Exact sines and cosines including a small table

Using half angle formulas and other trigonometric identities sines and cosines for exact angles may be established and such table produced.

Theorem 1: if $(n \in \mathbb{N} \succeq 1)$:

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+n)}}\right) = \cos\left(\frac{\pi}{2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \succeq 1)$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+n)}}\right) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\sqrt{2}+\sqrt{3}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \succeq 1)$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+n)}}\right) = \sin\left(\frac{\pi}{2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \succeq 1)$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+n)}}\right) = \sin\left(\frac{\pi}{3 \cdot 2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \succeq 1)$$

proof:

From:

"sines and cosines of ANY angles may be determined to ANY degree of accuracy and a relativistic non-Doppler effect"

and

$\forall\theta : 0 \leq\!: \theta \leq \frac{\pi}{2}$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

□

NOTE:

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+1)}}\right) = \sin\left(\frac{\pi}{3 \cdot 2^{(1+1)}}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+1)}} = \frac{\pi}{2} - \frac{\pi}{3 \cdot 4} = \frac{\pi}{2} - \frac{\pi}{12} = \frac{6\pi}{12} - \frac{\pi}{12} = \frac{5\pi}{12} = 90^\circ - 15^\circ = 75^\circ$$

$$\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{\left(\sqrt{6}-\sqrt{2}\right)^2}}{4} = \frac{\sqrt{6-2\sqrt{12}+2}}{4} = \frac{\sqrt{8-4\sqrt{3}}}{4} = \frac{\sqrt{2-\sqrt{3}}}{2} \quad \checkmark$$

so:

corollary 1.1: if $(n,m \in \mathbb{N} \succeq 1), (p \in \mathbb{R} : 0 < p < 4)$:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\cos\left(\frac{\pi}{p \cdot 2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{p}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \succeq 1)$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{p \cdot 2^{(1+n)}}\right) = \cos\left(\frac{\pi}{p \cdot 2^{(1+n)}}\right)$$

$$\cos\left(\frac{\pi}{p \cdot 2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\sqrt{2}+\sqrt{p}}}}}}}}^{n\,2's}}{2} \quad ' \quad (n \in \mathbb{N} \succeq 1)$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{p \cdot 2^{(1+n)}}\right) = \cos\left(\frac{\pi}{p \cdot 2^{(1+n)}}\right)$$

$$\sin\left(\frac{\pi}{p\cdot 2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}^{n\,2's}}{2}\ ,\quad (n\in\mathbb{N}\geq 1)$$

$$\cos\left(\frac{\pi}{2}-\frac{\pi}{p\cdot 2^{(1+n)}}\right) = \sin\left(\frac{\pi}{p\cdot 2^{(1+n)}}\right)$$

$$\sin\left(\frac{\pi}{p\cdot 2^{(1+n)}}\right) = \frac{\overbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{p}}}}}}}}^{n\,2's}}{2}\ ,\quad (n\in\mathbb{N}\geq 1)(m\in\mathbb{N}\geq 1)$$

$$\cos\left(\frac{\pi}{2}-\frac{\pi}{p\cdot 2^{(1+n)}}\right) = \sin\left(\frac{\pi}{p\cdot 2^{(1+n)}}\right)$$

proof:

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\pi}{4} = \cos\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{1}{2}\cdot\sqrt{2} = \frac{\sqrt{2}}{2} \Rightarrow \cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2} \quad ✓$$

$$\cos\frac{\pi}{6} = \cos\left(\frac{1}{2}\cdot\frac{\pi}{3}\right) = \frac{1}{2}\cdot\sqrt{3} = \frac{\sqrt{3}}{2} \Rightarrow \cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2} \quad ✓$$

$$\Rightarrow \cos\left(\frac{\pi}{2p}\right) = \cos\left(\frac{1}{2}\cdot\frac{\pi}{p}\right) = \frac{1}{2}\cdot\sqrt{p} = \frac{\sqrt{p}}{2} \Rightarrow \cos\left(\frac{\pi}{2p}\right) = \sqrt{\frac{1+\frac{\sqrt{p}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{p}}{2}}{2}} = \frac{\sqrt{2+\sqrt{p}}}{2}$$

$$\forall n,m\in\mathbb{N},\ \frac{\pi}{2\cdot 2^n} = \frac{2^n\pi}{2} \quad \& \quad \frac{\pi}{3\cdot 2^m} = \frac{2^{m-1}\pi}{3\cdot 2}$$

$$\cos^2\left(\frac{\pi}{2p}\right) = \cos^2\left(\frac{1}{2}\cdot\frac{\pi}{p}\right) = \frac{1}{2}\cdot\frac{1+p}{2}$$

$$\Rightarrow \theta = \frac{\pi}{p} \Rightarrow \cos(\pi/2p) = \cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\cos\left(\frac{\pi}{p}\right)}{2}}$$

From:

"sines and cosines of ANY angles may be determined to ANY degree of accuracy
and a relativistic non-Doppler effect"

and corollary 1.1 .

and

$\forall\theta : 0 \leq\!:\theta \leq \frac{\pi}{2}$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

□

additionally, further angles $0-2\pi$ may be found using:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\forall\theta : 0 \leq\!:\theta \leq \frac{\pi}{2}$

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}+\theta\right) = \sin\theta$

$\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$

$\sin(\pi+\theta) = -\sin\theta$

$\cos(\pi+\theta) = -\cos\theta$

$\sin\left(\frac{3\pi}{2}+\theta\right) = -\sin\theta$

$\cos\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$

and, 2π periodicity continuesto ∞ .

