A Proof of Riemann Hypothesis by Symmetry and Circular Properties of Riemann Zeta Function

Tae Beom Lee

Abstract: Riemann zeta function(RZF) $\zeta(s)$ is a function of a complex variable $s = x + iy$. Riemann hypothesis(RH) states that all the non-trivial zeros of RZF lie on the critical line, $0.5 + iy$. The symmetricity of RZF zeros implies that if $\zeta(\alpha + i\beta) = 0, 0 < \alpha < 0.5,$ then $\zeta(1 - \alpha + i\beta) = 0$, too. The graphs of RZF are similar to the graphs of circles with non-uniform radius and argument. These two, symmetry and circular properties of RZF, are the basis of our proof.

1. Introduction

RZF [1][2][3][4][5] $\zeta(s)$ and Dirichlet eta function(DEF) [6] $\eta(s)$ are functions of a complex variable $s = x + iy$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{1.1}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \tag{1.2}$$

RZF converges for $x > 1$ and DEF converges for $x > 0$. It will be reasonable to use DEF when dealing domain $0 < x < 1$. But, because zeros of RZF and DEF are same, we used RZF without loss of generality.

RH [1][7][8] states that all the non-trivial zeros of RZF are of the form $s = 0.5 + iy$ and still remains unsolved.

The symmetricity of RZF zeros implies that if $\zeta(\alpha + i\beta) = 0,$ then $\zeta(1 - \alpha + i\beta) = 0$, too. So, the two zeros should be on the two edge lines of a strip $\alpha \le x \le 1 - \alpha, 0 < \alpha < 0.5$. (From now on, suppose $0 < \alpha < 0.5$, otherwise specified.)

To satisfy $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$, two curves, $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$, must intersect at the origin, when $y = \beta$. This means that, if we RZF-map the linear movement of x from α to $1 - \alpha$ for $y = \beta$, the image should be to a closed loop. This loop starts from the origin, where $\zeta(\alpha + i\beta) = 0$, and ends at the origin, where $\zeta(1 - \alpha + i\beta) = 0$.

The graphs of RZF are similar to the graphs of circles with non-uniform radius and argument. For a circle, the rotational motion is always orthogonal to the radial motion. For RZF graphs, too, the rotational motion is always orthogonal to the radial motion.

Our proof is based on the symmetry and the circular properties of RZF. We showed that, without breaching the orthogonal relationships between the rotational and the radial motions, the closed loop from $\zeta(\alpha + i\beta) = 0$ to $\zeta(1 - \alpha + i\beta) = 0$ can't be drawn.

2. Terminologies

Definition 2.1. Domain strip: A strip $\alpha \le x \le 1 - \alpha,$ $-\infty \le y \le \infty$.

Definition 2.2. Domain edge lines: Two edge lines of a domain strip.

Definition 2.3. Range strip: A strip generated through a mapping of a domain strip.

Definition 2.4. Range edge lines: Two edge lines of a range strip.

Definition 2.5. Contour *C*: A closed curve in lemma 3.2.

3. Symmetry Properties of RZF Zeros

The following three equations are well known. $\xi(s)$ is Riemann Xi function [8][9].

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{\frac{-s}{2}}. \tag{3.1}$$

$$\xi(s) = \xi(1-s). \tag{3.2}$$

$$\zeta(\overline{s}) = \overline{\zeta(s)}. \tag{3.3}$$

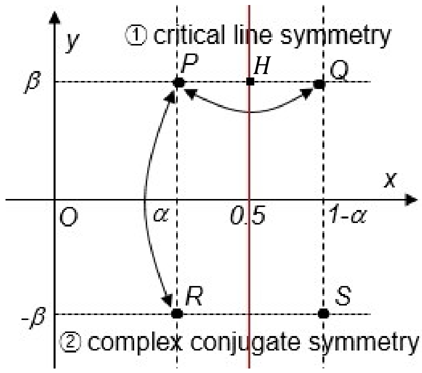
The right side of (3.1) includes $\zeta(s)$, so, the zeros of $\zeta(s)$ are also the zeros $\xi(s)$.

