Proof of Fermat's Last Theorem for odd primes

Minho Baek

ABSTRACT. It was already proved right that $x^n + y^n = z^n$, $(n > 2)$ has no solutions in positive integers which we called Fermat's Last Theorem (FLT) by Andrew Wiles. But his proof would be impossible in the 17th century. Since Fermat showed he proved n=even by leaving proof for n=4, many people have tried to prove the odd primes. I took the idea from Euler proof and proved in case of n=odd primes by simple method.

1. Introduction

Pierre de Fermat claimed he had proof that no three positive integer x, y and z satisfy the equation $x^n + y^n = z^n$ for n greater than 2 which we called Fermat's Last Theorem (FLT). For about 3 centuries, many people have tried to prove FLT. Finally, FLT was proved right by Andrew Wiles in 1995. However, the proof of Wiles is a modern math that is difficult to understand and complex. So, some people, who think Fermat prove FLT by himself, still believe elementary method exist. We know that FLT can be proved by proving n=odd primes because case of n=even proved by Fermat himself. I got an idea to solve for n=odd primes from Euler's proof. In this paper, I proved the case of n=odd primes by simple method.

2. Proof for n=3

$$x^3 + y^3 = z^3, (x < y < z)$$

Where x, y and $z =$ positive integer, relatively prime

This equation can be classified into three categories as follows.

Case 1. (x, y, z) = (even, odd, odd)

Case 2. (x, y, z) = (odd, even, odd)

Case 3. (x, y, z) = (odd, odd, even)

Case 1. (x, y, z) = (even, odd, odd)

Let y=(u−v), z=(u+v).

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$y = f(U - V)$$

$$z = f(U + V)$$

But this contradicts because y and z are relatively prime.

So, u and v are relatively prime.

Also u and v are opposite parity because y and z are odd.

$$x^3 = (u + v)^3 - (u - v)^3$$

$$x^3 = 2v(v^2 + 3u^2)$$

Assume u=even, v=odd.

$v^2 + 3u^2$ is odd.

Assume u=odd, v=even.

$v^2 + 3u^2$ is odd.

So, $v^2 + 3u^2$ is always odd.

Thus the greatest common factor of 2v and $v^2 + 3u^2$ is odd.

Assume the common factor is odd except 1 and 3.

Let $v = fV$ and $v^2 + 3u^2 = fN$.

$$3u^2 = f(N - fV^2)$$

u and v have common factor of f.

But this contradicts because u and v are relatively prime.

So, the greatest common factor of 2v and $v^2 + 3u^2$ is either 1 or 3.

Assume the greatest common factor of 2v and $v^2 + 3u^2$ is 1.

It is possible that $2v = p^3$ and $v^2 + 3u^2 = q^3$.

Where p and q are relatively prime

$$x^3 = p^3q^3$$

Assume the greatest common factor of 2v and $v^2 + 3u^2$ is 3.

Let v=3r.

u and r are relatively prime because u and v are relatively prime.

$$x^3 = 6r(9r^2 + 3u^2)$$

$$x^3 = 18r(u^2 + 3r^2)$$

The greatest common factor of 18r and $u^2 + 3r^2$ must be 1.

It is possible that $18r = p'^3$ and $u^2 + 3r^2 = q'^3$.

$$x^3 = p'^3q'^3$$

Assume z=y+i.

From $y=(u-v)$, $z=(u+v)$,

$$z - y = (u + v) - (u - v) = 2v$$

$$2v = i$$

Assume y and i are not relatively prime.

Let $y=fY$, $i=fI$

$$z = y + i$$

$$z = f(Y + I)$$

But this contradicts because y and z are relatively prime.

So, y and i are relatively prime.

Assume $y=k+i$.

k is odd because y is odd and i is even.

Assume k and y are not relatively prime.

Let $k=fK$, $y=fY$.

$$i = y - k$$

$$i = f(Y - K)$$

But this contradicts because y and i are relatively prime.

So, k and y are relatively prime.

Let $k=(u'-v)$, $y=(u'+v)$.

Assume u' and v are not relatively prime.

Let $u'=fU'$, $v=fV$

$$k = f(U' - V)$$

$$y = f(U' + V)$$

But this contradicts because k and y are relatively prime.

So, u' and v are relatively prime.

Also u' and v are opposite parity because k and y are odd.

$$y^3 - k^3 = (u' + v)^3 - (u' - v)^3$$

$$y^3 - k^3 = 2v(v^2 + 3u'^2)$$

Assume $u'=$even, $v=$odd.

$v^2 + 3u'^2$ is odd.

Assume u'=odd, v=even.

$v^2 + 3u'^2$ is odd.

So, $v^2 + 3u'^2$ is always odd.

Thus the greatest common factor of 2v and $v^2 + 3u'^2$ is odd.

Assume the common factor is odd except 1 and 3.

Let v=fV and $v^2 + 3u'^2$=fN.

$$3u'^2 = f(N - fV^2)$$

u' and v have common factor of f.

But this contradicts because u' and v are relatively prime.

So, the greatest common factor of 2v and $v^2 + 3u'^2$ is either 1 or 3.

Assume the greatest common factor of 2v and $v^2 + 3u'^2$ is 1,

It is possible that $2v = p^3$ and $v^2 + 3u'^2 = \gamma$.

$$y^3 - k^3 = \gamma p^3$$

Assume $y^3 = \gamma y'$, $k^3 = \gamma k'$ or $y^3 = \alpha p^3$, $k^3 = \beta p^3$.

But this contradicts because k and y are relatively prime.

So, γ must be s^3.

$$y^3 - k^3 = s^3 p^3$$

Where s and p are relatively prime

Let $y^3 = c^3, k^3 = b^3, s^3 p^3 = a^3$

$$a^3 + b^3 = c^3$$

But this equation contradicts by the method of infinite descent.

