = \sum_{n=1}^{\infty} \frac{1}{2^{8n}} \binom{2n}{n}^3 \left(3n + 16\left(\frac{n}{2n-1}\right)^3\right)$$

Entry 10.

$$\frac{63}{20\pi} = \sum_{n=0}^{\infty} (-1)^n\, 63^{-n}\;\; {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, \frac{47}{42}, \frac{(-1)^n}{2} - \left[\frac{n}{2}\right], -\left[\frac{n}{2}\right];\; \frac{5}{42}, 1, 1, 1;\; 64\right)$$

$$\frac{65}{20\pi} = \sum_{n=0}^{\infty} 65^{-n}\;\; {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, \frac{47}{42}, \frac{(-1)^n}{2} - \left[\frac{n}{2}\right], -\left[\frac{n}{2}\right];\; \frac{5}{42}, 1, 1, 1;\; 64\right)$$

Entry 11.

$$\frac{2^{16}}{4095\pi} = \sum_{n=0}^{\infty} \frac{42n+5}{2^{12n}} \sum_{k=0}^{n} \binom{2k}{k}^3 - \frac{8192}{798\,525} \left({}_2F_1\left(\frac{1}{2}, \frac{1}{2}\,; 1; \frac{8-3\sqrt{7}}{16}\right)\right)^2$$

$$\frac{2^{10}}{255\pi} = \sum_{n=0}^{\infty} \frac{6n+1}{2^{8n}} \sum_{k=0}^{n} \binom{2k}{k}^3 - \frac{512}{21\,675} \left({}_2F_1\left(\frac{1}{2}, \frac{1}{2}\,; 1; \frac{2-\sqrt{3}}{4}\right)\right)^2$$

Entry 12.

$$\frac{9\sqrt{3}}{4\pi} = \sum_{n=0}^{\infty} \frac{8n+1}{3^{2n}} \sum_{k=0}^{n} \frac{(4k)!}{4^{4k}(k!)^4} - \frac{9}{8}\, {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{9}\right)$$

$$\frac{4608\sqrt{3}}{2303\pi} = \sum_{n=0}^{\infty} \frac{8n+1}{3^{2n} \cdot 4^{4n}} \sum_{k=0}^{n} \frac{(4k)!}{(k!)^4} - \frac{18\,432}{5\,303\,809}\, {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{9}\right)$$

Entry 13.

$$\frac{16}{\pi} + 21 = 26\left({}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{8-3\sqrt{7}}{16}\right)\right)^2 +$$

$$\frac{21}{512}\, {}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1, 2; \frac{1}{64}\right) - \frac{567}{2\,097\,152}\, {}_5F_4\left(2, 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}; 1, 3, 3, 3; \frac{1}{64}\right)$$

Entry 14.

$$\sum_{n=0}^{\infty} \frac{(1+n)(5+21n)}{2^{12n}} \binom{2n}{n}^3 = \frac{16}{\pi} + \sum_{n=0}^{\infty} \frac{(1+n)(5+21n)}{2^{12n+12}} \binom{2n+2}{n+1}^3$$

$$\sum_{n=0}^{\infty} \frac{(1+n)(1+3n)}{2^{8n}} \binom{2n}{n}^3 = \frac{4}{\pi} + \sum_{n=0}^{\infty} \frac{(1+n)(1+3n)}{2^{8n+8}} \binom{2n+2}{n+1}^3$$

$$\sum_{n=0}^{\infty} \frac{(1+n)(1+4n)}{3^{2n}} \left(\frac{(4n)!}{4^{4n}(n!)^4}\right) =$$

$$\frac{2\sqrt{3}}{\pi} + \sum_{n=0}^{\infty} \frac{(1+n)(1+4n)}{3^{2n+2}} \left(\frac{(4n+4)!}{4^{4n+4}((n+1)!)^4}\right)$$

Entry 15.

$$\frac{27}{8\pi} = \sum_{n=0}^{\infty} 6^{-2n} \sum_{k=0}^{[n/2]} \frac{(4k)!(10k+1)}{4^{4k}(k!)^4} \binom{2n-4k}{n-2k}$$

Entry 16.