Lemma 3.1. Equations (3.2) and (3.3) means that there exist two types of symmetries of RZF zeros, as in figure 1.

① **Critical line symmetry:** Symmetry of (3.2), which means that if $s = \alpha + i\beta$ is a zero, then $s = 1 - \alpha + i\beta$ is also a zero.

② **Complex conjugate symmetry:** Symmetry of (3.3), which means that if $s = \alpha + i\beta$ is a zero, then $s = \alpha - i\beta$ is also a zero.

Figure 1. Zero symmetries of RZF.



Proof. First, in (3.3), $\zeta(\alpha - i\beta) = \overline{\zeta(\alpha + i\beta)} = 0$, that corresponds $\zeta(R) = \overline{\zeta(P)} = 0$ in figure 1, which is the complex conjugate symmetry. Second, in (3.2), $\xi(\alpha + i\beta) = \xi\{1 - (\alpha + i\beta)\} = 0$, that corresponds $\xi(P) = \xi(S) = 0$ in figure 1. By the complex conjugate symmetry, $\xi(S) = \xi(Q) = 0$. So, $\xi(P) = \xi(Q) = 0$, which is the critical line symmetry. ■

Lemma 3.2. To satisfy a critical line symmetry, $\zeta(\alpha + iy) = \zeta(1 - \alpha + iy) = 0, y = \beta$, a closed curve(contour) must be drawn by the movement of x in $\alpha \le x \le 1 - \alpha, y = \beta$.

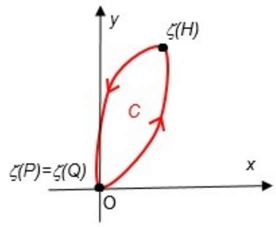
Proof. In figure 1, *P(α, β)* and *Q(1 - α, β)* are critical line symmetry zeros, and *H(0.5, β)* lies on $x = 0.5$. Then a contour must be drawn by the following 3 steps.

① **Initial state at (α, β):** At *P(α, β)*, graph remains at the origin as $\zeta(P)$ in figure 2.

② **Movement to $H(0.5, \beta)$:** Graph leaves the origin and reaches $\zeta(H)$ in figure 2.

③ **Movement to $Q(1 - \alpha, \beta)$:** Graph leaves $\zeta(H)$ and reaches back to the origin $\zeta(Q)$ in figure 2.

Figure 2. Contour *C* for $\alpha \le x \le 1-\alpha$.



So, the RZF image graph for a line segment $\alpha \le x \le 1-\alpha$ should be a closed curve, as in Figure 2. Let's call it a *contour C*. ■

4. Orthogonal Properties of RZF Graphs

RZF is the infinite sum of trigonometric functions with different amplitude and argument.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, s = x + iy.$$

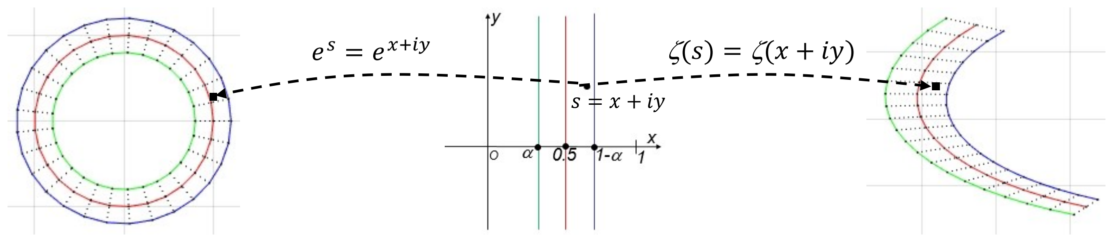
$$\mathrm{v}\frac{1}{n^s} = e^{-xlnn} e^{-iylnn}.$$

$$\zeta(s) = \sum_{n=1}^{\infty} e^{-xlnn} e^{-iylnn}$$

$$= \sum_{n=1}^{\infty} e^{-xlnn} cos(-ylnn) + i\sum_{n=1}^{\infty} e^{-xlnn} sin(-ylnn).$$

Figure 3 shows how a domain strip is mapped to a circular or a RZF range strip.