Assume the greatest common factor of 2v and $v^2 + 3u'^2$ is 3.

Let v=3r.

u' and r are relatively prime because u' and v are relatively prime.

$$y^3 - k^3 = 6r(9r^2 + 3u'^2)$$

$$y^3 - k^3 = 18r(u'^2 + 3r^2)$$

The greatest common factor of 18r and $u'^2 + 3r^2$ must be 1.

It is possible that $18r = p'^3$ and $u'^2 + 3r^2 = \gamma'$.

$$y^3 - k^3 = \gamma' p'^3$$

Assume $y^3 = \gamma y'$, $k^3 = \gamma' k'$ or $y^3 = \alpha' p'^3$, $k^3 = \beta' p'^3$

But this contradicts because k and y are relatively prime.

So, γ' must be s'^3.

$$y^3 - k^3 = s'^3 p'^3$$

Where s' and p' are relatively prime

Let $y^3 = c^3, k^3 = b^3, s'^3 p'^3 = a^3$.

$$a^3 + b^3 = c^3$$

But this equation contradicts by the method of infinite descent.

Thus Case 1 has a contradiction.

Case 2. (x, y, z) = (odd, even, odd)

Let x=(u−v), z=(u+v).

Case 2 has a contradiction because it can be proved in the same form as Case 1.

Case 3. (x, y, z) = (odd, odd, even)

Let x=(u−v), y=(u+v).

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$x = f(U - V)$$

$$y = f(U + V)$$

But this contradicts because x and y are relatively prime.

So, u and v are relatively prime.

Also u and v are opposite parity because x and y are odd.

$$z^3 = (u + v)^3 + (u - v)^3$$

$$z^3 = 2u(u^2 + 3v^2)$$

Assume u=even, v=odd.

$u^2 + 3v^2$ is odd.

Assume u=odd, v=even.

$u^2 + 3v^2$ is odd.

So, $u^2 + 3v^2$ is always odd.

Thus the greatest common factor of 2u and $u^2 + 3v^2$ is odd.

Assume the common factor is odd except 1 and 3.

Let u=fU and $u^2 + 3v^2$=fN.

$$3v^2 = f(N - fU^2)$$

u and v have common factor of f.

But this contradicts because u and v are relatively prime.

So, the greatest common factor of 2u and $u^2 + 3v^2$ is either 1 or 3.

Assume the greatest common factor of 2u and $u^2 + 3v^2$ is 1.

It is possible that $2u = p^3$ and $u^2 + 3v^2 = q^3$.

Where p and q are relatively prime

$$z^3 = p^3q^3$$

Assume the greatest common factor of 2u and $u^2 + 3v^2$ is 3.

Let u=3r.

v and r are relatively prime because u and v are relatively prime.

$$z^3 = 6r(9r^2 + 3v^2)$$

$$z^3 = 18r(v^2 + 3r^2)$$

The greatest common factor of 18r and $v^2 + 3r^2$ must be 1.

It is possible that $18r = p'^3$ and $v^2 + 3r^2 = q'^3$.

$$z^3 = p'^3q'^3$$

Let k=u−v', l=u+v'.

Where $v' = 3v$

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$x = f(U - V)$$

$$y = f(U + V)$$

But this contradicts because x and y are relatively prime.

So, u and v are relatively prime.

Assume x and u are not relatively prime.

Let x=fX, u=fU.

$$v = u - x$$

$$v = f(U + X)$$

But this contradicts because u and v are relatively prime.

So, x and u are relatively prime.

Assume u and v’ are not relatively prime.

Let u=fU, v’=fV’.

$$x = u - \frac{1}{3}v'$$

$$x = f(U - \frac{1}{3}V')$$

But this contradicts because x and u are relatively prime.

So, u and v’ are relatively prime.

Assume k and v’ are not relatively prime.

Let k=fK, v’=fV’

$$u = k - v'$$

$$u = f(K - V')$$

But this contradicts because u and v’ are relatively prime.

So, k and v’ are relatively prime.

Assume l and k are not relatively prime.

Let l=fL, k=fK.

$$2v' = l - k$$

$$2v' = f(L - K)$$

But this contradicts because k and v' are relatively prime.

So, l and k are relatively prime.

u and v' are opposite parity because l and k are odd.

$$k^3 + l^3 = (u - v')^3 + (u + v')^3$$

$$k^3 + l^3 = 2u(u^2 + 3v'^2)$$

Assume u=even, v'=odd.

$u^2 + 3v'^2$ is odd.

Assume u=odd, v'=even.

$u^2 + 3v'^2$ is odd.

So, $u^2 + 3v'^2$ is always odd.

Thus the greatest common factor of 2u and $u^2 + 3v'^2$ is odd.

Assume the common factor is odd except 1 and 3.

Let u=fU and $u^2 + 3v'^2$=fN.

$$3v'^2 = f(N - fU^2)$$

u and v' have common factor of f.

But this contradicts because u and v' are relatively prime.

So, the greatest common factor of 2u and $u^2 + 3v'^2$ is either 1 or 3.

Assume the greatest common factor of 2u and $u^2 + 3v'^2$ is 1.

It is possible that $2u = p^3$ and $u^2 + 3v'^2 = \gamma$.

$$k^3 + l^3 = \gamma p^3$$

Assume $k^3 = \gamma k'$, $l^3 = \gamma l'$ or $k^3 = \alpha p^3$, $l^3 = \beta p^3$.

But this contradicts because k and l are relatively prime.