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} 2^{-4n} \sum_{k=0}^{n} (6k+1) \left(\binom{2n-2k}{n-k} \binom{2k}{k} \right)^{3/2} - \sum_{n=0}^{\infty} \frac{3n+4}{2^{4n+3}} \sum_{k=0}^{[n/2]} \left(\binom{2n-2k+2}{n-k+1} \binom{2k}{k} \right)^{3/2}$$

Entry 17.

$$\frac{4}{\pi} = 1 + \sum_{n=0}^{\infty} \binom{2^{n+1}}{2^n}^3 \left(\frac{1 + 3 \cdot 2^{n+1}}{2^{2^{n+3}}} \right) + \sum_{n=1}^{\infty} 2^{-2^{n+3}} \sum_{k=1}^{2^n - 1} \binom{2^{n+1} + 2k}{2^n + k}^3 \left(\frac{1 + 3 \cdot 2^{n+1} + 2k}{2^{8k}} \right)$$

Entry 18.

$$\frac{4}{\pi} = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2^{-8k(n+1)} \left(2^{8k} (6nk+1) \binom{2nk}{nk}^3 - (6(n+1)k+1) \binom{2nk+2k}{nk+k}^3 \right)$$

Entry 19.

$$\frac{5}{\pi \sqrt{2}} = \sum_{n=0}^{\infty} 10^{-n} \sum_{k=0}^{[n/2]} \frac{(4k)!\,(10k+1)}{4^{4k}\,(k!)^4} \binom{n}{n-2k}$$

$$\frac{80}{\pi} = \sum_{n=0}^{\infty} 10^{-n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (4k)!\,(260k+23)}{2^{10k}\,(k!)^4} \binom{n}{n-2k}$$

Entry 20.

$$\frac{63}{4\pi} = 5 + \sum_{n=0}^{\infty} 2^{-12n-9} \binom{2n}{n}^3 \left(\frac{7 - 132n - 312n^2 - 168n^3}{(n+1)^3} \right)$$

Entry 21.

$$\frac{2560}{\pi} = \sum_{n=0}^{\infty} 10^{-n} \left(736\, {}_4F_3\left(\frac{1}{4}, \frac{3}{4}, \frac{(-1)^n}{2} - \left[\frac{n}{2}\right], -\left[\frac{n}{2}\right]; 1, 1, 1; -\frac{1}{4} \right) + \left((-1)^n 195 \left[\frac{n}{2}\right] - 390 \left(\left[\frac{n}{2}\right] \right)^2 \right) {}_4F_3\left(\frac{5}{4}, \frac{7}{4}, 1 - \frac{n}{2}, \frac{3-n}{2}; 2, 2, 2; -\frac{1}{4} \right) \right)$$

Entry 22.

$$\frac{6\sqrt{7}}{5\pi} = \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} (-63)^{-n} {}_4F_3\left(-n, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}; 1, 1, \frac{5}{42}; 1\right)$$

$$\frac{128}{15\sqrt{7}\,\pi} = \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-8n} {}_5F_4\left(-n, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}; \frac{1}{2} - n, 1, 1, \frac{5}{42}; 1\right)$$

Entry 23.

$$\frac{16}{5\pi} = 2\,{}_8F_7\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{47}{84}, \frac{89}{84}; \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{5}{84}, \frac{47}{84}; \frac{1}{2^{12}}\right) - {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}; 1, 1, \frac{5}{42}; -\frac{1}{2^6}\right)$$

$$\frac{16}{5\pi} = \frac{47}{1280}\,{}_8F_7\left(\frac{3}{4}, \frac{5}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4}, \frac{5}{4}, \frac{89}{84}, \frac{131}{84}; \frac{3}{2}, 1, \frac{3}{2}, 1, \frac{3}{2}, \frac{47}{84}, \frac{89}{84}; \frac{1}{2^{12}}\right) + {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}; 1, 1, \frac{5}{42}; -\frac{1}{2^6}\right)$$

Entry 24.

