Abraham Sharp's Formula

Edgar Valdebenito

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ABSTRACT: In 1699 , Abraham Sharp calculates π to 72 decimal places by taking $x = 1/\sqrt{3}$ in Gregory's series.

Key words: Sharp's pi formula , computational science , pi formulas, mathematical reasoning.

I. Introduction: Gregory's series

In mathematics, Gregory's series for the inverse tangent function is its infinite taylor series expansion at the origin:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \ |x| \leq 1, \ x \neq \pm i \, , \ i = \sqrt{-1} \tag{1}$$

It was firts discovered in the 14th century by Madhava of Sangamagrama (c. 1340 - c. 1425), as credited by Madhava's Kerala school follower Jyesthadeva Yuktibhasa (c. 1530). In recent literature it is sometimes called the Madhava-Gregory series to recognize Madhava's priority. It was independently rediscovered by James Gregory in 1671 and by Gottfried Leibniz in 1673, who obtained the Leibniz formula for Pi as the special case.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \tag{2}$$

By the beginning of the eighteenth century Abraham Sharp under the direction of the English astronomer and mathematician E. Halley had obtained 72 correct digits of Pi using Gregory's series with $x = 1/\sqrt{3}$, namely

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-n}}{2n+1} \tag{3}$$

In this note we give some series related to (3).

Remark:

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-n}}{2n+1} = 6\sqrt{3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^{-n}}{2n-1} = 6 \arctan\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{a_n}$$

where

$$a_{n+2} = 6\, a_{n+1} - 9\, a_n \quad , \ a_0 = 1 \, , \ a_1 = 9$$

Notations:

Binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \quad , \ n \geq k \quad , \ \binom{n}{k} = 0 \quad , \ n < k$$

Gauss hypergeometric function:

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad , \ |x| < 1 $$

where

$$ (a)_n = a(a+1)(a+2)\ldots(a+n-1) \ , \ (a)_0 = 1 $$

Generalized hypergeometric function:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!} \quad , \ |x| < 1 $$

Appell hypergeometric function:

$$ \text{F1}(a, b_1, b_2, c, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c)_{m+n} m! n!} x^m y^n $$

Floor function: Floor(x) = $\lfloor x \rfloor$

Ceiling function: Ceil(x) = Ceiling(x) = $\lceil x \rceil$

Integer Part: $[x]$

II. Series

Entry 1. For $s = 0, 1, 2, 3, \ldots$, we have

$$ \frac{\pi}{2\sqrt{3}} = \sum_{m=0}^{s} \frac{(-1)^m 3^{-m}}{2m+1} + \sum_{n=1}^{\infty} (-1)^{(s+1)n} \left(\frac{1}{3}\right)^{\frac{(2s+3)^n - 1}{2}} \sum_{k=0}^{(s+1)(2s+3)^n - 1} \frac{(-1)^k 3^{-k}}{(2s+3)^n + 2k} \tag{4} $$

Examples:

Example 1: $s = 0$

$$ \frac{\pi}{2\sqrt{3}} = 1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{3}\right)^{\frac{3^n - 1}{2}} \sum_{k=0}^{3^n - 1} \frac{(-1)^k 3^{-k}}{3^n + 2k} \tag{5} $$

Example 2: $s = 1$

$$ \frac{\pi}{2\sqrt{3}} = \frac{8}{9} + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{\frac{5^n - 1}{2}} \sum_{k=0}^{2 \cdot 5^n - 1} \frac{(-1)^k 3^{-k}}{5^n + 2k} \tag{6} $$

Example 3: $s = 2$

$$ \frac{\pi}{2\sqrt{3}} = \frac{41}{45} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{3}\right)^{\frac{7^n - 1}{2}} \sum_{k=0}^{3 \cdot 7^n - 1} \frac{(-1)^k 3^{-k}}{7^n + 2k} \tag{7} $$

Entry 2.

$$ \pi = \frac{3}{2} \cdot \sum_{n=0}^{\infty} 2^{-4n} \cdot \sum_{k=0}^{n} 2^{3k} \binom{2n-2k}{n-k} \sum_{m=0}^{k} \binom{k}{m} \frac{(-3)^{-m}}{2m+1} \tag{8} $$

$$\pi = \frac{3}{2} \cdot \sum_{n=0}^{\infty} 2^{-n} \cdot \sum_{k=0}^{n} 2^{-3k} \binom{2k}{k} \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{(-3)^{-m}}{2m+1} \tag{9}$$

Entry 3.