Figure 3. Strip mapping examples.



(a) Circular range strip. (b) Domain strip. (c) RZF range strip.

For the circular range strip in figure 3 (a), let

$f(s) = e^s = e^{x+iy} = e^x e^{iy}$, then the derivatives [10][11] w.r.t. x and y are

$$\frac{\partial f(s)}{\partial x} = e^x e^{iy} = f(s),$$

$$\frac{\partial f(s)}{\partial y} = ie^x e^{iy} = i\frac{\partial f(s)}{\partial x}, |\frac{\partial f(s)}{\partial x}| = |\frac{\partial f(s)}{\partial y}|. \quad (4.1)$$

For RZF range strip in figure 3 (c), the derivatives w.r.t. x and y are

$$\frac{\partial \zeta(s)}{\partial x} = -\sum_{n=1}^{\infty}(lnn)e^{-xlnn}e^{-iylnn},$$

$$\frac{\partial \zeta(s)}{\partial y} = -i\sum_{n=1}^{\infty}(lnn)e^{-xlnn}e^{-iylnn} = i\frac{\partial \zeta(s)}{\partial x}, |\frac{\partial \zeta(s)}{\partial x}| = |\frac{\partial \zeta(s)}{\partial y}|. \quad (4.2)$$

In (4.1), $\frac{\partial f(s)}{\partial x}$ is the radial variation(motion) of $f(s)$ and $\frac{\partial f(s)}{\partial y}$ is the rotational variation of $f(s)$, and they are orthogonal to each other with same magnitude.

In (4.2), $\frac{\partial \zeta(s)}{\partial x}$ is the radial variation of $\zeta(s)$, and $\frac{\partial \zeta(s)}{\partial y}$ is the rotational variation of $\zeta(s)$, and they are orthogonal to each other with same magnitude.

5. Illustrative Edge Line Intersections

Figure 4 shows illustrative edge line intersections. We forged zeros by multiplying $(s - s_1)(s - s_2)$ and $cos\varphi$ to $\zeta(s)$. Let $g(s) = (s - s_1)(s - s_2)\zeta(s)$ and $h(s) = cos\varphi\zeta(s)$, where $s_1 = \alpha + i\beta$, $s_2 = 1 - \alpha + i\beta$ and $\varphi = \frac{y\pi}{\beta}$. When $s = s_1$ or $s = s_2$, $g(s) = 0$, and when $y = \frac{\beta}{2}$, $h(s) = 0$.

Figure 4. Illustrative edge line intersections.

(a) $g(s) = (s - s_1)(s - s_2)\zeta(s)$. (b) $h(s) = cos\varphi\zeta(s)$.

Figure 4 (a) depicts $g(s) = (s - s_1)(s - s_2)\zeta(s), 7.5 \leq y \leq 8, s_1 \approx 0.1 + 7.74i, s_2 \approx 0.9 + 7.74i$, with following observations.

① Two range edge lines for $x = 0.1$ and $x = 0.9$ intersect at the origin $(0,0)$ when $s = s_1$ or $s = s_2$.

② When two range edge lines intersect, the line segment $0.1 \leq x \leq 0.9$ is mapped to the red contour *C*, which starts from $(0,0)$ and returns back to $(0,0)$.

③ RZF graphs for $0.1 < x < 0.5$ and $0.5 < x < 0.9$ intersect contour *C* at two points.

④ RZF graph for $x = 0.5$ contact contour *C* tangentially at *T*.

⑤ The orthogonal properties of RZF graphs in (4.2) is not kept.