So, γ must be s^3.

$$k^3 + l^3 = s^3 p^3$$

Where s and p are relatively prime

Let $k^3 = a^3, l^3 = b^3, s^3p^3 = c^3$

$$a^3 + b^3 = c^3$$

But this equation contradicts by the method of infinite descent.

Assume the greatest common factor of 2u and $u^2 + 3v'^2$ is 3.

Let u=3r.

u and r are relatively prime because u and v' are relatively prime.

$$k^3 + l^3 = 6r(9r^2 + 3v'^2)$$

$$k^3 + l^3 = 18r(v'^2 + 3r^2)$$

The greatest common factor of 18r and $v'^2 + 3r^2$ must be 1.

It is possible that $18r = p'^3$ and $u'^2 + 3r^2 = \gamma'$.

$$k^3 + l^3 = \gamma' p'^3$$

Assume $k^3 = \gamma' k'$, $l^3 = \gamma' l'$ or $k^3 = \alpha' p'^3$, $l^3 = \beta' p'^3$.

But this contradicts because k and l are relatively prime.

So, γ' must be s'^3.

$$k^3 + l^3 = s'^3 p'^3$$

Where s' and p' are relatively prime.

Let $k^3 = a^3, l^3 = b^3, p'^3 s'^3 = c^3$

$$a^3 + b^3 = c^3$$

But this equation contradicts by the method of infinite descent.

Thus Case 3 has a contradiction.

Therefore, there are no positive integers in case of n=3 since all of Case 1, 2 and 3 have a contradiction.

3. Proof for n=5

$$x^5 + y^5 = z^5, (x < y < z)$$

Where x, y and $z =$ positive integer, relatively prime

This equation can be classified into three categories as follows.

Case 1. (x, y, z) = (even, odd, odd)

Case 2. $(x, y, z) = (\text{odd}, \text{even}, \text{odd})$

Case 3. $(x, y, z) = (\text{odd}, \text{odd}, \text{even})$

Case 1. $(x, y, z) = (\text{even}, \text{odd}, \text{odd})$

Let $y=(u-v)$, $z=(u+v)$.

Assume u and v are not relatively prime.

Let $u=fU$, $v=fV$.

$$y = f(U - V)$$

$$z = f(U + V)$$

But this contradicts because y and z are relatively prime.

So, u and v are relatively prime.

Also u and v are opposite parity because y and z are odd.

$$x^5 = (u + v)^5 - (u - v)^5$$

$$x^5 = 2v(v^4 + 5(u^4 + 2u^2v^2))$$

Assume u=even, v=odd.

$v^4 + 5(u^4 + 2u^2v^2)$ is odd.

Assume u=odd, v=even.

$v^4 + 5(u^4 + 2u^2v^2)$ is odd.

So, $v^4 + 5(u^4 + 2u^2v^2)$ is always odd.

Thus the greatest common factor of 2v and $v^4 + 5(u^4 + 2u^2v^2)$ is odd.

Assume the common factor is odd except 1 and 5.

Let $v = fV$ and $v^4 + 5(u^4 + 2u^2v^2) = fN$.

$$f^4V^4 + 5(u^4 + 2u^2f^2V^2) = fN$$

$$5u^4 = f(N - f^3V^4 - 2u^2fV^2)$$

u and v have common factor of f.

But this contradicts because u and v are relatively prime.

So, the greatest common factor of 2v and $v^4 + 5(u^4 + 2u^2v^2)$ is either 1 or 5.

Assume the greatest common factor of 2v and $v^4 + 5(u^4 + 2u^2v^2)$ is 1.

It is possible that $2v = p^5$ and $v^4 + 5(u^4 + 2u^2v^2) = q^5$.

Where p and q are relatively prime

$$x^5 = p^5q^5$$

Assume the greatest common factor of 2v and $v^4 + 5(u^4 + 2u^2v^2)$ is 5.

Let v=5r.

u and r are relatively prime because u and v are relatively prime.

$$x^5 = 10r(5^4r^4 + 5(u^4 + 2u^25^2r^2))$$

$$x^5 = 50r(5^3r^4 + (u^4 + 2u^25^2r^2))$$

The greatest common factor of 50r and $5^3r^4 + (u^4 + 2u^25^2r^2)$ must be 1.

It is possible that $50r = p'^5$ and $5^3r^4 + (u^4 + 2u^25^2r^2) = q'^5$.

$$x^5 = p'^5q'^5$$

Assume z=y+i.

From y=(u−v), z=(u+v),

$$z - y = (u + v) - (u - v) = 2v$$

$$2v = i$$

Assume y and i are not relatively prime.

Let y=fY, i=fI

$$z = y + i$$

$$z = f(Y + I)$$

But this contradicts because y and z are relatively prime.

So, y and i are relatively prime.

Assume y=k+i.

k is odd because y is odd and i is even.

Assume k and y are not relatively prime.

Let k=fK, y=fY.

$$i = y - k$$

$$i = f(Y - K)$$

But this contradicts because y and i are relatively prime.

So, k and y are relatively prime.

Let k=(u'−v), y=(u'+v).

Assume u' and v are not relatively prime.

Let u'=fU', v=fV

$$k = f(U' - V)$$

$$y = f(U' + V)$$

But this contradicts because k and y are relatively prime.

So, u' and v are relatively prime.

Also u' and v are opposite parity because k and y are odd.

$$y^5 - k^5 = (u' + v)^5 - (u' - v)^5$$

$$y^5 - k^5 = 2v(v^4 + 5(u'^4 + 2u'^2v^2))$$

Assume u'=even, v=odd.

$v^4 + 5(u'^4 + 2u'^2v^2)$ is odd.

Assume u'=odd, v=even.

$v^4 + 5(u'^4 + 2u'^2v^2)$ is odd.

So, $v^4 + 5(u'^4 + 2u'^2v^2)$ is always odd.