$$\frac{1}{\pi} = \frac{2^8}{2^{12}+1} \sum_{n=0}^{\infty} \left(2^{12}+1\right)^{-n} \sum_{k=0}^{n} (42k+5) \binom{2k}{k}^3 \binom{n}{n-k}$$

$$\frac{1}{\pi} = \frac{5\cdot 2^8}{2^{12}+1} \sum_{n=0}^{\infty} \left(2^{12}+1\right)^{-n} {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}, -n; \frac{5}{42}, 1, 1, 1; -64\right)$$

$$\frac{1}{\pi} = \frac{2^8}{2^{12}-1} \sum_{n=0}^{\infty} (-1)^n \left(2^{12}-1\right)^{-n} \sum_{k=0}^{n} (-1)^k (42k+5) \binom{2k}{k}^3 \binom{n}{n-k}$$

$$\frac{1}{\pi} = \frac{2^8}{2^{12}-1} \sum_{n=0}^{\infty} (-1)^n \left(2^{12}-1\right)^{-n} \left(5\,{}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -n; 1, 1, 1; 64\right) - 336\,n\,{}_4F_3\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1-n; 2, 2, 2; 64\right)\right)$$

Entry 25.

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} (-1)^n 2^{-12n} \cdot \sum_{k=0}^{[n/2]} 2^{12k} (42n - 42k + 5) \binom{2n-2k}{n-k}^2 \binom{2n-2k+1}{k}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} 2^{-12 n} \cdot \sum_{k=0}^{[n/2]} 2^{12 k} (42 n - 42 k + 5) \binom{2 n - 2 k}{n - k}^2 \binom{n - k}{k}^2$$

Entry 26.

$$\frac{16}{\pi} = 5 \sum_{n=0}^{\infty} \binom{2 n}{n}^3 2^{-12 n} + 42 \cdot \sum_{n=1}^{\infty} 2^{-12 n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{2 n - 2 k}{n - k}^3 2^{12 k}$$

$$\frac{16}{\pi} = 5 + \sum_{n=1}^{\infty} 2^{-12 n} \left(5 \binom{2 n}{n}^3 + 42 \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{2 n - 2 k}{n - k}^3 2^{12 k} \right)$$

Entry 27.

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} 2^{-6 n} \left(5 \sqrt{41} - 32\right)^n \sum_{k=0}^{[n/2]} (42 n - 42 k + 5) \binom{2 n - 2 k}{n - k}^3 \binom{n - k}{k}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} 2^{-4 n} \left(8 + \sqrt{65}\right)^{-n} \sum_{k=0}^{[n/2]} (6 n - 6 k + 1) \binom{2 n - 2 k}{n - k}^3 \binom{n - k}{k}$$

Entry 28.

$$\frac{4}{\pi} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{1}{4}\right) + 6 \sum_{n=0}^{\infty} \left({}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{1}{4}\right) - \sum_{k=0}^{n} \binom{2 k}{k}^3 2^{-8 k} \right)$$

$$\frac{16}{\pi} = 5\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{1}{64}\right) +$$
$$42 \sum_{n=0}^{\infty} \left({}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{1}{64}\right) - \sum_{k=0}^{n} \binom{2 k}{k}^3 2^{-12 k} \right)$$

Entry 29.

$$\frac{28\,900}{\pi} = 7424 + \sum_{n=0}^{\infty} \frac{199 + 170 n}{2^{8 n}} \left(\binom{2 n + 2}{n + 1}^3 - \binom{2 n}{n}^3 \right)$$

$$\frac{12\,776\,400}{\pi} = 4\,001\,792 + \sum_{n=0}^{\infty} \frac{9167 + 8190 n}{2^{12 n}} \left(\binom{2 n + 2}{n + 1}^3 - \binom{2 n}{n}^3 \right)$$

Entry 30.

$$\frac{4}{\pi} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; \frac{1}{4}\right) + \frac{9}{32}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}; 2, 3; \frac{1}{4}\right) - \frac{3}{32}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1, 3; \frac{1}{4}\right) - \frac{1}{64}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, 3; \frac{1}{4}\right)$$

$$\frac{16}{\pi} = 5\,{}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; \frac{1}{64}\right) + \frac{63}{512}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}; 2, 3; \frac{1}{64}\right) - \frac{21}{512}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1, 3; \frac{1}{64}\right) - \frac{5}{1024}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, 3; \frac{1}{64}\right)$$

