Proof for specific type of continued fraction

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Abstract

I am going to prove the following result:

$$f(n) = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

I am going to use telescoping series and then a proof by induction
i am using Lambert's continued fraction for the base case

(i am providing the proof people asked me to give about one of my formulas)

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=?$$

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\frac{2k}{(n+1-k)(n+k)}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=$$

we can rewrite the 1st term in this way:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!}{(n-(k-1))!(k-1)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=$$

This is a telescoping series

Side note:

for the first term (when k=0) I will use this form:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]$$

inorder to a void the (-1)!

$$k=0::\left(2^{n-(0-1)}\frac{n!}{(2n)!}\frac{(n+(0-1))!0}{(n-(0-1))!(0)!}-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}\right)$$

$$k=1::\left(2^{n-(1-1)}\frac{n!}{(2n)!}\frac{(n+(1-1))!}{(n-(1-1))!(1-1)!}-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}\right)$$

$$k=2::\left(2^{n-(2-1)}\frac{n!}{(2n)!}\frac{(n+(2-1))!}{(n-(2-1))!(2-1)!}-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}\right)$$

$$k=n-1::\left(2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}-2^{1}\frac{n!}{(2n)!}\frac{(n+n-1)!}{(n-n+1)!(n-1)!}\right)$$

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=\left(0-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}\right)+$$
$$\left(2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}\right)+\left(2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}\right)+$$
$$\left(2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}-2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}\right)+\ldots+\left(2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}-2^{1}\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}\right)$$

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=0+$$

$$\left(-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}+2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}\right)+\left(-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}+2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}\right)+$$

$$\left(-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}+2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}\right)+\left(-2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}+2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}\right)+$$

$$\ldots+\left(-2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}+2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}\right)-2^{1}\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}$$

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=0-2^{1}\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}=-1$$

$$\boxed{\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=-1}$$

the same way we can prove for:

$$\boxed{\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^{n}}{(-2)^{k}}\right)\frac{(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]=-1}$$

you need to change the above form to this form:

$$\sum_{k=0}^{n-1}\left[(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}+(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!}{(n-(k-1))!(k-1)!}\right]=-1$$

This is also a telescoping series! (just like before)

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]=-1$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{1}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{1}{(-2)^n}\right)\frac{(2n)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=-1$$

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{2^n}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]=-1$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{1}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{1}{2^n}\right)\frac{(2n)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^n}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

now lets combin both

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

and we will get this:

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\boxed{\begin{aligned}&\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=\\&\left(\frac{e^2}{(-2)^n}-\frac{(-1)^n}{2^n}+\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}-\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=\\&\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(2n+1)!}{n!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}+2\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(2n+1)!}{n!}\end{aligned}}$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}\right]+$$

$$(2n+1)\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(2n)!}{n!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}+\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(2n+2)!}{n+1!}=0$$

$$\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+(2n+1)\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(n+n)!}{(n-n)!n!}+$$

$$\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(n+1+n)!}{(n+1-n)!n!}+\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(n+1+n+1)!}{(n+1-n-1)!n+1!}=0$$

$$(2n+1)\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}+\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}=\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}$$

$$2n+1+\frac{\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}=\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

$$\boxed{2n+1+\frac{1}{\dfrac{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}{\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}}=\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}}$$

$$\boxed{f(n)=\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}}$$

$$2n+1+\frac{1}{f(n+1)}=f(n)$$

this is what we wanted to get! (remember this because we will use this result later in the Induction step)

$$f(1)=\frac{\sum_{k=0}^{0}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^1}{2^k}\right)\frac{(1-1+k)!}{(1-1-k)!k!}}{\sum_{k=0}^{1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1-1}}{2^k}\right)\frac{(1+k)!}{(1-k)!k!}}=\frac{\left(\frac{e^2}{(-2)^0}-\frac{1}{2^0}\right)\frac{(0)!}{(0)!0!}}{\left(\frac{e^2}{(-2)^0}+\frac{1}{2^0}\right)\frac{(1)!}{(1)!0!}+\left(\frac{e^2}{(-2)^1}+\frac{1}{2^1}\right)\frac{(1+1)!}{(1-1)!1!}}=\frac{e^2-1}{\left(e^2+1\right)+\left(-e^2+1\right)}=\frac{e^2-1}{2}$$

Lambert's continued fraction

$$\tanh(x)=\cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\cfrac{x^2}{9+\cfrac{x^2}{11+\ddots}}}}}}=\frac{e^{2x}-1}{e^{2x}+1}$$

$$\tanh(1)=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}}=\frac{e^2-1}{e^2+1}$$

$$\frac{e^2+1}{e^2-1}=1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}$$

$$\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}=\frac{e^2+1}{e^2-1}-1=\frac{2}{e^2-1}$$

$$\boxed{f(1)=3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}=\frac{e^2-1}{2}}$$

this is what we wanted!

(This is known result but I am showing how to get to it anyway)

A proof by induction

Proposition:

$$f(n) = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

Base Case: n=1

$$f(1) = 3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}} = \frac{\sum_{k=0}^{0}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^1}{2^k}\right)\frac{(1-1+k)!}{(1-1-k)!k!}}{\sum_{k=0}^{1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1-1}}{2^k}\right)\frac{(1+k)!}{(1-k)!k!}} = \frac{e^2-1}{2}$$ (I already showed that above!)

Induction step

$$f(n+1) = 2(n+1)+1+\cfrac{1}{2(n+1)+3+\cfrac{1}{2(n+1)+5+\cfrac{1}{2(n+1)+7+\cfrac{1}{2(n+1)+9+\ddots}}}}$$

$$f(n+1) = 2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\cfrac{1}{2n+11+\ddots}}}}$$

$$\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}} = f(n) = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = 2n+1+\frac{1}{f(n+1)}$$

liked we showed already above (at the top of the previous page)

QED