Thus the greatest common factor of 2v and $v^4 + 5(u'^4 + 2u'^2v^2)$ is odd.

Assume the common factor is odd except 1 and 5.

Let $v = fV$ and $v^4 + 5(u'^4 + 2u'^2v^2) = fN$.

$$f^4V^4 + 5(u'^4 + 2u'^2f^2V^2) = fN$$

$$5u'^4 = f(N - f^3V^4 - 2u'^2fV^2)$$

u' and v have common factor of f.

But this contradicts because u' and v are relatively prime.

So, the greatest common factor of 2v and $v^4 + 5(u'^4 + 2u'^2v^2)$ is either 1 or 5.

Assume the greatest common factor of 2v and $v^4 + 5(u'^4 + 2u'^2v^2)$ is 1,

It is possible that $2v = p^5$ and $v^4 + 5(u'^4 + 2u'^2v^2) = \gamma$.

$$y^5 - k^5 = \gamma p^5$$

Assume $y^5 = \gamma y'$ $k^5 = \gamma k'$ or $y^5 = \alpha p^5$, $k^5 = \beta p^5$.

But this contradicts because k and y are relatively prime.

So, γ must be s^5.

$$y^5 - k^5 = s^5 p^5$$

Where s and p are relatively prime

Let $y^5 = c^5, k^5 = b^5, s^5 p^5 = a^5$

$$a^5 + b^5 = c^5$$

But this equation contradicts by the method of infinite descent.

Assume the greatest common factor of 2v and $v^4 + 5(u'^4 + 2u'^2 v^2)$ is 5.

Let v=5r.

u' and r are relatively prime because u' and v are relatively prime.

$$y^5 - k^5 = 10r(5^4 r^4 + 5(u'^4 + 2u'^2 5^2 r^2))$$

$$y^5 - k^5 = 50r(5^3 r^4 + (u'^4 + 2u'^2 5^2 r^2))$$

The greatest common factor of 50r and $5^3 r^4 + (u'^4 + 2u'^2 5^2 r^2)$ must be 1.

It is possible that $50r = p'^5$ and $5^3 r^4 + (u'^4 + 2u'^2 5^2 r^2) = \gamma'$.

$$y^5 - k^5 = \gamma' p'^5$$

Assume $y^5 = \gamma' y'$, $k^5 = \gamma' k'$ or $y^5 = \alpha' p'^5$, $k^5 = \beta' p'^5$

But this contradicts because k and y are relatively prime.

So, γ' must be s'^5.

$$y^5 - k^5 = s'^5 p'^5$$

Where s' and p' are relatively prime.

Let $y^5 = c^5, k^5 = b^5, s'^5 p'^5 = a^5$.

$$a^5 + b^5 = c^5$$

But this contradicts by the method of infinite descent.

Thus Case 1 has a contradiction.

Case 2. (x, y, z) = (odd, even, odd)

Let $x=(u-v)$, $z=(u+v)$.

Case 2 has a contradiction because it can be proved in the same form as Case 1.

Case 3. (x, y, z) = (odd, odd, even)

Let $x=(u-v)$, $y=(u+v)$.

Assume u and v are not relatively prime.

Let $u=fU$, $v=fV$.

$$x = f(U - V)$$

$$y = f(U + V)$$

But this contradicts because x and y are relatively prime.

So, u and v are relatively prime.

Also u and v are opposite parity because x and y are odd.

$$z^5 = (u + v)^5 + (u - v)^5$$

$$z^5 = 2u(u^4 + 5(v^4 + 2v^2u^2))$$

Assume u=even, v=odd.

$u^4 + 5(v^4 + 2v^2u^2)$ is odd.

Assume u=odd, v=even.

$u^4 + 5(v^4 + 2v^2u^2)$ is odd.

So, $u^4 + 5(v^4 + 2v^2u^2)$ is always odd.

Thus the greatest common factor of 2u and $u^4 + 5(v^4 + 2v^2u^2)$ is odd.

Assume the common factor is odd except 1 and 5.

Let $u = fU$ and $u^4 + 5(v^4 + 2v^2u^2) = fN$.

$$f^4U^4 + 5(v^4 + 2v^2f^2U^2) = fN$$

$$5v^4 = f(N - f^3U^4 - 2v^2fU^2)$$

u and v have common factor of f.

But this contradicts because u and v are relatively prime.

So, the greatest common factor of 2u and $u^4 + 5(v^4 + 2v^2u^2)$ is either 1 or 5.

Assume the greatest common factor of 2u and $u^4+5(v^4+2v^2u^2)$ is 1.

It is possible that $2u=p^5$ and $u^4+5(v^4+2v^2u^2)=q^5$.

Where p and q are relatively prime

$$z^5=p^5q^5$$

Assume the greatest common factor of 2v and $v^4+5(u^4+2u^2v^2)$ is 5.

Let u=5r.

u and r are relatively prime because u and v are relatively prime.

$$x^5=10r(5^4r^4+5(v^4+2v^25^2r^2))$$

$$x^5=50r(5^3r^4+(v^4+2v^25^2r^2))$$

The greatest common factor of 50r and $5^3r^4+(v^4+2v^25^2r^2)$ must be 1.

It is possible that $50r=p'^5$ and $5^3r^4+(v^4+2v^25^2r^2)=q'^5$.

$$x^5=p'^5q'^5$$

Let k=u−v', l=u+v'.

Where $v'=3v$

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$x=f(U-V)$$

$$y=f(U+V)$$

But this contradicts because x and y are relatively prime.

So, u and v are relatively prime.

Assume x and u are not relatively prime.

Let x=fX, u=fU.

$$v=u-x$$

$$v=f(U+X)$$

But this contradicts because u and v are relatively prime.