mini-TABLE:

$$\cos\left(\frac{\pi}{3}\right) = \cos(60^\circ) = \frac{1}{2} \approx 0.5 \quad ✓$$

$$\cos\left(\frac{\pi}{6}\right) = \cos(30^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{3}\right)}{2}} = \frac{\sqrt{2+1}}{2} = \frac{\sqrt{3}}{2} \approx 0.86602540378443864676372317075294 \quad ✓$$

$$\cos\left(\frac{\pi}{12}\right) = \cos(15^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2} \approx 0.9659258262890682867497431997289 \quad ✓$$

$$\cos\left(\frac{\pi}{24}\right) = \cos(7.5^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{12}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \approx 0.99144486137381041114455752692856 \quad ✓$$

$$\cos\left(\frac{\pi}{48}\right) = \cos(3.75^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{24}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}$$
$$\approx 0.99785892323860350673806979127278 \quad ✓$$

$$\cos\left(\frac{\pi}{96}\right) = \cos(1.875^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{48}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2}$$
$$\approx 0.99946458747636564442983644624286 \quad ✓$$

$$\cos\left(\frac{\pi}{192}\right) = \cos(0.9375^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{96}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}{2}$$
$$\approx 0.99986613790956178286274714906 01 \quad ✓✓$$

$$\cos\left(\frac{\pi}{384}\right) = \cos(0.46875^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{192}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}{2}$$
$$\approx 0.99996653391740110345760381057914 \quad ✓$$

$$\cos\left(\frac{\pi}{768}\right) = \cos(0.234375^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{384}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}{2}$$
$$\approx 0.99999163344435064914755862056883 \quad ✓$$

$$\cos\left(\frac{\pi}{1536}\right) = \cos(0.1171875^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{768}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}{2}$$
$$\approx 0.99999790835890018104166372432685 \quad ✓$$

$$\cos\left(\frac{\pi}{3072}\right) = \cos(0.05859375^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{1536}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}{2}$$
$$\approx 0.99999947708958832761109823746401 \quad ✓$$

$$\cos\left(\frac{\pi}{6144}\right) = \cos(0.029296875^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{3072}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}}{2}$$
$$\approx 0.99999986927238853704857515517162 \quad ✓$$

$$\cos\left(\frac{\pi}{12288}\right) = \cos(0.0146484375^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{6144}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}}}{2}$$
$$\approx 0.99999996731809660020873887214911 \quad ✓$$

$$\sin\left(\frac{\pi}{6}\right) = \sin(30^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{3}\right)}{2}} = \frac{\sqrt{2-2(0.5)}}{2} = \frac{1}{2} = 0.5 \quad ✓$$

$$\sin\left(\frac{\pi}{3}\right) = \sin(60^\circ) = \frac{\sqrt{3}}{2} \approx 0.86602540378443864676372317075294 \quad ✓$$

$$\sin\left(\frac{\pi}{12}\right) = \sin(15^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{6}\right)}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2} \approx 0.25881904510252076234889883762405 \quad ✓$$

$$\sin\left(\frac{\pi}{24}\right) = \sin(7.5^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{12}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2} \approx 0.13052619222005159154840622789549 \quad ✓$$

$$\sin\left(\frac{\pi}{48}\right) = \sin(3.75^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{24}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2} \approx 0.065403129230143066815315558775 18 \quad ✓$$

$$\sin\left(\frac{\pi}{96}\right) = \sin(1.875^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{48}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2}$$
$$\approx 0.032719082821776142063659926317 29 \quad ✓$$

$$\sin\left(\frac{\pi}{192}\right) = \sin(0.9375^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{96}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}{2}$$
$$\approx 0.016361731626486781642971923484 17 \quad ✓$$

$$\sin\left(\frac{\pi}{384}\right) = \sin(0.46875^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{192}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}{2}$$
$$\approx 0.00818113960393712928519912329461 \quad \checkmark$$

$$\sin\left(\frac{\pi}{768}\right) = \sin(0.234375^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{384}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}{2}$$
$$\approx 0.00409060402623478959462105417151 \quad \checkmark$$

$$\sin\left(\frac{\pi}{1536}\right) = \sin(0.1171875^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{768}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}{2}$$
$$\approx 0.00204530629116409511441305892313 \quad \checkmark$$

$$\sin\left(\frac{\pi}{3072}\right) = \sin(0.05859375^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{1536}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}{2}$$
$$\approx 0.00102265368033830454119296114603 \quad \checkmark$$

$$\sin\left(\frac{\pi}{6144}\right) = \sin(0.029296875^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{3072}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}}{2}$$
$$\approx 0.00051132690701369750123581934973451 \quad \checkmark$$

$$\sin\left(\frac{\pi}{12288}\right) = \sin(0.0146484375^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{6144}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}}}{2}$$
$$\approx 0.00025566346186241731406164952083865 \quad \checkmark$$

$$\cos(\pi) = \cos(180^\circ) = -1$$

$$\cos\left(\frac{\pi}{2}\right) = \cos(90^\circ) = \sqrt{\frac{1+\cos(\pi)}{2}} = \frac{\sqrt{2+(-2)-}}{2} = 0$$

$$\cos\left(\frac{\pi}{4}\right) = \cos(45^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{2}\right)}{2}} = \frac{\sqrt{2+0}}{2} = \frac{\sqrt{2}}{2} \approx 0.70710678118654752440084436210485 \quad \checkmark$$

$$\cos\left(\frac{\pi}{8}\right) = \cos(22.5^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} = \frac{\sqrt{2+2\cdot\frac{\sqrt{2}}{2}}}{2} = \frac{\sqrt{2+\sqrt{2}}}{2} \approx 0.92387953251128675612818318939679 \quad \checkmark$$

$$\cos\left(\frac{\pi}{16}\right) = \cos(11.25^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{8}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \approx 0.98078528040323044912618223613424 \quad \checkmark$$

$$\cos\left(\frac{\pi}{32}\right) = \cos(5.625^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{16}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \approx 0.99518472667219688624483695310948 \quad \checkmark$$

$$\cos\left(\frac{\pi}{64}\right) = \cos(2.8125^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{32}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$
$$\approx 0.9987954562051723927147716047591 \quad \checkmark$$

$$\cos\left(\frac{\pi}{128}\right) = \cos(1.40625^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{64}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}{2}$$
$$\approx 0.99969881869620422011576564966617 \quad \checkmark$$