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n 3^{-n} \sum_{k=0}^{n} \frac{2^{-2k}}{2n-2k+1} \binom{2k}{k} \tag{10}$$

$$\pi = \sum_{n=0}^{\infty} (-1)^n 12^{-n} \binom{2n+2}{n+1} {}_3F_2\left(\frac{1}{2}, 1, \frac{3}{2}+n; \frac{3}{2}, 2+n; 1\right) \tag{11}$$

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n 12^{-n} \sum_{k=0}^{n} \frac{2^{2k}}{2k+1} \binom{2n-2k}{n-k} \tag{12}$$

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n 12^{-n} \binom{2n}{n} {}_3F_2\left(\frac{1}{2}, 1, -n; \frac{3}{2}, \frac{1}{2}-n; 1\right) \tag{13}$$

Entry 4.

$$\pi = 3 \sum_{n=0}^{\infty} (-1)^n 3^{-n} \sum_{k=0}^{n} \frac{2^{-2k}}{(2n-2k+1)(1-2k)} \binom{2k}{k} \tag{14}$$

$$\pi = 3 \sum_{n=0}^{\infty} (-1)^n 12^{-n} \sum_{k=0}^{n} \frac{2^{2k}}{(2k+1)(1-2n+2k)} \binom{2n-2k}{n-k} \tag{15}$$

Entry 5.

$$\pi = 3 \cdot \sum_{n=0}^{\infty} 2^{-4n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 2^{2k}}{(2k+1)\binom{2k}{k}} \tag{16}$$

$$\pi = 3 \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 2^{-2k}}{(2n-2k+1)\binom{2n-2k}{n-k}} \tag{17}$$

Entry 6.

$$\pi = 3 \sum_{n=0}^{\infty} \left(\frac{2-\sqrt{3}}{2}\right)^n \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)} \binom{n+k}{n-k} \left(\frac{2+\sqrt{3}}{2}\right)^k \tag{18}$$

$$\pi = 3 \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \sum_{k=0}^{n} \frac{(-1)^k}{(2n-2k+1)} \binom{2n-k}{k} \left(4-2\sqrt{3}\right)^k \tag{19}$$

$$\pi = 3 \sum_{n=0}^{\infty} (-1)^n \frac{2^{-2n}}{2n+1} {}_2F_1\left(-\frac{1}{2}-n, -n; -2n; 8\left(2-\sqrt{3}\right)\right) \tag{20}$$

Entry 7.

$$\frac{\pi}{6\sqrt{3}} = \sum_{n=1}^{\infty} (-1)^{n-1} 3^{-n^2} \sum_{k=0}^{2n} \frac{(-3)^{-k}}{2k+2n^2-1} \tag{21}$$

Entry 8.

$$\frac{\pi}{6\sqrt{3}} = \frac{8}{27} - \sum_{n=4}^{\infty} \left(-\frac{1}{3}\right)^{F_n} \sum_{k=0}^{F_{n-1}-1} \frac{(-3)^{-k}}{2k+2F_n-1} \tag{22}$$

Remark: F_n is the Fibonacci number:

$$F_{n+2} = F_{n+1} + F_n \ , \ F_1 = F_2 = 1$$

Entry 9.

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{3} - \sum_{n=2}^{\infty} 3^{-n!} \sum_{k=0}^{n \cdot n! - 1} \frac{(-3)^{-k}}{2k + 2n! - 1} \tag{23}$$

Entry 10.

$$\pi = \sqrt{3} \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \frac{(-3)^{-k}}{2k+1} \binom{n}{k} \tag{24}$$

$$\pi = \sqrt{3} \sum_{n=0}^{\infty} 2^{-n} \, {}_2F_1\left(\frac{1}{2}, -n ; \frac{3}{2} ; \frac{1}{3}\right) \tag{25}$$

$$\pi = \frac{3}{\sqrt{2}} \sum_{n=0}^{\infty} 2^{-n} \, {}_2F_1\left(\frac{1}{2}, n + \frac{3}{2} ; \frac{3}{2} ; -\frac{1}{2}\right) \tag{26}$$

$$\pi = \sqrt{3} \sum_{n=0}^{\infty} 3^{-n} \, {}_2F_1\left(-n, 1 ; \frac{3}{2} ; -\frac{1}{2}\right) \tag{27}$$

$$\pi = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} 3^{-n} \, {}_2F_1\left(1, n + \frac{3}{2} ; \frac{3}{2} ; \frac{1}{3}\right) \tag{28}$$