So, x and u are relatively prime.

Assume u and v' are not relatively prime.

Let u=fU, v'=fV'.

$$x = u - \frac{1}{3}v'$$

$$x = f(U - \frac{1}{3}V')$$

But this contradicts because x and u are relatively prime.

So, u and v' are relatively prime.

Assume k and v' are not relatively prime.

Let k=fK, v'=fV'

$$u = k - v'$$

$$u = f(K - V')$$

But this contradicts because u and v' are relatively prime.

So, k and v' are relatively prime.

Assume l and k are not relatively prime.

Let l=fL, k=fK.

$$2v' = l - k$$

$$2v' = f(L - K)$$

But this contradicts because k and v' are relatively prime.

So, l and k are relatively prime.

u and v' are opposite parity because l and k are odd.

$$k^5 + l^5 = (u - v')^5 + (u + v')^5$$

$$k^5 + l^5 = 2u(u^4 + 5(v'^4 + 2v'^2u^2))$$

Assume u=even, v'=odd.

$u^4 + 5(v'^4 + 2v'^2u^2)$ is odd.

Assume u=odd, v'=even.

$u^4 + 5(v'^4 + 2v'^2u^2)$ is odd.

So, $u^4 + 5(v'^4 + 2v'^2u^2)$ is always odd.

Thus the greatest common factor of 2u and $u^4 + 5(v'^4 + 2v'^2u^2)$ is odd.

Assume the common factor is odd except 1 and 5.

Let $u = fU$ and $u^4 + 5\left(v'^4 + 2v'^2u^2\right) = fN$.

$$f^4U^4 + 5(v'^4 + 2v'^2f^2U^2) = fN$$

$$5v'^4 = f(N - f^3U^4 - 2v'^2fU^2)$$

u and v' have common factor of f.

But this contradicts because u and v' are relatively prime.

So, the greatest common factor of 2u and $u^4 + 5(v'^4 + 2v'^2u^2)$ is either 1 or 5.

Assume the greatest common factor of 2u and $u^4 + 5(v'^4 + 2v'^2u^2)$ is 1.

It is possible that $2u = p^5$ and $u^4 + 5(v'^4 + 2v'^2u^2) = \gamma$.

$$k^5 + l^5 = \gamma p^5$$

Assume $k^5 = \gamma k'$, $l^5 = \gamma l'$ or $k^5 = \alpha p^5$, $l^5 = \beta p^5$.

But this contradicts because k and l are relatively prime.

So, γ must be s^5.

$$k^5 + l^5 = s^5p^5$$

Where s and p are relatively prime

Let $k^5 = a^5, l^5 = b^5, s^5p^5 = c^5$

$$a^5 + b^5 = c^5$$

Assume the greatest common factor of 2u and $u^4 + 5(v'^4 + 2v'^2u^2)$ is 5.

Let u=5r.

u and r are relatively prime because u and v' are relatively prime.

$$k^5 + l^5 = 10r(5^4r^4 + 5(v'^4 + 2v'^2 5^2r^2))$$

$$k^5 + l^5 = 50r(5^3r^4 + (v'^4 + 2v'^2 5^2r^2))$$

The greatest common factor of 50r and $5^3r^4 + (v'^4 + 2v'^2 5^2r^2)$ must be 1.

It is possible that $50r = p'^5$ and $5^3r^4 + (v'^4 + 2v'^2 5^2r^2) = \gamma'$.

$$k^5 + l^5 = \gamma' p'^5$$

Assume $k^5 = \gamma' k'$, $l^5 = \gamma' l'$ or $k^5 = \alpha' p'^5$, $l^5 = \beta' p'^5$.

But this contradicts because k and l are relatively prime.

So, γ' must be s'^5.

$$k^5 + l^5 = s'^5 p'^5$$

Where s' and p' are relatively prime.

Let $k^5 = a^5, l^5 = b^5, p'^5 s'^5 = c^5$

$$a^5 + b^5 = c^5$$

But this equation contradicts by the method of infinite descent.

Thus Case 3 has a contradiction.

Therefore, there are no positive integers in case of n=5 since all of Case 1, 2 and 3 have a contradiction.

4. Proof for n=odd primes

$$x^n + y^n = z^n, (x < y < z)$$

Where x, y and $z =$ positive integer, relatively prime

This equation can be classified into three categories as follows.

Case 1. (x, y, z) = (even, odd, odd)

Case 2. (x, y, z) = (odd, even, odd)

Case 3. (x, y, z) = (odd, odd, even)

Case 1. (x, y, z) = (even, odd, odd)

Let y=(u−v), z=(u+v).

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$y = f(U - V)$$

$$z = f(U + V)$$

But this contradicts because y and z are relatively prime.

So, u and v are relatively prime.

Also u and v are opposite parity because y and z are odd.

$$x^n = (u + v)^n - (u - v)^n$$

$$x^n = (u^n + C_1 u^{n-1} v^1 + C_2 u^{n-2} v^2 + \cdots + C_{n-2} u^2 v^{n-2} + C_{n-1} u^1 v^{n-1} + v^n)$$

$$-(u^n - C_1u^{n-1}v^1 + C_2u^{n-2}v^2 + \cdots - C_{n-2}u^2v^{n-2} + C_{n-1}u^1v^{n-1} - v^n)$$

Where $C = \{C_1, C_2, \cdots, C_{n-2}, C_{n-1}\} = \{\frac{n}{1}, \frac{n(n-1)}{1\cdot 2}, \cdots, \frac{n(n-1)\cdots 4\cdot 3}{1\cdot 2\cdots(n-3)(n-2)}, \frac{n(n-1)\cdots\cdot 2\cdot 1}{1\cdot 2\cdots(n-2)(n-1)}\}$

$$x^n = 2v(v^{n-1} + n(u^{n-1} + C'_3u^{n-3}v^2 + \cdots + C'_{n-2}u^2v^{n-3}))$$