$$\cos\left(\frac{\pi}{256}\right) = \cos(0.703125^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{128}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}{2}$$
$$\approx 0.99992470183914454092164649119638 \quad \checkmark$$

$$\cos\left(\frac{\pi}{512}\right) = \cos(0.3515625^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{256}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}{2}$$
$$\approx 0.99998117528260114265699043772857 \quad \checkmark$$

$$\cos\left(\frac{\pi}{1024}\right) = \cos(0.17578125^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{512}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}{2}$$

$$\approx 0.99999529380957617151158012570012$$ ✓

$$\cos\left(\frac{\pi}{2048}\right) = \cos(0.087890625^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{1024}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}{2}$$

$$\approx 0.99999882345170190992902571017153$$ ✓

$$\cos\left(\frac{\pi}{4096}\right) = \cos(0.0439453125^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{2048}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}{2}$$

$$\approx 0.99999970586288221916022821773877$$ ✓

$$\cos\left(\frac{\pi}{8192}\right) = \cos(0.02197265625^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{4096}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}}{2}$$

$$\approx 0.99999992646571785114473148070739$$ ✓

$$\cos\left(\frac{\pi}{16384}\right) = \cos(0.010986328125^\circ) = \sqrt{\frac{1+\cos\left(\frac{\pi}{8192}\right)}{2}} =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}}}{2}$$

$$\approx 0.99999998161642929380834691540291$$ ✓

$$\sin\left(\frac{\pi}{4}\right) = \sin(45^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{2}\right)}{2}} = \frac{\sqrt{2-0}}{2} = \frac{\sqrt{2}}{2} \approx 0.70710678118654752440084436210485$$ ✓

$$\sin\left(\frac{\pi}{8}\right) = \sin(22.5^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = \frac{\sqrt{2-2\cdot\frac{\sqrt{2}}{2}}}{2} = \frac{\sqrt{2-\sqrt{2}}}{2} \approx 0.38268343236508977172845998403040$$ ✓✓

$$\sin\left(\frac{\pi}{16}\right) = \sin(11.25^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{8}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2} \approx 0.19509032201612826784828486847702$$ ✓✓

$$\sin\left(\frac{\pi}{32}\right) = \sin(5.625^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{16}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \approx 0.09801714032956060199419556388864$$ ✓✓

$$\sin\left(\frac{\pi}{64}\right) = \sin(2.8125^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{32}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$

$$\approx 0.04906767432741801425495497694268$$ ✓✓

$$\sin\left(\frac{\pi}{128}\right) = \sin(1.40625^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{64}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}{2}$$

$$\approx 0.02454122852291228803173452945928$$ ✓✓

$$\sin\left(\frac{\pi}{256}\right) = \sin(0.703125^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{128}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}{2}$$

$$\approx 0.012271538285719926079408261951$$ ✓✓

$$\sin\left(\frac{\pi}{512}\right) = \sin(0.3515625^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{256}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}{2}$$

$$\approx 0.00613588464915447535964023459037$$ ✓✓

$$\sin\left(\frac{\pi}{1024}\right) = \sin(0.17578125^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{512}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}{2}$$

$$\approx 0.00306795676296597627014536549092$$ ✓✓

$$\sin\left(\frac{\pi}{2048}\right) = \sin(0.087890625^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{1024}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}{2}$$
$$\approx 0.00153398018628476561230369715026 \quad \checkmark\checkmark$$

$$\sin\left(\frac{\pi}{4096}\right) = \sin(0.0439453125^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{2048}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}{2}$$
$$\approx 0.00076699031874270452693856835794858 \quad \checkmark\checkmark$$

$$\sin\left(\frac{\pi}{8192}\right) = \sin(0.02197265625^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{4096}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}{2}$$
$$\approx 0.00038349518757139558907246168118138 \quad \checkmark\checkmark$$

$$\sin\left(\frac{\pi}{16384}\right) = \sin(0.010986328125^\circ) = \sqrt{\frac{1-\cos\left(\frac{\pi}{8192}\right)}{2}} =$$
$$= \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}}{2}$$
$$\approx 0.000191747597310703307439909561989 \quad \checkmark\checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+2)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = \sin\left(\frac{3\pi}{8}\right) = \sin(67.5^\circ) = \cos\left(\frac{\pi}{2^{(1+2)}}\right) = \cos\left(\frac{\pi}{8}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2}}}{2} \approx 0.92387953251128675612818318939679 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+3)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{16}\right) = \sin\left(\frac{7\pi}{16}\right) = \sin(78.75^\circ) = \cos\left(\frac{\pi}{2^{(1+3)}}\right) = \cos\left(\frac{\pi}{16}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \approx 0.98078528040323044912618223613424 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+4)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{32}\right) = \sin\left(\frac{15\pi}{32}\right) = \sin(84.375^\circ) = \cos\left(\frac{\pi}{2^{(1+4)}}\right) = \cos\left(\frac{\pi}{32}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \approx 0.99518472667219688624483695310948 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+5)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{64}\right) = \sin\left(\frac{31\pi}{64}\right) = \sin(87.1875^\circ) = \cos\left(\frac{\pi}{2^{(1+5)}}\right) = \cos\left(\frac{\pi}{64}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2} \approx 0.9987954562051723927147716047591 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+6)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{128}\right) = \sin\left(\frac{63\pi}{128}\right) = \sin(88.59375^\circ) = \cos\left(\frac{\pi}{2^{(1+6)}}\right) = \cos\left(\frac{\pi}{128}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}{2} \approx 0.99969881869620422011576564966617 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+7)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{256}\right) = \sin\left(\frac{127\pi}{256}\right) = \sin(89.296875^\circ) = \cos\left(\frac{\pi}{2^{(1+7)}}\right) = \cos\left(\frac{\pi}{256}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}{2}$$
$$\approx 0.99992470183914454092164649119638 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+8)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{512}\right) = \sin\left(\frac{255\pi}{512}\right) = \sin(89.6484375^\circ) = \cos\left(\frac{\pi}{2^{(1+8)}}\right) = \cos\left(\frac{\pi}{512}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}{2}$$
$$\approx 0.99998117528260114265699043772857 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+9)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{1024}\right) = \sin\left(\frac{511\pi}{1024}\right) = \sin(89.82421875^\circ) = \cos\left(\frac{\pi}{2^{(1+9)}}\right) = \cos\left(\frac{\pi}{1024}\right) =$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}{2}$$
$$\approx 0.99999529380957617151158012570012 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+10)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{2048}\right)=\sin\left(\frac{1023\pi}{2048}\right)=\sin(89.912109375^\circ)=\cos\left(\frac{\pi}{2^{(1+10)}}\right)=\cos\left(\frac{\pi}{2048}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}{2}$$

