Zeta Function

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Abstract

This article delves into the properties of the Riemann zeta function, providing a demonstration of the existence of a sequence of zeros $\boldsymbol{z_k}$, where $\mathbf{\lim Re(z_k) = 1}$. The exploration of these mathematical phenomena contributes to our understanding of complex analysis and the behavior of the zeta function on the critical line.

Keywords: Zeta function, Functional equation, Zeros of zeta function.

MSC Classification: 11-11 , 11Mxx

1 Introduction

In this article, I present a demonstration revealing that the Riemann zeta function possesses a sequence of zeros z_k with the property $\lim \text{Re}(z_k) = 1$.

This is established by assuming the convergence of the series in the region $\text{Re}(s) > \rho$:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} < +\infty \tag{1.1}$$

This assumption is equivalent to the absence of zeros of the Riemann zeta function ($\zeta(s)$) in the region where $\text{Re}(s) > \rho$, a fact proven in [1].

Under the condition, I prove that the following implication holds:

$$\frac{\zeta(s)}{\zeta(1-s)} = s \int_0^{\infty} \frac{1}{x^{s+1}} \frac{\sin(2\pi x)}{\pi} dx \tag{1.2}$$

leading to a contradiction.

The proof involves the observation that:

$$-\frac{\zeta(s)}{s(s+1)(s+2)}\frac{\mu(n)}{n^{1-s}} = \int_0^\infty \frac{\theta_n(x)}{x^{s+3}}dx \tag{1.3}$$

where

$$\phi_n(x) = \int_0^x nu\frac{\mu(n)}{n}du \tag{1.4}$$

$$\theta_n(x) = \int_0^x \phi_n(u)du \tag{1.5}$$

and

$$\sum_{n=1}^{\infty}\theta_n(x) = \frac{1}{2\pi^2}(\frac{\sin(2\pi x)}{2\pi} - x) \tag{1.6}$$

To establish this result, I utilize the inverse Mellin transform to estimate an upper bound for

$$\sum_{n=M}^{\infty}\theta_n(x) \tag{1.7}$$

This yields:

$$\sum_{n=M}^{\infty}\theta_n(x) \leq x^{\rho+2}\max|\frac{1}{(\rho+2+it)}\sum_{n=M}^{\infty}\frac{\mu(n)}{n^{1-\rho-it}}|, t\in\mathbb{R}\int_{-\infty}^{\infty}|\frac{\zeta(\rho+t)}{(\rho+it)(\rho+1+it)}|dt \tag{1.8}$$

Consequently, by comparing upper bounds on both sides of the equality, we deduce the contradiction in (1.6). The proof of the inconsistency in (1.2) is straightforward, as it involves a comparison of upper bounds for the functions on either side of the equation, revealing a mismatch.

2 Fundamental Theorems

In this section, I will list some theorems used throughout the article.

Theorem 2.1. *If $\varphi(s)$ is analytic in the strip $a < \mathrm{Re}(s) < b$, and if it tends to zero uniformly as $Im(s) \to \pm\infty$ for any real value c between a and b, with its integral along such a line converging absolutely, then if*

$$f(x) = \mathcal{M}^{-1}\varphi = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\,ds,$$

we have that

$$\varphi(s) = \mathcal{M}f = \int_0^\infty x^{s-1}f(x)\,dx.$$

Conversely, suppose $f(x)$ is piecewise continuous on the positive real numbers, taking a value halfway between the limit values at any jump discontinuities, and suppose the integral

$$\varphi(s) = \int_0^\infty x^{s-1}f(x)\,dx$$

is absolutely convergent when $a < \text{Re}(s) < b$. Then f is recoverable via the inverse Mellin transform from its Mellin transform φ.

Proof. [2] $\square$

Theorem 2.2. *If* $\text{Re} > 1$*, we have:*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

If the zeta function has no zeros in the region $Re(s) > \rho$, we can extend the equality above to such a region.

Proof. [3] $\square$

Theorem 2.3. *If* $0 < \text{Re}(s) < 1$*, we have:*

$$-\frac{\zeta(s)}{s} = \int_0^\infty \frac{\{x\}}{x^{s+1}}\,dx$$

Proof. [4] $\square$

Theorem 2.4. *For any natural number $n > 1$, the sum of the values of the Möbius function $\mu(d)$ over all positive divisors of n is given by:*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \textit{if } n = 1, \\ 0 & \textit{if } n > 1. \end{cases}$$

Proof. [1] $\square$

3 Proof

In the case where $0 < \text{Re}(s) < 1$:

$$\zeta(s) = -s \int_0^\infty \frac{\{y\}}{y^{s+1}} dy \tag{3.1}$$

Thus:

$$-\frac{\zeta(s)}{s} \frac{\mu(n)}{n^{1-s}} = \int_0^\infty \frac{\{nx\}}{x^{s+1}} \frac{\mu(n)}{n} dx \tag{3.2}$$

where $n \in \mathbb{Z}$.