Where $C' = \{C'_1, C'_2, \cdots, C'_{n-2}, C'_{n-1}\} = \{\frac{1}{1}, \frac{(n-1)}{1\cdot 2}, \cdots, \frac{(n-1)\cdots 4\cdot 3}{1\cdot 2\cdots(n-3)(n-2)}, \frac{(n-1)\cdots\cdot 2\cdot 1}{1\cdot 2\cdots(n-2)(n-1)}\}$

$$x^n = 2v(v^{n-1} + n(u^{n-1} + M))$$

Where $M = C'_3u^{n-3}v^2 + C'_5u^{n-5}v^4 + \cdots + C'_{n-4}u^4v^{n-5} + C'_{n-2}u^2v^{n-3}$

Assume u=even, v=odd.

$v^{n-1} + n(u^{n-1} + M)$ is odd.

Assume u=odd, v=even.

$v^{n-1} + n(u^{n-1} + M)$is odd.

So, $v^{n-1} + n(u^{n-1} + M)$ is always odd.

Thus the greatest common factor of 2v and $v^{n-1} + n(u^{n-1} + M)$ is odd.

Assume the common factor is odd except 1 and n.

Let $v = fV$ and $v^{n-1} + n(u^{n-1} + M) = fN$.

$$f^{n-1}V^{n-1} + n(u^{n-1} + fM') = fN$$

Where $fM' = f(C'_3u^{n-3}f^1V^2 + C'_5u^{n-5}f^3V^4 + \cdots + C'_{n-4}u^4f^{n-6}V^{n-5} + C'_{n-2}u^2f^{n-4}V^{n-3})$

$$nu^{n-1} = f(N - f^{n-2}V^{n-1} - nM')$$

u and v have common factor of f.

But this contradicts because u and v are relatively prime.

So, the greatest common factor of 2v and $v^{n-1} + n(u^{n-1} + M)$ is either 1 or n.

Assume the greatest common factor of 2v and $v^{n-1} + n(u^{n-1} + M)$ is 1.

It is possible that $2v = p^n$ and $v^{n-1} + n(u^{n-1} + M) = q^n$.

Where p and q are relatively prime

$$x^n = p^nq^n$$

Assume the greatest common factor of 2v and $v^{n-1} + n(u^{n-1} + M)$ is n.

Let v=nr.

u and r are relatively prime because u and v are relatively prime.

$$x^n = 2nv(v^{n-1} + n(u^{n-1} + M))$$

$$x^n = 2n^2v(n^{n-2}r^{n-1} + (u^{n-1} + M))$$

The greatest common factor of $2n^2v$ and $n^{n-2}r^{n-1} + (u^{n-1} + M)$ must be 1.

It is possible that $2n^2v = p'^n$ and $n^{n-2}r^{n-1} + (u^{n-1} + M) = q'^n$.

$$x^n = p'^n q'^n$$

Assume z=y+i.

From y=(u−v), z=(u+v),

$$z - y = (u + v) - (u - v) = 2v$$

$$2v = i$$

Assume y and i are not relatively prime.

Let y=fY, i=fI

$$z = y + i$$

$$z = f(Y + I)$$

But this contradicts because y and z are relatively prime.

So, y and i are relatively prime.

Assume y=k+i.

k is odd because y is odd and i is even.

Assume k and y are not relatively prime.

Let k=fK, y=fY.

$$i = y - k$$

$$i = f(Y - K)$$

But this contradicts because y and i are relatively prime.

So, k and y are relatively prime.

Let k=(u'−v), y=(u'+v).

Assume u' and v are not relatively prime.

Let u'=fU', v=fV

$$k = f(U' - V)$$

$$y = f(U' + V)$$

But this contradicts because k and y are relatively prime.

So, u' and v are relatively prime.

Also u' and v are opposite parity because k and y are odd.

$$y^n - k^n = (u' + v)^n - (u' - v)^n$$

$$y^n - k^n = 2v(v^{n-1} + n(u'^{n-1} + Q))$$

Where $Q = C'_3 u'^{n-3} v^2 + C'_5 u'^{n-5} v^4 + \cdots + C'_{n-4} u'^4 v^{n-5} + C'_{n-2} u'^2 v^{n-3}$

Assume u'=even, v=odd.

$v^{n-1} + n(u'^{n-1} + Q)$ is odd.

Assume u'=odd, v=even.

$v^{n-1} + n(u'^{n-1} + Q)$ is odd.

So, $v^{n-1} + n(u'^{n-1} + Q)$ is always odd.

Thus the greatest common factor of 2v and $v^{n-1} + n(u'^{n-1} + Q)$ is odd.

Assume the common factor is odd except 1 and n.

Let $v = fV$ and $v^{n-1} + n(u'^{n-1} + Q) = fN$.

$$f^{n-1}V^{n-1} + n(u'^{n-1} + fQ') = fN$$

Where $fQ' = f(C'_3 u'^{n-3} f^1 V^2 + C'_5 u'^{n-5} f^3 V^4 + \cdots + C'_{n-4} u'^4 f^{n-6} V^{n-5} + C'_{n-2} u'^2 f^{n-4} V^{n-3})$

$$nu'^{n-1} = f(N - f^{n-2}V^{n-1} - nQ')$$

u' and v have common factor of f.

But this contradicts because u' and v are relatively prime.

So, the greatest common factor of 2v and $v^{n-1} + n(u'^{n-1} + Q)$ is either 1 or n.