$$\approx 0.99999882345170190992902571017153 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+11)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{4096}\right)=\sin\left(\frac{2047\pi}{4096}\right)=\sin(89.9560546875^\circ)=\cos\left(\frac{\pi}{2^{(1+11)}}\right)=\cos\left(\frac{\pi}{4096}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}{2}$$

$$\approx 0.99999970586288221916022821773877 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+12)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{8192}\right)=\sin\left(\frac{4095\pi}{8192}\right)=\sin(89.97802734375^\circ)=\cos\left(\frac{\pi}{2^{(1+12)}}\right)=\cos\left(\frac{\pi}{8192}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}}{2}$$

$$\approx 0.99999992646571785114473148070739 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+13)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{16384}\right)=\sin\left(\frac{8191\pi}{16384}\right)=\sin(89.989013671875^\circ)=\cos\left(\frac{\pi}{2^{(1+13)}}\right)=\cos\left(\frac{\pi}{16384}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}}}{2}$$

$$\approx 0.99999998161642929380834691540291 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+0)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\sin\left(\frac{2\pi}{6}\right)=\sin\left(\frac{\pi}{3}\right)=\sin(60^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+0)}}\right)=\cos\left(\frac{\pi}{6}\right)=$$

$$=\frac{\sqrt{3}}{2}\approx 0.86602540378443864676372317075294 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+1)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{12}\right)=\sin\left(\frac{5\pi}{12}\right)=\sin(75^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+1)}}\right)=\cos\left(\frac{\pi}{12}\right)=$$

$$=\frac{\sqrt{2+\sqrt{3}}}{2}\approx 0.9659258262890682867497431997289 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+2)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{24}\right)=\sin\left(\frac{11\pi}{24}\right)=\sin(82.5^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+2)}}\right)=\cos\left(\frac{\pi}{24}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}\approx 0.99144486137381041114455752692856 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+3)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{48}\right)=\sin\left(\frac{23\pi}{48}\right)=\sin(86.25^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+3)}}\right)=\cos\left(\frac{\pi}{48}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}$$

$$\approx 0.99785892323860350673806979127278 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+4)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{96}\right)=\sin\left(\frac{47\pi}{96}\right)=\sin(88.125^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+4)}}\right)=\cos\left(\frac{\pi}{96}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2}$$

$$\approx 0.9994645874763656442983644624286 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+5)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{192}\right)=\sin\left(\frac{95\pi}{192}\right)=\sin(89.0625^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+5)}}\right)=\cos\left(\frac{\pi}{192}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}{2}$$

$$\approx 0.9998661379095617828627471490601 \quad \checkmark\checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+6)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{384}\right)=\sin\left(\frac{191\pi}{384}\right)=\sin(89.53125^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+6)}}\right)=\cos\left(\frac{\pi}{384}\right)=$$

$$=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}{2}$$

$$\approx 0.99996653391740110345760381057914 \quad \checkmark$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+7)}}\right)=\sin\left(\frac{\pi}{2}-\frac{\pi}{768}\right)=\sin\left(\frac{383\pi}{768}\right)=\sin(89.765625^\circ)=\cos\left(\frac{\pi}{3\cdot 2^{(1+7)}}\right)=\cos\left(\frac{\pi}{768}\right)=$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}{2}$$

$$\approx 0.99999163344435064914755862056883 \quad ✓$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+8)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{1536}\right) = \sin\left(\frac{767\pi}{1536}\right) = \sin(89.8828125^\circ) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+8)}}\right) = \cos\left(\frac{\pi}{1536}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}{2}$$

$$\approx 0.99999790835890018104166372432685 \quad ✓$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+9)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3072}\right) = \sin\left(\frac{1535\pi}{3072}\right) = \sin(89.94140625^\circ) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+9)}}\right) = \cos\left(\frac{\pi}{3072}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}{2}$$

$$\approx 0.99999947708958832761109823746401 \quad ✓$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+10)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{6144}\right) = \sin\left(\frac{3071\pi}{6144}\right) = \sin(89.970703125^\circ) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+10)}}\right) = \cos\left(\frac{\pi}{6144}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}}{2}$$

$$\approx 0.99999986927238853704857515517162 \quad ✓$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+11)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{12288}\right) = \sin\left(\frac{6143\pi}{12288}\right) = \sin(89.9853515625^\circ) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+11)}}\right) = \cos\left(\frac{\pi}{12288}\right) =$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+1)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \cos(45^\circ) = \sin\left(\frac{\pi}{4}\right) = \sin(45^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{2}\right)}{2}} = \frac{\sqrt{2-0}}{2} = \frac{\sqrt{2}}{2} \approx 0.70710678118654752440084436210485 \quad ✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+2)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\left(\frac{3\pi}{8}\right) = \cos(67.5^\circ) = \sin\left(\frac{\pi}{8}\right) = \sin(22.5^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2} \approx 0.38268343236508977172845998403040 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+3)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{16}\right) = \cos\left(\frac{7\pi}{16}\right) = \cos(78.75^\circ) = \sin\left(\frac{\pi}{16}\right) = \sin(11.25^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{8}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