Entry 31.

$$\frac{4}{\pi} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 2; \frac{1}{4}\right) + \frac{3}{32}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1, 3; \frac{1}{4}\right) + \frac{7}{64}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, 3; \frac{1}{4}\right)$$

$$\frac{16}{\pi} = 5\,{}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 2; \frac{1}{64}\right) + \frac{21}{512}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1, 3; \frac{1}{64}\right) + \frac{47}{1024}\,{}_3F_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, 3; \frac{1}{64}\right)$$

Entry 32.

$$\frac{2\sqrt{3}}{\pi} = {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{9}\right) + 8\sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^4\, 4^{4n} \cdot 3^{2n}}\,{}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; \frac{1}{9}\right)$$

$$\frac{9}{2\pi\sqrt{2}} = {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{81}\right) + 10\sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^4\, 4^{4n} \cdot 3^{4n}}\,{}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; \frac{1}{81}\right)$$

$$\frac{49}{3\pi\sqrt{3}} = 3\,{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{2401}\right) +$$

$$40\sum_{n=1}^{\infty}\frac{(4n)!}{(n!)^4\,4^{4n}\cdot 7^{4n}}\,{}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; \frac{1}{2401}\right)$$

$$\frac{8}{\pi} = 3\,{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; -\frac{1}{4}\right) +$$

$$20\sum_{n=1}^{\infty}\frac{(4n)!\,(-1)^n}{(n!)^4\,2^{10n}}\,{}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; -\frac{1}{4}\right)$$

Entry 33.

$$\frac{2\sqrt{3}}{\pi} = {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{3}\right) +$$

$$\sum_{n=1}^{\infty}\frac{(4n)!\,(22-16n)}{(n!)^4\,4^{4n}\cdot 3^{2n}}\,{}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; \frac{1}{3}\right)$$

$$\frac{9}{2\pi\sqrt{2}} = {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{9}\right) +$$

$$2\sum_{n=1}^{\infty}\frac{(4n)!\,(41-40n)}{(n!)^4\,4^{4n}\cdot 9^{2n}}\,{}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; \frac{1}{9}\right)$$

$$\frac{49}{3\pi\sqrt{3}} = 3\,{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{49}\right) + 8\sum_{n=1}^{\infty}\frac{(4n)!\,(227-240n)}{(n!)^4\,4^{4n}\cdot 7^{4n}}$$

$${}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; \frac{1}{49}\right)$$

$$\frac{8}{\pi} = 3\,{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; -\frac{1}{2}\right) + \sum_{n=1}^{\infty}\frac{(4n)!\,(-1)^n\,(37-20n)}{(n!)^4\,2^{10n}}$$

$${}_4F_3\left(1, \frac{1}{4}+n, \frac{1}{2}+n, \frac{3}{4}+n; 1+n, 1+n, 1+n; -\frac{1}{2}\right)$$

Entry 34.

$$\frac{16}{\pi} = \frac{32}{3\sqrt{7}} + \sum_{n=1}^{\infty}\frac{(6n+1)}{2^{8n}}\binom{2n}{n}^3\sum_{k=0}^{n-1}\left(\frac{27+128k+195k^2+90k^3}{(1+2k)^2(1+6k)(7+6k)}\right)\binom{2k}{k}^{-2}$$

$$\frac{64}{5\pi} =$$

$$\frac{128}{\sqrt{1023}} + \sum_{n=1}^{\infty} \frac{(42 n + 5)}{2^{12 n}} \binom{2 n}{n}^3 \sum_{k=0}^{n-1} \left(\frac{183 + 868 k + 1335 k^2 + 630 k^3}{(1 + 2 k)^2 (5 + 42 k) (47 + 42 k)} \right) \binom{2 k}{k}^{-2}$$

Entry 35.

$$\frac{16}{3\pi} = \sum_{n=0}^{\infty} \frac{(6 n - 1)}{2^{8 n}} \binom{2 n}{n}^3 {}_4F_3\left(1, -n, -n, -n\,;\, \frac{1}{2} - n, \frac{1}{2} - n, \frac{1}{2} - n; 1\right)$$

$$\frac{1024}{21\pi} = \sum_{n=0}^{\infty} \frac{(126 n + 13)}{2^{12 n}} \binom{2 n}{n}^3 {}_4F_3\left(1, -n, -n, -n\,;\, \frac{1}{2} - n, \frac{1}{2} - n, \frac{1}{2} - n; 1\right)$$