Assume the greatest common factor of 2v and $v^{n-1} + n(u'^{n-1} + Q)$ is 1,

It is possible that $2v = p^n$ and $v^{n-1} + n(u'^{n-1} + Q) = \gamma$.

$$y^n - k^n = \gamma p^n$$

Assume $y^n = \gamma y'$, $k^n = \gamma k'$ or $y^n = \alpha p^n$, $k^n = \beta p^n$.

But this contradicts because k and y are relatively prime.

So, γ must be s^n.

$$y^n - k^n = s^n p^n$$

Where s and p are relatively prime

Let $y^n = c^n, k^n = b^n, s^n p^n = a^n$

$$a^n + b^n = c^n$$

But this equation contradicts by the method of infinite descent.

Assume the greatest common factor of 2v and $v^{n-1} + n(u'^{n-1} + Q)$ is n.

Let v=nr.

u' and r are relatively prime because u' and v are relatively prime.

$$y^n - k^n = 2nr(v^{n-1} + n(u'^{n-1} + Q))$$

$$y^n - k^n = 2n^2 r(n^{n-2} r^{n-1} + (u'^{n-1} + Q))$$

The greatest common factor of $2n^2 r$ and $n^{n-2} r^{n-1} + (u'^{n-1} + Q)$ must be 1.

It is possible that $2n^2 r = p'^n$ and $n^{n-2} r^{n-1} + (u'^{n-1} + Q) = \gamma'$.

$$y^n - k^n = \gamma' p'^n$$

Assume $y^n = \gamma' y'$, $k^n = \gamma' k'$ or $y^n = \alpha' p'^n$, $k^n = \beta' p'^n$

But this contradicts because k and y are relatively prime.

So, γ' must be s'^n.

$$y^n - k^n = s'^n p'^n$$

Where s' and p' are relatively prime.

Let $y^n = c^n, k^n = b^n, s'^n p'^n = a^n$.

$$a^n + b^n = c^n$$

But this equation contradicts by the method of infinite descent.

Thus Case 1 has a contradiction.

Case 2. (x, y, z) = (odd, even, odd)

Let x=(u−v), z=(u+v).

Case 2 has a contradiction because it can be proved in the same form as Case 1.

Case 3. (x, y, z) = (odd, odd, even)

Let x=(u−v), y=(u+v).

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$x = f(U - V)$$

$$y = f(U + V)$$

But this contradicts because x and y are relatively prime.

So, u and v are relatively prime.

Also u and v are opposite parity because x and y are odd.

$$z^n = (u - v)^n + (u + v)^n$$

$$z^n = (u^n - C_1 u^{n-1} v^1 + C_2 u^{n-2} v^2 + \cdots - C_{n-2} u^2 v^{n-2} + C_{n-1} u^1 v^{n-1} - v^n) + (u^n + C_1 u^{n-1} v^1 + C_2 u^{n-2} v^2 + \cdots + C_{n-2} u^2 v^{n-2} + C_{n-1} u^1 v^{n-1} + v^n)$$

Where $C = \{C_1, C_2, \cdots, C_{n-2}, C_{n-1}\} = \{\frac{n}{1}, \frac{n(n-1)}{1\cdot 2}, \cdots, \frac{n(n-1)\cdots 4\cdot 3}{1\cdot 2\cdots(n-3)(n-2)}, \frac{n(n-1)\cdots\cdot 2\cdot 1}{1\cdot 2\cdots(n-2)(n-1)}\}$

$$x^n = 2u(u^{n-1} + n(v^{n-1} + C'_{n-3} v^{n-3} u^2 + \cdots + C'_2 v^2 u^{n-3}))$$

Where $C' = \{C'_1, C'_2, \cdots, C'_{n-2}, C'_{n-1}\} = \{\frac{1}{1}, \frac{(n-1)}{1\cdot 2}, \cdots, \frac{(n-1)\cdots 4\cdot 3}{1\cdot 2\cdots(n-3)(n-2)}, \frac{(n-1)\cdots\cdot 2\cdot 1}{1\cdot 2\cdots(n-2)(n-1)}\}$

$$x^n = 2u(u^{n-1} + n(v^{n-1} + M))$$

Where $M = C'_{n-3} v^{n-3} u^2 + C'_{n-5} v^{n-5} u^4 + \cdots + C'_4 v^4 u^{n-5} + C'_2 v^2 u^{n-3}$

Assume u=even, v=odd.

$u^{n-1} + n(v^{n-1} + M)$ is odd.

Assume u=odd, v=even.

$u^{n-1} + n(v^{n-1} + M)$ is odd.

So, $u^{n-1} + n(v^{n-1} + M)$ is always odd.

Thus the greatest common factor of 2u and $u^{n-1} + n(v^{n-1} + M)$ is odd.

Assume the common factor is odd except 1 and n.

Let $u = fU$ and $u^{n-1} + n(v^{n-1} + M) = fN$.

$$f^{n-1} U^{n-1} + n(u^{n-1} + fM') = fN$$

Where $fM' = f(C'_{n-3} v^{n-3} f^1 U^2 + C'_{n-5} v^{n-5} f^3 U^4 + \cdots + C'_4 v^4 f^{n-6} U^{n-5} + C'_2 v^2 f^{n-4} U^{n-3})$

$$nu^{n-1} = f(N - f^{n-2} U^{n-1} - nM')$$

u and v have common factor of f.

But this contradicts because u and v are relatively prime.

So, the greatest common factor of 2u and $u^{n-1} + n(v^{n-1} + M)$ is either 1 or n.

Assume the greatest common factor of 2u and $u^{n-1} + n(v^{n-1} + M)$ is 1.

It is possible that $2u = p^n$ and $u^{n-1} + n(v^{n-1} + M) = q^n$.