Entry 11.

$$\frac{2\pi}{\sqrt{3}} + \frac{1}{9} \, {}_2F_1\left(\frac{3}{4}, 1 ; \frac{7}{4} ; \frac{1}{36}\right) = \sum_{n=0}^{\infty} 2^{-2n} \sum_{k=0}^{2n} \left(2\binom{2n}{k} + \binom{2n+1}{k}\right) \frac{(-3)^{-k}}{2k+1} \tag{29}$$

Entry 12.

$$\frac{\pi}{6\sqrt{3}} = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{\frac{n(n-1)}{2}} \sum_{k=1}^{n} \frac{(-1)^{k-1} \, 3^{-k}}{2k - 1 + n(n-1)} \tag{30}$$

Entry 13.

$$\frac{\pi}{18\sqrt{3}} = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{\binom{2n}{n}} \sum_{k=1}^{\left(\frac{3n+1}{n+1}\right)\binom{2n}{n}} \frac{(-3)^{-k}}{2k - 3 + 2\binom{2n}{n}} \tag{31}$$

Entry 14.

$$\pi = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} 3^{-2n} \left(\frac{3}{4n+1} - \frac{1}{4n+3}\right) \tag{32}$$

$$\pi = \frac{3\sqrt{3}}{5} \sum_{n=0}^{\infty} 10^{-n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{3}{4k+1} - \frac{1}{4k+3}\right) \tag{33}$$

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} (-1)^n \, 2^{-3n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{3}{4k+1} - \frac{1}{4k+3}\right) \tag{34}$$

Entry 15.

$$\frac{\pi}{\sqrt{3}} = 1 + 3 \cdot \sum_{n=1}^{\infty} 2^{-n} \left(\sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{3^{-2k-1}}{4k+1} - \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \frac{3^{-2k-2}}{4k+3} \right) \tag{35}$$

Entry 16.

$$\pi = \frac{48}{5} \sum_{n=0}^{\infty} (-50)^{-n} \sum_{k=0}^{n} \left(-\frac{50}{9}\right)^k \binom{2n-2k}{n-k} \left(\frac{k+1}{(4k+1)(4k+3)} \right) \tag{36}$$

Entry 17.

$$\frac{\pi}{6\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{6} \left(\frac{1}{12}\right)^n + \frac{1}{7} \left(\frac{3}{49}\right)^n \right) \tag{37}$$

$$\frac{\pi}{2\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{2} \left(\frac{1}{12}\right)^n + \frac{1}{3} \left(\frac{1}{27}\right)^n + \frac{1}{11} \left(\frac{1}{363}\right)^n \right) \tag{38}$$

Entry 18.

$$\frac{\pi}{2\sqrt{3}} = 1 - \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{3}\right)^{\frac{3^n-1}{2}+n} + \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{\frac{3^n-1}{2}} \sum_{k=1}^{3^n-1} \frac{(-3)^{-k}}{2k+3^n} \tag{39}$$

Entry 19.

$$\pi = 2\sqrt{3} - \sum_{n=0}^{\infty} 2^{-4n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(-1)^k}{(2k+3)} \left(\frac{16}{3}\right)^k \tag{40}$$

$$\pi = 2\sqrt{3} - \frac{1}{3} \cdot \sum_{n=0}^{\infty} 2^{-4n} \binom{2n}{n} {}_3F_2\left(1, \frac{3}{2}, -n;\ \frac{5}{2}, \frac{1}{2} - n;\ -\frac{4}{3}\right) \tag{41}$$

Entry 20.