Figure 4 (b) depicts $g(s) = cos\varphi\zeta(s), 7.5 \leq y \leq 8, \varphi = \frac{(y-7.5)\pi}{8-7.5}$, with following observations.

① Two range edge lines for $x = 0.1$ and $x = 0.9$ intersect at the origin $(0,0)$ when $y = 7.75$ and $\varphi = \frac{(7.75-7.5)\pi}{8-7.5} = \frac{\pi}{2}$.

② There is no contour *C* because all other lines also pass the origin.

③ This pattern is meaningless because there exist infinitely many zeros.

The MATLAB [11] coding for figure 4 is provided in appendix A.

6. Proving Lemmas

Lemma 6.1. The contour *C*, $\alpha \le x \le 1-\alpha, y=\beta$, should be tangential to the graph $\zeta(0.5+iy)$.

Proof. Suppose the contour *C* is not tangential to the graph $\zeta(0.5+iy)$, as in figure 5 (a).

Figure 5. Contour *C* intersections with the $\zeta(0.5+iy)$ graph.

(a) Non-tangential intersection.. (b) Tangential intersection..

RZF is analytic and continuous function. So, the movement of x in $\alpha \le x \le 0.5, y=\beta$ will draw a portion of contour *C,* which starts from $(0{,}0)$ and ends at T_1 or T_2. It must end at T_1 or T_2 because T_1 or T_2 is a point where $x=0.5$ and the graph $\zeta(0.5+iy)$ intersects the contour *C*. Let's suppose it ends at T_1.

Likewise, the movement of x in $1-\alpha \ge x \ge 0.5, y=\beta$ will draw the other portion of contour *C,* which starts from $(0{,}0)$ and ends at T_2. It must end at T_2 because T_2 is a point where $x=0.5$ and the graph $\zeta(0.5+iy)$ intersects the contour *C*.

So, the contour *C* can't be closed which contradicts lemma 3.2. So, the contour *C* must be tangential to the graph $\zeta(0.5+iy)$, as in figure 5 (b). ∎

Lemma 6.2. The graph $\zeta(0.5+iy)$ and the contour *C* can't satisfy the orthogonal properties of RZF in (4.2) at the tangential point.

Proof. Any two tangentially contacting curves can't be orthogonal to each other at the point of contact because they have common slope. That is to say, at the tangential point the rotational and the radial variations have same direction. So, the orthogonal properties in (4.2) can't be kept at the tangential point. ∎

6. Conclusion

The symmetricity of RZF zeros implies that if $\zeta(\alpha+i\beta)=0$, then $\zeta(1-\alpha+i\beta)=0$, too. So, the two zeros should be on the two edge lines of a strip $\alpha \le x \le 1-\alpha$. To satisfy $\zeta(\alpha+i\beta)=\zeta(1-\alpha+i\beta)=0$, two curves, $\zeta(\alpha+iy)$ and $\zeta(1-\alpha+iy)$, must intersect at the origin, when $y=\beta$. This means that, if we RZF-map the linear movement of x from α to $1-\alpha$ for $y=\beta$, the image should be to a closed loop, what we called contour *C*.

For a circle, the rotational motion is always orthogonal to the radial motion. For RZF graphs, too, the rotational motion is always orthogonal to the radial motion. And we proved that, without breaching these orthogonal properties of RZF graphs, the contour *C* can't be drawn.