Integrating by parts in equation (3.2), we obtain:

$$-\frac{\zeta(s)}{s(s+1)} \frac{\mu(n)}{n^{1-s}} = \int_0^\infty \frac{\phi_n(x)}{x^{s+2}} dx \tag{3.3}$$

Where:

$$\phi_n(x) = \int_0^x \{nu\}\frac{\mu(n)}{n}du \tag{3.4}$$

Doing one more integration by parts, we have:

$$-\frac{\zeta(s)}{s(s+1)(s+2)}\frac{\mu(n)}{n^{1-s}} = \int_0^\infty \frac{\theta_n(x)}{x^{s+3}}dx \tag{3.5}$$

And

$$\theta_n(x) = \int_0^x \phi_n(u)du \tag{3.6}$$

Using the fact that, for every $0 < x < 1$, we have:

$$\{x\} = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n} \tag{3.7}$$

It follows that:

$$\sum_{n=1}^{\infty}\theta_n(x) = \frac{1}{2\pi^2}(\frac{\sin(2\pi x)}{2\pi} - x) \tag{3.8}$$

Indeed, by (3.7) we have:

$$\phi_n(x) = \frac{x}{2} + \frac{1}{2n\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi nkx) - 1}{k^2} \tag{3.9}$$

And

$$\theta_n(x) = \frac{x^2}{4} + \frac{1}{4n^2\pi^3}\sum_{k=1}^{\infty}\frac{\sin(2\pi nkx)}{k^3} - \frac{x}{2n\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \tag{3.10}$$

Thus:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\theta_n(x) = \frac{x^2}{4}\sum_{n=1}^{\infty}\frac{\mu(n)}{n} - \frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{1}{4\pi^3}\sum_{n,k=1}^{\infty}\frac{\sin(2\pi nkx)\mu(n)}{n^3k^3} \tag{3.11}$$

For:

$$\sum_{n,k=1}^{\infty}\frac{\sin(2\pi nkx)\mu(n)}{n^3k^3} = \sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^3}\sum_{n|l}\mu(n) = \sin(2\pi x) \tag{3.12}$$

(The rearrangement of the summations is justified by the uniform convergence of the series)

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0 \tag{3.13}$$

And

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^3} = 1 \tag{3.14}$$

we conclude (3.8).

Using the inverse Mellin transform on (3.5):

$$\theta_n(x) = -\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s+2}\frac{\zeta(s)}{s(s+1)(s+2)}\frac{\mu(n)}{n^{1-s}}ds \tag{3.15}$$

where $\sigma = \mathrm{Re}(s)$ and $0 < \sigma < 1$. With this:

$$\sum_{n=M}^{M+P}\theta_n(x) = -\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s+1}\frac{\zeta(s)}{s(s+1)(s+2)}\sum_{n=M}^{M+P}\frac{\mu(n)}{n^{1-s}}ds \tag{3.16}$$

Assume that:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{1-s}} < +\infty \tag{3.17}$$

for $\sigma = \mathrm{Re}\, s \leq \rho$, where it is known that $\rho < \frac{1}{2} - \epsilon$, $\epsilon > 0$.

In this case, we have:

$$\sum_{n=M}^{\infty}\theta_n(x) \leq x^{\gamma+2}\max\{\|\frac{1}{(\gamma+3+it)}\sum_{n=M}^{\infty}\frac{\mu(n)}{n^{1-\gamma-it}}\|, t\in\mathbb{R}\}\int_{-\infty}^{\infty}\|\frac{\zeta(\gamma+t)}{(\gamma+it)(\gamma+1+it)}\|dt \tag{3.18}$$

with $0 < \gamma < \rho$.

Where, by hypothesis:

$$\psi_\gamma(M) = \max\{\|\frac{1}{(\gamma+3+it)}\sum_{n=M}^{\infty}\frac{\mu(n)}{n^{1-\gamma-it}}\|, t\in\mathbb{R}\} \tag{3.19}$$

$$\lim_{M\to\infty}\psi_\gamma(M) = 0 \tag{3.20}$$

Since:

$$\sum_{n=k}^{\infty}\frac{\mu(n)}{n^s} = \frac{M(k)}{k^s} - s\int_k^{\infty}\frac{M(x)}{x^{s+1}}dx$$

Where:

$$M(x) = \sum_{n=1}^{x}\mu(n)$$

By equation (3.5), we have:

$$-\frac{\zeta(s)}{s(s+1)(s+2)}\sum_{n=1}^{M}\frac{\mu(n)}{n^{1-s}} = \int_0^{\infty}\frac{1}{x^{s+3}}\sum_{n=1}^{M}\theta_n(x)dx \tag{3.21}$$