$$\pi = \frac{6q}{p} \sum_{n=0}^{\infty} (-1)^n \left(\frac{q\sqrt{3} - p}{p} \right)^n \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{2k+1} \left(\frac{q^2}{p\left(q\sqrt{3} - p\right)} \right)^k \tag{42}$$

where

$$q < p\ ,\quad \left| \frac{q\sqrt{3}}{p} - 1 \right| < 1\ ,\quad p, q \in \mathbb{N} \tag{43}$$

Entry 21.

$$\pi = \frac{12\sqrt{3}}{7} \sum_{n=0}^{\infty} 7^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^k}{2k+1} \tag{44}$$

$$\pi = \frac{12\sqrt{3}}{7} \sum_{n=0}^{\infty} 7^{-n}\ {}_2F_1\left(\frac{1}{2}, -n;\ \frac{3}{2};\ 2\right) \tag{45}$$

Entry 22.

$$\pi = 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{1}{\sqrt{3}} \right)^{n+1} \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{n-k} \tag{46}$$

Entry 23.

$$\pi = 3 \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{2}{\sqrt{3}}\right)^{n+1} {}_2F_1\left(2\,n+2,\, n+1\,;\; n+2\;;\; -\frac{1}{\sqrt{3}}\right) \tag{47}$$

$$\pi = 3 \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\sqrt{3}-1\right)^{n+1} {}_2F_1\left(-n,\, n+1\,;\; n+2\;;\; \frac{1}{1+\sqrt{3}}\right) \tag{48}$$

$$\pi = 3 \sum_{n=0}^{\infty} \frac{1}{n+1} \left(2\sqrt{3}-3\right)^{n+1} {}_2F_1\left(2\,n+2,\, 1\,;\; n+2\;;\; \frac{1}{1+\sqrt{3}}\right) \tag{49}$$

$$\pi = 3\left(\sqrt{3}-1\right) \sum_{n=0}^{\infty} \frac{1}{n+1} \left(2\sqrt{3}-3\right)^{n} {}_2F_1\left(-n,\, 1\,;\; n+2\;;-\frac{1}{\sqrt{3}}\right) \tag{50}$$

Entry 24.

$$\pi = 6 \sum_{n=0}^{\infty} (-2)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k-2\,n-1} \left(\left(1+\frac{1}{\sqrt{3}}\right)^{k-2\,n-1} - 1\right) \tag{51}$$

Entry 25.

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sqrt{21}-3}{6}\right)^n \sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{(-1)^k}{2\,n-2\,k+1} \tag{52}$$

Entry 26.

$$\pi = \left(\sqrt{9+12\sqrt{3}} - 3\right) + 6 \sum_{n=1}^{\infty} \left(\frac{\sqrt{9+12\sqrt{3}}-3}{6}\right)^{n+1} \sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]} \frac{(-1)^k}{2\,k+1} \binom{2\,k+1}{n-2\,k} \tag{53}$$

Entry 27. for $N = 1, 2, 3, \ldots$, we have

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\left[\frac{N-1}{2}\right]} \frac{(-1)^n\, 3^{-n}}{2\,n+1} + \left(-\frac{1}{\sqrt{3}}\right)^N \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+N+1} \left(\frac{1}{\sqrt{3}}\right)^{k+1} \operatorname{Im}\left(i^{k+N+1}\right) \tag{54}$$

Entry 28.

$$\frac{\pi}{6} = \sum_{n=1}^{\infty} \frac{2^{-2\,n}}{n} \operatorname{Im}\left(\left(1+i\sqrt{3}\right)^n\right) = \sqrt{3} \sum_{n=1}^{\infty} \frac{2^{-2\,n}}{n}\, s(n) \tag{55}$$

$$s(n+2) = 2\,s(n+1) - 4\,s(n)\;,\;\; s(1) = 1\;,\;\; s(2) = 2 \tag{56}$$

$$s(n) = \{1,\, 2,\, 0,\, -8,\, -16,\, 0,\, \ldots\} \tag{57}$$

Remark: $\operatorname{Im}(z)$ is the imaginary part of z , $i = \sqrt{-1}$.