Where p and q are relatively prime

$$z^n = p^n q^n$$

Assume the greatest common factor of 2v and $u^{n-1} + n(v^{n-1} + M)$ is n.

Let u=nr.

u and r are relatively prime because u and v are relatively prime.

$$x^n = 2nr(u^{n-1} + n(v^{n-1} + M))$$

$$x^n = 2n^2 r(n^{n-2} r^{n-1} + (v^{n-1} + M))$$

The greatest common factor of $2n^2 r$ and $n^{n-2} r^{n-1} + (v^{n-1} + M)$ must be 1.

It is possible that $2n^2 r = p'^n$ and $n^{n-2} r^{n-1} + (v^{n-1} + M) = q'^n$.

$$x^n = p'^n q'^n$$

Let k=u−v', l=u+v'.

Where $v' = 3v$

Assume u and v are not relatively prime.

Let u=fU, v=fV.

$$x = f(U - V)$$

$$y = f(U + V)$$

But this contradicts because x and y are relatively prime.

So, u and v are relatively prime.

Assume x and u are not relatively prime.

Let x=fX, u=fU.

$$v = u - x$$

$$v = f(U + X)$$

But this contradicts because u and v are relatively prime.

So, x and u are relatively prime.

Assume u and v' are not relatively prime.

Let u=fU, v'=fV'.

$$x = u - \frac{1}{3}v'$$

$$x = f(U - \frac{1}{3}V')$$

But this contradicts because x and u are relatively prime.

So, u and v' are relatively prime.

Assume k and v' are not relatively prime.

Let k=fK, v'=fV'

$$u = k - v'$$

$$u = f(K - V')$$

But this contradicts because u and v' are relatively prime.

So, k and v' are relatively prime.

Assume l and k are not relatively prime.

Let l=fL, k=fK.

$$2v' = l - k$$

$$2v' = f(L - K)$$

But this contradicts because k and v' are relatively prime.

So, l and k are relatively prime.

u and v' are opposite parity because l and k are odd.

$$k^n + l^n = (u - v')^n + (u + v')^n$$

$$k^n + l^n = 2u(u^{n-1} + n(v'^{n-1} + Q))$$

Where $Q = C'_{n-3}v'^{n-3}u^2 + C'_{n-5}v'^{n-5}u^4 + \cdots + C'_4v'^4u^{n-5} + C'_2v'^2u^{n-3}$

Assume u=even, v'=odd.

$u^{n-1} + n(v'^{n-1} + Q)$ is odd.

Assume u=odd, v'=even.

$u^{n-1} + n(v'^{n-1} + Q)$ is odd.

So, $u^{n-1} + n(v'^{n-1} + Q)$ is always odd.

Thus the greatest common factor of 2u and $u^{n-1} + n(v'^{n-1} + Q)$ is odd.

Assume the common factor is odd except 1 and n.

Let $u = fU$ and $u^{n-1} + n(v'^{n-1} + Q) = fN$.

$$f^{n-1}U^{n-1} + n(v'^{n-1} + fQ') = fN$$

Where $fQ' = f(C'_{n-3}v'^{n-3}f^1U^2 + C'_{n-5}v'^{n-5}f^3U^4 + \cdots + C'_4v'^4f^{n-6}U^{n-5} + C'_2v'^2f^{n-4}U^{n-3})$

$$nv'^{n-1} = f(N - f^{n-2}U^{n-1} - nQ')$$

u and v' have common factor of f.

But this contradicts because u and v' are relatively prime.

So, the greatest common factor of 2u and $u^{n-1} + n(v'^{n-1} + Q)$ is either 1 or n.

Assume the greatest common factor of 2u and $u^{n-1} + n(v'^{n-1} + Q)$ is 1.

It is possible that $2u = p^n$ and $u^{n-1} + n(v'^{n-1} + Q) = \gamma$.

$$k^n + l^n = \gamma p^n$$

Assume $k^n = \gamma k'$ $l^n = \gamma l'$ or $k^n = \alpha p^n$, $l^n = \beta p^n$.

But this contradicts because k and l are relatively prime.

So, γ must be s^n.

$$k^n + l^n = s^n p^n$$

Where s and p are relatively prime

Let $k^n = a^n, l^n = b^n, s^n p^n = c^n$

$$a^n + b^n = c^n$$

But this equation contradicts by the method of infinite descent.

Assume the greatest common factor of 2u and $u^{n-1} + n(v'^{n-1} + Q)$ is n.

Let u=nr.

u and r are relatively prime because u and v' are relatively prime.

$$k^n + l^n = 2nr(u^{n-1} + n(v'^{n-1} + Q))$$

$$k^n + l^n = 2n^2r(n^{n-2}r^{n-1} + (v'^{n-1} + Q))$$

The greatest common factor of $2n^2r$ and $n^{n-2}r^{n-1} + (v'^{n-1} + Q)$ must be 1.

It is possible that $2n^2r = p'^n$ and $n^{n-2}r^{n-1} + (v'^{n-1} + Q) = \gamma'$.

$$k^n + l^n = \gamma' p'^n$$

Assume $k^n = \gamma' k'$, $l^n = \gamma' l'$ or $k^n = \alpha' p'^n$, $l^n = \beta' p'^n$.

But this contradicts because k and l are relatively prime.

So, γ' must be s'^n.

$$k^n + l^n = s'^n p'^n$$

Where s' and p' are relatively prime.

Let $k^n = a^n, l^n = b^n$, $s'^n p'^n = c^n$

$$a^n + b^n = c^n$$

But this equation contradicts by the method of infinite descent.

Thus Case 3 has a contradiction.

Therefore, there are no positive integers in case of n=odd primes since all of Case 1, 2 and 3 have a contradiction.

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