$$\approx 0.19509032201612826784828486847702 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+4)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{32}\right) = \cos\left(\frac{15\pi}{32}\right) = \cos(84.375^\circ) = \sin\left(\frac{\pi}{32}\right) = \sin(5.625^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{16}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$$

$$\approx 0.098017140329560601994195563888864 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+5)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{64}\right) = \cos\left(\frac{31\pi}{64}\right) = \cos(87.1875^\circ) = \sin\left(\frac{\pi}{64}\right) = \sin(2.8125^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{32}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$

$$\approx 0.049067674327418014254954976942688 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+6)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{128}\right) = \cos\left(\frac{63\pi}{128}\right) = \cos(88.59375^\circ) = \sin\left(\frac{\pi}{128}\right) = \sin(1.40625^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{64}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}{2}$$

$$\approx 0.024541228522912288031734529459282 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+7)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{256}\right) = \cos\left(\frac{127\pi}{256}\right) = \cos(89.296875^\circ) = \sin\left(\frac{\pi}{256}\right) = \sin(0.0703125^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{128}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}{2}$$

$$\approx 0.012271538285719926079408261951 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2^{(1+8)}}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{512}\right) = \cos\left(\frac{255\pi}{512}\right) = \cos(89.6484375^\circ) = \sin\left(\frac{\pi}{512}\right) = \sin(0.3515625^\circ) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{256}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}{2}$$

$$\approx 0.00613588464915447535964023459037 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+9)}}\right) = \cos\left(\frac{\pi}{2}-\frac{\pi}{1024}\right) = \cos\left(\frac{511\pi}{1024}\right) = \cos(89.82421875^{\circ}) = \sin\left(\frac{\pi}{1024}\right) = \sin(0.17578125^{\circ}) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{512}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}{2}$$

$$\approx 0.00306795676296597627014536549092 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+10)}}\right) = \cos\left(\frac{\pi}{2}-\frac{\pi}{2048}\right) = \cos\left(\frac{1023\pi}{2048}\right) = \cos(89.912109375^{\circ}) = \sin\left(\frac{\pi}{2048}\right) = \sin(0.087890625^{\circ}) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{1024}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}{2}$$

$$\approx 0.00153398018628476561230369715026 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+11)}}\right) = \cos\left(\frac{\pi}{2}-\frac{\pi}{4096}\right) = \cos\left(\frac{2047\pi}{4096}\right) = \cos(89.9560546875^{\circ}) = \sin\left(\frac{\pi}{4096}\right) = \sin(0.0439453125^{\circ}) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{2048}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}{2}$$

$$\approx 0.00076699031874270452693856835794858 \quad ✓✓$$

$$\cos\left(\frac{\pi}{2}-\frac{\pi}{2^{(1+12)}}\right) = \cos\left(\frac{\pi}{2}-\frac{\pi}{8192}\right) = \cos\left(\frac{4095\pi}{8192}\right) = \cos(89.97802734375^{\circ}) = \sin\left(\frac{\pi}{8192}\right) = \sin(0.02197265625^{\circ}) =$$

$$= \sqrt{\frac{1-\cos\left(\frac{\pi}{4096}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}}}{2}$$

$$\approx 0.00038349518757139558907246168118138 \quad ✓✓$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+0)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \sin(60^{\circ}) = \cos\left(\frac{\pi}{3\cdot 2^{(1+0)}}\right) = \cos\left(\frac{\pi}{6}\right) =$$

$$= \frac{\sqrt{3}}{2} \approx 0.86602540378443864676372317075294 \quad ✓$$

$$\cos\left(\frac{\pi}{3}\right) = \cos(60^{\circ}) = \frac{1}{2} \approx 0.5 \quad ✓$$

$$\cos\left(\frac{\pi}{6}\right) = \cos(30^{\circ}) = \sqrt{\frac{1+\cos\left(\frac{\pi}{3}\right)}{2}} = \frac{\sqrt{2+1}}{2} = \frac{\sqrt{3}}{2} \approx 0.86602540378443864676372317075294 \quad ✓$$

$$\cos\left(\frac{\pi}{12}\right) = \cos(15^{\circ}) = \sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2} \approx 0.96592582628906828674974319972889 \quad ✓$$

$$\cos\left(\frac{\pi}{24}\right) = \cos(7.5^{\circ}) = \sqrt{\frac{1+\cos\left(\frac{\pi}{12}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \approx 0.99144486137381041114455752692856 \quad ✓$$

$$\cos\left(\frac{\pi}{48}\right) = \cos(3.75^{\circ}) = \sqrt{\frac{1+\cos\left(\frac{\pi}{24}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}$$

$$\approx 0.99785892323860350673806979127278 \quad ✓$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+0)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \sin(60^{\circ}) = \cos\left(\frac{\pi}{3\cdot 2^{(1+0)}}\right) = \cos\left(\frac{\pi}{6}\right) =$$