Entry 36.

$$\frac{1024}{3\pi} = \sum_{n=0}^{\infty} \sum_{k=\mathrm{Ceil}(n/2)}^{n} \frac{83 + 510 k}{2^{8 k}} \binom{2 n - 2 k}{n - k}^3$$

$$\frac{4608\sqrt{3}}{3\pi} = \sum_{n=0}^{\infty} \sum_{k=\mathrm{Ceil}(n/2)}^{n} \frac{2295 + 18\,424 k}{2304^k} \binom{2 n - 2 k}{n - k}^2 \binom{4 n - 4 k}{2 n - 2 k}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} (-1)^n 2^{-6 n} \sum_{k=\mathrm{Ceil}(n/2)}^{n} (-1)^k 2^{5 k} (4 n - 4 k + 1) \binom{2 n - 2 k}{n - k}^3 \binom{k}{n - k}$$

where Ceil(x) is the Ceiling function.

Entry 37.

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} 2^{-3 n} \binom{2 n}{n} {}_4F_3\left(-n, \frac{1}{2}, \frac{1}{2}, \frac{5}{4}\,;\, \frac{1}{4}, 1, 1; 1\right)$$

Entry 38.

$$\frac{16}{5\pi} = e^{-1} \sum_{n=0}^{\infty} \frac{1}{n!}\, {}_5F_3\left(-n, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}\,;\, \frac{5}{42}, 1, 1;\, -\frac{1}{64}\right)$$

$$\frac{16}{5\pi} = e \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, {}_5F_3\left(-n, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42}\,;\, \frac{5}{42}, 1, 1;\, \frac{1}{64}\right)$$

Entry 39.

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 2^{-12n} \left((42n+5)\, {}_4F_3 \left(1, \frac{1}{2}+n, \frac{1}{2}+n, \frac{1}{2}+n; 1+n, 1+n, 1+n; \frac{1}{64}\right) - n(21n-16) \right)$$

Entry 40.

$$\frac{2^{11}}{5\pi} = \frac{2^{14}\sqrt{15}}{375} - \frac{1}{25} \sum_{n=0}^{\infty} \left(64 \binom{2n}{n}^2 - \binom{2n+2}{n+1}^2 \right) \binom{2n+2}{n+1} 2^{-12n} \left(7 + 7n + {}_2F_1\left(1, \frac{3}{2}+n; 2+n; \frac{1}{16}\right)\right)$$

Entry 41. for $k = 1, 2, 3, 4, \ldots$, we have

$$\frac{2^{k+4}}{(2^k - 1)\pi} = \sum_{n=0}^{\infty} (42n+5)\, 2^{-12n} \binom{2n}{n}^3 {}_5F_4 \left(1, \frac{37}{42}, -n, -n, -n; -\frac{5}{42}-n, \frac{1}{2}-n, \frac{1}{2}-n, \frac{1}{2}-n; 2^{6-k}\right)$$

Entry 42. for $i = \sqrt{-1}$, we have

$$\frac{16}{\pi} = \frac{5}{2} + \int_0^{\infty} \left(\frac{\Gamma(1+2x)}{(\Gamma(1+x))^2}\right)^3 \left(\frac{42x+5}{2^{12x}}\right) dx + i \int_0^{\infty} \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}\, dx$$

where

$$f(x) = \left(\frac{\Gamma(1+2x)}{(\Gamma(1+x))^2}\right)^3 \left(\frac{42x+5}{2^{12x}}\right)$$

and $\Gamma(x)$ is the Gamma function.

7. References

[1] Ramanujan Aiyangar, Srinivasa: Notebooks, Bombay: Tata Institute of Fundamental Research, 1957.

[2] Ramanujan Aiyangar, Srinivasa: Modular Equations and Approximations to π, Quart. J. Pure. Appl. Math. 45, 350-372, 1913-1914.