Entry 29.

$$\frac{\pi}{6} = 2 \sum_{n=0}^{\infty} \frac{13^{-2\,n-1}}{2\,n+1} \operatorname{Im}\left(\left(1+2\,i\sqrt{3}\right)^{2\,n+1}\right) = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{13^{-2\,n-1}}{2\,n+1}\, s(n) \tag{58}$$

$$s(n+2) = -22\,s(n+1) - 169\,s(n)\;\;,\;\; s(0) = 2\;,\;\; s(1) = -18 \tag{59}$$

$$s(n) = \{2,\, -18,\, 58,\, 1766,\, -48\,654,\, 771\,934,\, \ldots\} \tag{60}$$

Remark: $\operatorname{Im}(z)$ is the imaginary part of z , $i = \sqrt{-1}$.

Entry 30.

$$\pi = 3\, i \sum_{n=1}^{\infty} \frac{2^n - \left(1 + i\sqrt{3}\right)^n}{n\left(3 + i\sqrt{3}\right)^n} \quad , \; i = \sqrt{-1} \tag{61}$$

Entry 31.

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{\left(1/\sqrt{3}\right)^{n+1}}{n+1} \operatorname{F1}\left(1+n,\, -n,\, 1+n,\, 2+n,\, \frac{1}{\sqrt{3}},\, -\frac{1}{\sqrt{3}}\right) \tag{62}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{\left(1/\sqrt{3}\right)^{n+1}}{n+1} \operatorname{F1}\left(1+n,\, 1+n,\, 1+n,\, 2+n,\, -\frac{2}{\sqrt{3}+3\,i},\, -\frac{2}{\sqrt{3}-3\,i}\right) \tag{63}$$

Entry 32. for $a > 0$, $i = \sqrt{-1}$, we have

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(a\,i)^n}{(1+a\,i)^{n+1}}\, {}_2F_1\left(1+n,\, \frac{1}{2};\, \frac{3}{2};\, -\frac{1}{3+3\,a\,i}\right) \tag{64}$$

Entry 33. for $m \in \{0, 1, 2, 3, \ldots\}$, we have

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(-3)^{-n}}{2\,n+1}\, {}_2F_1\left(-m,\, n+\frac{1}{2};\, n+\frac{3}{2};\, -\frac{1}{3}\right) \tag{65}$$

Entry 34.

$$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \cosh(2\,n+1)}{(2\,n+1)\left(\sqrt{3}\,\cosh(1) + \sqrt{1 + 3\,(\cosh(1))^2}\right)^{2\,n+1}} \tag{66}$$

$$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \sinh(2\,n+1)}{(2\,n+1)\left(\sqrt{3}\,\sinh(1) + \sqrt{3\,(\sinh(1))^2 - 1}\right)^{2\,n+1}} \tag{67}$$

Entry 35.

$$\pi = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \left(\frac{k+2}{k+1}\right)^{n-k} \tag{68}$$

Entry 36.

$$\pi = 2\sqrt{3} \sum_{n=1}^{\infty} (-1)^{n-1}\, 3^{-n} \left\lfloor \frac{n+1}{2} \right\rfloor \left(\frac{4\,n^2 + 10\,n + 3}{n\,(n+1)\left(4\,n^2 - 1\right)}\right) \tag{69}$$

Entry 37.

$$\pi = 4\sqrt{3} \sum_{n=1}^{\infty} (-1)^{n-1}\, 3^{-n}\, n \left(\frac{4\,n^2 + 8\,n + 1}{\left(4\,n^2 - 1\right)^2}\right) \tag{70}$$

Entry 38.

$$\pi = 2\sqrt{3} \sum_{n=1}^{\infty} (-1)^{n-1}\, 3^{-n} \left(\frac{8\,n^2 + 4\,n - 1}{4\,n^2 - 1}\right) H_n \tag{71}$$

Remark: $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the harmonic number.

Entry 39.

$$\pi = 2\sqrt{3}\sum_{n=1}^{\infty}\frac{3^{-n}\left(6n^2+7n+4\right)}{(n+1)\left(4n^2-1\right)}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{\binom{n}{k}(n-k)!} \tag{72}$$

$$\pi = \frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{3^{-n}(11n+17)}{(2n+1)(2n+3)}\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}^2}{\binom{2n}{2k}\binom{2n-2k}{n-k}} \tag{73}$$

$$\pi = 2\sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{-n}\left(6n^3+17n^2+11n+3\right)}{n(n+1)\left(4n^2-1\right)}\left(1-\frac{1}{(n+1)!}\right) \tag{74}$$

$$\pi = \frac{2}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{3^{-n}\left(12n^2+30n+17\right)}{(2n+1)(2n+2)(2n+3)}\sum_{k=0}^{n}\frac{(-1)^k}{\binom{2n}{2k}(2n-2k)!} \tag{75}$$

$$\pi = 4\sqrt{3}\sum_{n=1}^{\infty}\frac{3^{-n}\left(4n^2+12n+5\right)\left(1+(-1)^{n+1}(n+1)\right)}{(2n-1)(2n+1)^2(2n+3)} \tag{76}$$