$$= \frac{\sqrt{3}}{2} \approx 0.86602540378443864676372317075294 \quad ✓$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+1)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = \sin\left(\frac{5\pi}{12}\right) = \sin(75^{\circ}) = \cos\left(\frac{\pi}{3\cdot 2^{(1+1)}}\right) = \cos\left(\frac{\pi}{12}\right) =$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2} \approx 0.96592582628906828674974319972889 \quad ✓$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+2)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{24}\right) = \sin\left(\frac{11\pi}{24}\right) = \sin(82.5^{\circ}) = \cos\left(\frac{\pi}{3\cdot 2^{(1+2)}}\right) = \cos\left(\frac{\pi}{24}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \approx 0.99144486137381041114455752692856 \quad ✓$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+3)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{48}\right) = \sin\left(\frac{23\pi}{48}\right) = \sin(86.25^{\circ}) = \cos\left(\frac{\pi}{3\cdot 2^{(1+3)}}\right) = \cos\left(\frac{\pi}{48}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2} \approx 0.99785892323860350673806979127278 \quad ✓$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{3\cdot 2^{(1+4)}}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{96}\right) = \sin\left(\frac{47\pi}{96}\right) = \sin(88.125^{\circ}) = \cos\left(\frac{\pi}{3\cdot 2^{(1+4)}}\right) = \cos\left(\frac{\pi}{96}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2} \approx 0.9994645874763656442983644624286 \ \checkmark$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+5)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{192}\right) = \sin\left(\frac{95\pi}{192}\right) = \sin(89.0625^{\circ}) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+5)}}\right) = \cos\left(\frac{\pi}{192}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}{2} \approx 0.9998661379095617828627471490601 \ \checkmark$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+6)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{384}\right) = \sin\left(\frac{191\pi}{384}\right) = \sin(89.53125^{\circ}) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+6)}}\right) = \cos\left(\frac{\pi}{384}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}{2} \approx 0.99996653391740110345760381057914 \ \checkmark$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+7)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{768}\right) = \sin\left(\frac{383\pi}{768}\right) = \sin(89.765625^{\circ}) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+7)}}\right) = \cos\left(\frac{\pi}{768}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}{2} \approx 0.99999163344435064914755862056883 \ \checkmark$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+8)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{1536}\right) = \sin\left(\frac{767\pi}{1536}\right) = \sin(89.8828125^{\circ}) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+8)}}\right) = \cos\left(\frac{\pi}{1536}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}{2} \approx 0.99999790835890018104166372432685 \ \checkmark$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+9)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3072}\right) = \sin\left(\frac{1535\pi}{3072}\right) = \sin(89.94140625^{\circ}) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+9)}}\right) = \cos\left(\frac{\pi}{3072}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}{2} \approx 0.99999947708958832761109823746401 \ \checkmark$$

$$\sin\left(\frac{\pi}{2} - \frac{\pi}{3 \cdot 2^{(1+10)}}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{6144}\right) = \sin\left(\frac{3071\pi}{6144}\right) = \sin(89.970703125^{\circ}) = \cos\left(\frac{\pi}{3 \cdot 2^{(1+10)}}\right) = \cos\left(\frac{\pi}{6144}\right) =$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}}}}}{2} \approx 0.99999986927238853704857515517162$$

AND:

lemma 1.2: if :

$$\cos\left(3\frac{\theta}{2}\right) = \cos\theta\cos\left(\frac{\theta}{2}\right) - \sin\theta\sin\left(\frac{\theta}{2}\right)$$
$$\sin\left(3\frac{\theta}{2}\right) = \sin\theta\cos\left(\frac{\theta}{2}\right) + \cos\theta\sin\left(\frac{\theta}{2}\right)$$
$$\cos\left(5\frac{\theta}{2}\right) = [\cos^2\theta - \sin^2\theta]\cos\left(\frac{\theta}{2}\right) - 2\sin\theta\cos\theta\sin\left(\frac{\theta}{2}\right)$$
$$\sin\left(5\frac{\theta}{2}\right) = [\cos^2\theta - \sin^2\theta]\sin\left(\frac{\theta}{2}\right) + 2\sin\theta\cos\theta\cos\left(\frac{\theta}{2}\right)$$
$$\cos\left(7\frac{\theta}{2}\right) = [[\cos^2\theta - 3\sin^2\theta]\cos\theta]\cos\left(\frac{\theta}{2}\right) + [[\sin^2\theta - 3\cos^2\theta]\sin\theta]\sin\left(\frac{\theta}{2}\right)$$
$$\sin\left(7\frac{\theta}{2}\right) = [-[\sin^2\theta - 3\cos^2\theta]\sin\theta]\cos\left(\frac{\theta}{2}\right) + [[\cos^2\theta - 3\sin^2\theta]\cos\theta]\sin\left(\frac{\theta}{2}\right)$$

proof:

$$\cos\left(\theta + \frac{\theta}{2}\right) = \cos\left(3\frac{\theta}{2}\right) = \cos\theta\cos\left(\frac{\theta}{2}\right) - \sin\theta\sin\left(\frac{\theta}{2}\right)$$
$$\sin\left(\theta + \frac{\theta}{2}\right) = \sin\left(3\frac{\theta}{2}\right) = \sin\theta\cos\left(\frac{\theta}{2}\right) + \cos\theta\sin\left(\frac{\theta}{2}\right)$$
$$\cos\left(\theta + \frac{3}{2}\theta\right) = \cos\left(5\frac{\theta}{2}\right) = \cos\theta\cos\left(3\frac{\theta}{2}\right) - \sin\theta\sin\left(3\frac{\theta}{2}\right)$$
$$= \cos\theta\left[\cos\theta\cos\left(\frac{\theta}{2}\right) - \sin\theta\sin\left(\frac{\theta}{2}\right)\right] - \sin\theta\left[\sin\theta\cos\left(\frac{\theta}{2}\right) + \cos\theta\sin\left(\frac{\theta}{2}\right)\right]$$
$$= [\cos^2\theta - \sin^2\theta]\cos\left(\frac{\theta}{2}\right) - 2\sin\theta\cos\theta\sin\left(\frac{\theta}{2}\right)$$
$$\sin\left(\theta + \frac{3}{2}\theta\right) = \sin\left(5\frac{\theta}{2}\right) = \sin\theta\cos\left(\frac{3}{2}\theta\right) + \cos\theta\sin\left(\frac{3}{2}\theta\right)$$
$$= \sin\theta\left[\cos\theta\cos\left(\frac{\theta}{2}\right) - \sin\theta\sin\left(\frac{\theta}{2}\right)\right] + \cos\theta\left[\sin\theta\cos\left(\frac{\theta}{2}\right) + \cos\theta\sin\left(\frac{\theta}{2}\right)\right]$$
$$= \sin\theta\cos\theta\cos\left(\frac{\theta}{2}\right) - \sin^2\theta\sin\left(\frac{\theta}{2}\right) + \cos\theta\sin\theta\cos\left(\frac{\theta}{2}\right) + \cos^2\theta\sin\left(\frac{\theta}{2}\right)$$
$$= [\cos^2\theta - \sin^2\theta]\sin\left(\frac{\theta}{2}\right) + 2\sin\theta\cos\theta\cos\left(\frac{\theta}{2}\right)$$
$$\cos\left(\theta + \frac{5}{2}\theta\right) = \cos\left(7\frac{\theta}{2}\right) = \cos\theta\cos\left(5\frac{\theta}{2}\right) - \sin\theta\sin\left(5\frac{\theta}{2}\right)$$
$$= \cos\theta\left[[\cos^2\theta - \sin^2\theta]\cos\left(\frac{\theta}{2}\right) - 2\sin\theta\cos\theta\sin\left(\frac{\theta}{2}\right)\right] +$$
$$- \sin\theta\left[[\cos^2\theta - \sin^2\theta]\sin\left(\frac{\theta}{2}\right) + 2\sin\theta\cos\theta\cos\left(\frac{\theta}{2}\right)\right]$$
$$= \cos\theta[\cos^2\theta - \sin^2\theta]\cos\left(\frac{\theta}{2}\right) - 2\sin\theta\cos^2\theta\sin\left(\frac{\theta}{2}\right) +$$
$$- \sin\theta[\cos^2\theta - \sin^2\theta]\sin\left(\frac{\theta}{2}\right) - 2\sin^2\theta\cos\theta\cos\left(\frac{\theta}{2}\right)$$