Appendix A.

```
% Riemann Zeta Function: For illustrative edge line intersection
clear;
clc;
close all

amin = 0.1 %range of alpha
amax = 0.9

bmin = 7.5 %range of beta
bmax = 8

na = 11 %alpha sampling
nb = 20 %beta sampling

zi = 10 %index of zero, s1 and s2

a = linspace(amin, amax, na)
b = linspace(bmin, bmax, nb)

%make domain strip complex numbers
for i=1:length(a)
	for j=1:length(b)
		s(i, j) = complex(a(i), b(j))
	end
end

%map by RZF or other variations
for i=1:length(a)
	for j=1:length(b)
		%z(i, j) = zeta(s(i, j)) %RZF
		%z(i, j) = zeta(s(i, j))*(s(i, j) - s(1, zi))*(s(i, j) - s(na, zi))
		z(i, j) = zeta(s(i, j))*cos((j-1)*pi/(nb-1))
		%z(i, j) = s(i, j)
		x(i, j) = real(z(i, j))
		y(i, j) = imag(z(i, j))
	end
end

%figure id & window size
f = figure(1)
f.Position = [50 50 300 300] %[x y width height]

%plot trajectories for each alpha
for i=1:length(a) %for alpha values

	for j=1:length(b)
		xb(j) = real(z(i, j))
		yb(j) = imag(z(i, j))
	end

	if (a(i)>=0.5) && (a(i)<0.501) %alpha=0.5
		plot(xb, yb, '-black', linewidth=1.5)
	elseif a(i)==amin %alpha=a
		plot(xb, yb, '-green', linewidth=1)
	elseif a(i)==amax %alpha=1-a
		plot(xb, yb, '-blue', linewidth=1)
	elseif mod(i,3)==0 %selective plot

if a(i)<(amin+amax)/2 %for a<x<0.5
			plot(xb, yb, '-green', linewidth=0.5)
		else %for 0.5<x<1-a
			plot(xb, yb, '-blue', linewidth=0.5)
		end
	end
	hold on
	grid on
end
```

```
%plot trajectories for each beta values
for i=1:length(b) %for beta values
	for j=1:length(a)
		xa(j) = real(z(j, i))
		ya(j) = imag(z(j, i))
	end

	if b(i)==bmin %start
		plot(xa, ya, ":black o", linewidth=0.5, markersize=1)
	elseif b(i)==bmax %end
		plot(xa, ya, "-.black o", linewidth=0.5, markersize=1)
	elseif xa(1)==0 && ya(1)==0 %edge lines intersect at (0, 0)
		plot(xa, ya, "-o", "color", [1, 0, 0], linewidth=1, markersize=1)
	else %others
		if mod(i,2)==0
			%plot(xa, ya, "--x", "color", "#888888", Linewidth=1,
Markersize=1)
		else
			%plot(xa, ya, ":+", "color", "#666666", Linewidth=1,
Markersize=1)
		end
	end
end
```

References

[1] B. Riemann, On the number of prime numbers less than a given quantity, Monatsberichte der Berliner Akademie, November,1859.

[2] H.M. Edwards, Riemann's Zeta Function, Dover Publications, Inc., 1974.

[3] Aleksandar Ivic, The Riemann Zeta-Function, Theory and Applications, Dover Publications, Inc., 1985.

[4] https://en.wikipedia.org/wiki/Riemann_zeta_function

[5] https://www.youtube.com/watch?v=ZlYfEqdlhk0&list=PL32446FDD4DA932C9 Lectures on Euler-Riemann Zeta Function.

[6] https://en.wikipedia.org/wiki/Dirichlet_eta_function

[7] https://en.wikipedia.org/wiki/Riemann_hypothesis

[8] Peter Borwein, et al.: The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike, 2007, Springer

[9] https://en.wikipedia.org/wiki/Riemann_Xi_function

[10] Erwin Kreyszig, Advanced Engineering Mathematics, Tenth edition, Wiley, 2011.

[11] James Ward Brown, Ruel V. Churchill, Complex Variables and Applications, Eighth Edition, McGraw-Hill, 2009.

[12] https://www.mathworks.com.

Figures

Cover Letter

Dear Editors,
I wish to submit an original research article entitled “A Proof of Riemann Hypothesis by Symmetry and Circular Properties of Riemann Zeta Function” .

I confirm that this work is original and has not been published elsewhere, nor is it currently under consideration for publication elsewhere.

We have no conflicts of interest or sponsors to disclose.

Thank you for your consideration of this manuscript.

Sincerely Yours,

Tae Beom Lee, PE
join360@naver.com