Entry 40.

$$\pi = 6s\left(\frac{105-23\sqrt{3}}{48}+\frac{4337-2735\sqrt{3}}{560}s^4+\sum_{n=0}^{\infty}s^{4n+8}f(n)\right) \tag{77}$$

where

$$s = -\frac{1}{\sqrt{3}}+3^{-2/3}\left(9-\sqrt{3}+3\sqrt{6}\right)^{1/3}-\left(\sqrt{3}-1\right)\left(3\left(9-\sqrt{3}+3\sqrt{6}\right)\right)^{-1/3} \tag{78}$$

$$f(n) = \frac{1}{16}\left(\frac{(-1)^n\left(-3+\sqrt{3}\right)}{1+2n}+\frac{20(-1)^n\left(-5+3\sqrt{3}\right)}{3+2n}-\frac{5(-1)^n\left(-3+\sqrt{3}\right)}{5+2n}-\frac{4}{3+4n}+\frac{72-56\sqrt{3}}{7+4n}+\frac{16}{9+4n}+\frac{12-8\sqrt{3}}{11+4n}\right) \tag{79}$$

Entry 41.

$$\pi = 6\sum_{n=0}^{\infty}s^{n+1}\sum_{k=0}^{n}(-1)^k\sum_{m=\text{Ceiling}(k/3)}^{\text{floor}(k/2)}\frac{(-1)^m}{6m-2k+1}\binom{4m-k}{k-2m}\binom{6m-2k+1}{n-k} \tag{80}$$

where

$$s = -\frac{1}{\sqrt{3}}+3^{-2/3}\left(9-\sqrt{3}+3\sqrt{6}\right)^{1/3}-\left(\sqrt{3}-1\right)\left(3\left(9-\sqrt{3}+3\sqrt{6}\right)\right)^{-1/3} \tag{81}$$

Entry 42.

$$\frac{\pi}{\sqrt{3}} = 6\sum_{n=1}^{\infty}(-3)^{-\frac{3n(n-1)}{2}}\sum_{k=1}^{3n}\frac{(-1)^{k-1}3^{-k}}{3n(n-1)+2k-1} \tag{82}$$

Entry 43.

$$\frac{\pi}{2\sqrt{3}} = 1 + \sum_{n \in A} \sum_{k=1}^{\infty} (-3)^{-\left(\frac{n^k-1}{2}\right)} n^{-k} \tag{83}$$

where

$$A = \{3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 51, 53, 55, \ldots\} \tag{84}$$

$$A = \left\{n \geq 3\,,\ n \ \text{ odd}\,,\ n \neq m^k\,,\ m \ \text{odd}\,,\ k \in \mathbb{N} - \{1\}\right\} \tag{85}$$

Entry 44.

$$\frac{\pi}{2\sqrt{3}} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{k=1+\left[\frac{3^n-1}{4}\right]}^{3^n-1} (-1)^{n+k} \left(2\left(3^n - k\right)\right)^m \left(\frac{1}{3}\right)^{(n+1)(m+1)+k+\frac{3^n-1}{2}} \tag{86}$$

$$\frac{\pi}{2\sqrt{3}} = 1 - 3^{-2} + 3^{-4} - 3^{-5} + 5 \cdot 3^{-6} - 3 \cdot 3^{-7} + 17 \cdot 3^{-8} - 2 \cdot 3^{-9} + 59 \cdot 3^{-10} - 11 \cdot 3^{-11} + 271 \cdot 3^{-12} - 19 \cdot 3^{-13} + \ldots \tag{87}$$

Entry 45.

$$\pi = \frac{8}{3\sqrt{3}} \sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n} \frac{(-1)^k (n - k + 1)}{2k + 1} \tag{88}$$

III. References

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