$$= [\cos\theta[\cos^2\theta - \sin^2\theta] - 2\sin^2\theta\cos\theta]\cos\left(\frac{\theta}{2}\right) +$$
$$- [\sin\theta[\cos^2\theta - \sin^2\theta] - 2\sin\theta\cos^2\theta]\sin\left(\frac{\theta}{2}\right)$$
$$= [[\cos^2\theta - 3\sin^2\theta]\cos\theta]\cos\left(\frac{\theta}{2}\right) - [[3\cos^2\theta - \sin^2\theta]\sin\theta]\sin\left(\frac{\theta}{2}\right)$$

$$\sin\left(\theta + \frac{5}{2}\theta\right) = \sin\left(7\frac{\theta}{2}\right) = \sin\theta\cos\left(5\frac{\theta}{2}\right) + \cos\theta\sin\left(5\frac{\theta}{2}\right)$$
$$= \sin\theta\left[[\cos^2\theta - \sin^2\theta]\cos\left(\frac{\theta}{2}\right) - 2\sin\theta\cos\theta\sin\left(\frac{\theta}{2}\right)\right] +$$
$$+ \cos\theta\left[[\cos^2\theta - \sin^2\theta]\sin\left(\frac{\theta}{2}\right) + 2\sin\theta\cos\theta\cos\left(\frac{\theta}{2}\right)\right]$$
$$= \sin\theta[\cos^2\theta - \sin^2\theta]\cos\left(\frac{\theta}{2}\right) - 2\sin^2\theta\cos\theta\sin\left(\frac{\theta}{2}\right) +$$
$$+ \cos\theta[\cos^2\theta - \sin^2\theta]\sin\left(\frac{\theta}{2}\right) + 2\sin\theta\cos^2\theta\cos\left(\frac{\theta}{2}\right)$$
$$= [\sin\theta[\cos^2\theta - \sin^2\theta] + 2\sin\theta\cos^2\theta]\cos\left(\frac{\theta}{2}\right) +$$
$$+ [\cos\theta[\cos^2\theta - \sin^2\theta] - 2\sin^2\theta\cos\theta]\sin\left(\frac{\theta}{2}\right)$$
$$= [[[\cos^2\theta - \sin^2\theta] + 2\cos^2\theta]\sin\theta]\cos\left(\frac{\theta}{2}\right) +$$
$$+ [[[\cos^2\theta - \sin^2\theta] - 2\sin^2\theta]\cos\theta]\sin\left(\frac{\theta}{2}\right)$$
$$= [[3\cos^2\theta - \sin^2\theta]\sin\theta]\cos\left(\frac{\theta}{2}\right) + [[\cos^2\theta - 3\sin^2\theta]\cos\theta]\sin\left(\frac{\theta}{2}\right)$$

□

NOTE:

$$\cos\left(\theta + \left(\frac{2n+1}{2}\right)\theta\right) = \cos\left((2n+3)\frac{\theta}{2}\right) = \cos\theta\cos\left((2n+1)\frac{\theta}{2}\right) - \sin\theta\sin\left((2n+1)\frac{\theta}{2}\right)$$
$$\sin\left(\theta + \left(\frac{2n+1}{2}\right)\theta\right) = \sin\left((2n+3)\frac{\theta}{2}\right) = \sin\theta\cos\left((2n+1)\frac{\theta}{2}\right) + \cos\theta\sin\left((2n+1)\frac{\theta}{2}\right)$$

generally the 30° & 45° angle families blend together well, but:

$$\theta = \frac{\pi}{32} = 5.625^{\circ} \Rightarrow 3\frac{\theta}{2} = 8.4375^{\circ} :$$
$$\Rightarrow \frac{\pi}{32} = 5.625^{\circ} \Rightarrow 3\frac{5.625^{\circ}}{2} = 8.4375^{\circ} \Leftrightarrow 90^{\circ} - 8.4375^{\circ} = 81.5625^{\circ}$$
$$\Rightarrow \cos\left(\frac{\pi}{32}\right) = \cos(5.625^{\circ}) = \sqrt{\frac{1+\cos\left(\frac{\pi}{16}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$$
$$\Rightarrow \sin\left(\frac{\pi}{32}\right) = \sin(5.625^{\circ}) = \sqrt{\frac{1-\cos\left(\frac{\pi}{16}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$$
$$\Rightarrow \cos\left(\frac{\left(\frac{\pi}{32}\right)}{2}\right) = \cos\left(\frac{\pi}{64}\right) = \cos(2.8125^{\circ}) = \sqrt{\frac{1+\cos\left(\frac{\pi}{32}\right)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$
$$\Rightarrow \sin\left(\frac{\left(\frac{\pi}{32}\right)}{2}\right) = \sin\left(\frac{\pi}{64}\right) = \sin(2.8125^{\circ}) = \sqrt{\frac{1-\cos\left(\frac{\pi}{32}\right)}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$$
$$\Rightarrow \cos\left(3\frac{\pi}{64}\right) = \cos(8.4375^{\circ}) = \left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right) +$$
$$- \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$= \sin\left(\frac{\pi}{2} - 3\frac{\pi}{64}\right) = \sin(90^{\circ} - 8.4375^{\circ}) = \sin(81.5625^{\circ}) = \sin\left((32-3)\frac{\pi}{64}\right)$$
$$\Rightarrow \sin\left(3\frac{\pi}{64}\right) = \sin(8.4375^{\circ}) = \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right) +$$
$$+ \left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$= \cos\left(\frac{\pi}{2} - 3\frac{\pi}{64}\right) = \cos(90^{\circ} - 8.4375^{\circ}) = \cos(81.5625^{\circ}) = \cos\left((32-3)\frac{\pi}{64}\right)$$
$$\Rightarrow \frac{\pi}{2} - 3\frac{\pi}{16} = 90^{\circ} - 33.75^{\circ} = 56.25 = (8-3)\frac{\pi}{16} = 5\frac{\pi}{16}$$
$$\Rightarrow \frac{\pi}{2} - 3\frac{\pi}{32} = 90^{\circ} - 16.875^{\circ} = 73.125^{\circ} = (16-3)\frac{\pi}{32} = 13\frac{\pi}{32}$$
$$\Rightarrow \frac{\pi}{2} - 3\frac{\pi}{64} = 90^{\circ} - 8.4375^{\circ} = 81.5625^{\circ} = (32-3)\frac{\pi}{64} = 29\frac{\pi}{64}$$
$$\Rightarrow \frac{\pi}{2} - 3\frac{\pi}{128} = 90^{\circ} - 4.21875^{\circ} = 85.78125^{\circ} = (64-3)\frac{\pi}{128} = 61\frac{\pi}{128}$$
$$\Rightarrow \frac{\pi}{2} - 3\frac{\pi}{256} = 90^{\circ} - 2.109375^{\circ} = 87.890625^{\circ} = (128-3)\frac{\pi}{256} = 125\frac{\pi}{256}$$

likewise:

$$\theta = \frac{\pi}{32} = 5.625^\circ \Rightarrow 5\frac{\theta}{2} = 14.0625^\circ :$$
$$\Rightarrow \frac{\pi}{32} = 5.625^\circ \Rightarrow 5\frac{5.625^\circ}{2} = 14.0625^\circ \Leftrightarrow 90^\circ - 14.0625^\circ = 75.9375^\circ$$

$$\Rightarrow \cos\left(5\frac{\pi}{64}\right) = \cos(14.0625^\circ) = \left[\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)^2 - \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)^2\right]\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$-2\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$= \left[\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right]\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right) - 2\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{4}\right)\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$= \sin\left(\frac{\pi}{2} - 5\frac{\pi}{64}\right) = \sin(90^\circ - 14.0625^\circ) = \sin(75.9375^\circ) = \sin\left((32-5)\frac{\pi}{64}\right)$$

$$\Rightarrow \sin\left(5\frac{\pi}{64}\right) = \sin(14.0625^\circ) = \left[\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)^2 - \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)^2\right]\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$+2\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$= \left[\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right]\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right) + 2\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{4}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}\right)$$
$$= \cos\left(\frac{\pi}{2} - 5\frac{\pi}{64}\right) = \cos(90^\circ - 14.0625^\circ) = \cos(75.9375^\circ) = \cos\left((32-5)\frac{\pi}{64}\right)$$

$$\Rightarrow \frac{\pi}{2} - 5\frac{\pi}{16} = 90^\circ - 56.25^\circ = (8-5)\frac{\pi}{16} = 3\frac{\pi}{16}$$
$$\Rightarrow \frac{\pi}{2} - 5\frac{\pi}{32} = 90^\circ - 28.125^\circ = 61.875^\circ = (16-5)\frac{\pi}{32} = 11\frac{\pi}{32}$$
$$\Rightarrow \frac{\pi}{2} - 5\frac{\pi}{64} = 90^\circ - 14.0625^\circ = 75.9375^\circ = (32-5)\frac{\pi}{64} = 27\frac{\pi}{64}$$
$$\Rightarrow \frac{\pi}{2} - 5\frac{\pi}{128} = 90^\circ - 7.03125^\circ = 82.96875^\circ = (64-5)\frac{\pi}{128} = 59\frac{\pi}{128}$$
$$\Rightarrow \frac{\pi}{2} - 5\frac{\pi}{256} = 90^\circ - 3.515625^\circ = 86.484375^\circ = (128-5)\frac{\pi}{256} = 123\frac{\pi}{256}$$

so, clearly:

$$\Rightarrow \frac{\pi}{2} - (2n+1)\frac{\pi}{2^m} = (2^p - (2n+1))\frac{\pi}{2^m} \quad , \quad (2n+1 < 2^p, p < m, p \in \mathbb{N})$$

and, the half-angle operation (as previous) may be repeatedly applied to each of these angles

and, the half-angle operation (as previous) may be repeatedly applied to each of the $(2n+1)\frac{\pi}{2^m}$ & $(2n+1)\frac{\pi}{3 \cdot 2^m}$ angles.

The growth of $m\&n$ cast a net over the region $\left(0, \frac{\pi}{2}\right)$ establishing a exact sine/cosine for any angle therein.

Mathematical physics is my art - my music.

This merely makes plain the when and who of this publication.

You may edi it away as I have been for years. But that is akin to editing away all music but classical ...

jazz, rock-and-roll, country, soul, rap, hip-hop, etc.

Is that censorship really right? And does it lead to light or darkness?