Note that:

$$\int_0^{\infty}\frac{1}{x^{s+3}}\sum_{n=1}^{M}\theta_n(x)dx = \sum_{k=1}^{\infty}\int_0^1\frac{1}{(x+k)^{s+3}}\sum_{n=1}^{M}\theta_n(x+k)dx + \int_0^1\frac{1}{x^{s+3}}\sum_{n=1}^{M}\theta_n(x)dx \tag{3.22}$$

Using (3.18), we conclude that this difference tends to zero when $M \to \infty$, if $\mathrm{Re} < \rho - \epsilon$, for every $\epsilon > 0$. Indeed, using (3.18), we conclude that:

$$\int_0^1 G(x) \sum_{n=M}^{\infty} \theta_n(x)dx + \int_0^1 \frac{1}{x^{s+3}} \sum_{n=M}^{\infty} \theta_n(x)dx < C_\gamma \psi_\gamma(M) \int_0^1 \sum_{k=1}^{\infty} \frac{(x+k)^{\gamma+2}}{(x+k)^{\sigma+3}} dx + C_\rho \frac{\psi_\rho(M)}{\rho - \sigma} \tag{3.23}$$

Where

$$C_\sigma = \int_{-\infty}^{\infty} \| \frac{\zeta(\sigma+t)}{(\sigma+it)(\sigma+1+it)} \| dt \tag{3.24}$$

$$\gamma < \sigma$$

And the result follows from (3.19).

With this, taking the limit in (3.21), we conclude:

$$-\frac{\zeta(s)}{s(s+1)(s+2)\zeta(1-s)} = \int_0^{\infty} \frac{1}{x^{s+3}} \frac{1}{2\pi^2} \{ \frac{\sin(2\pi x)}{2\pi} - x \} dx \tag{3.25}$$

Where $0 < \mathrm{Re}\, s < \rho$.

However, by analytic continuation, it is concluded that this equality holds for all $0 < \mathrm{Re}(s) < 1$. Performing integrations by parts, we obtain:

$$\frac{\zeta(s)}{\zeta(1-s)} = s \int_0^{\infty} \frac{1}{x^{s+1}} \frac{\sin(2\pi x)}{\pi} dx \tag{3.26}$$

Defining:

$$F(s) = \pi \int_0^{\infty} \frac{\sin(2\pi x)}{x^{s+1}} dx \tag{3.27}$$

F is a holomorphic function in the region $0 < \mathrm{Re}(s) < 1$, and furthermore, $F(s) = O\left(\frac{1}{s}\right)$, indeed, writing:

$$F(s) = F_1(s) + F_2(s) \tag{3.28}$$

Where:

$$F_1(s) = \pi \int_0^{2} \frac{\sin(2\pi x)}{x^{s+1}} dx \tag{3.29}$$

$$F_2(s) = \pi \int_2^{\infty} \frac{\sin(2\pi x)}{x^{s+1}} dx \tag{3.30}$$

Note that:

$$\int_0^2 \cos 2\pi x x^{-s} dx = \frac{s}{2\pi} \int_0^2 \frac{\sin(2\pi x)}{x^{s+1}} dx \tag{3.31}$$

Hence, we conclude that $F_1(s) = O\left(\frac{1}{s}\right)$.

Now, observing that:

$$F_2(s) = 2^s \pi \int_1^{\infty} \frac{\sin(\pi x)}{x^{s+1}} dx \tag{3.32}$$

$$\int_1^{\infty} \frac{\sin(\pi x)}{x^{s+1}} dx = \frac{\pi}{s} \int_1^{\infty} \frac{\cos(\pi x)}{x^{s}} dx \tag{3.33}$$

And

$$\int_1^\infty \frac{\cos(\pi x)}{x^s}dx = \sum_{n=1}^{\infty}\int_n^{n+1}\frac{\cos(\pi x)}{x^s}dx = \int_0^1 \cos(\pi x)\sum_{n=1}^{\infty}\frac{(-1)^n}{(x+n)^s}dx \tag{3.34}$$

As the function

$$\psi(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{(x+n)^s} \tag{3.35}$$

is bounded for $Re(s) > 0$ and $x > -1$, we conclude that $F_2(s) = O\left(\frac{1}{s}\right)$.

With this result, it can be inferred from equation (3.31) that:

$$\frac{\zeta(s)}{\zeta(1-s)} = sF(s) = O(1) \tag{3.36}$$

By the Riemann functional equation:

$$\frac{\zeta(s)}{\zeta(1-s)} = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = sF(s) = O(1) \tag{3.37}$$

For every s in $0 < \mathrm{Re}(s) < 1$. Absurd, considering:

$$\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = O(\|s\|^{\frac{1}{2}-Re(s)}) \tag{3.38}$$

Therefore, it is concluded that:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \tag{3.39}$$

does not converge if $Re(s) < 1$, implying that the zeta function has a sequence of zeros $\{z_k\}$ such that $\lim \mathrm{Re}(z_k) = 1$.

4 Conclusion

In this article, I demonstrate that the Riemann zeta function possesses a sequence of zeros, with their real parts converging to 1.

References

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