Riemann Hypothesis

Bertrand Wong
Eurotech, S'pore Branch
Email: bwong8@singnet.com.sg

Abstract. This paper discusses the distribution of the non-trivial zeros of the Riemann zeta function ζ. It looks into the question of whether any non-trivial zeros would ever possibly be found off the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1, e.g., at Re(s) = 1/4, 1/3, 3/4, 4/5, etc., and why all the non-trivial zeros are always found at the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1 and not anywhere else on this critical strip, with the first 10^{13} non-trivial zeros having been found only at the critical line Re(s) = 1/2. It should be noted that a conjecture, or, hypothesis could possibly be proved by comparing it with a theorem that has been proven, which is one of the several deductions utilized in this paper. Through these several deductions presented, the paper shows how the Riemann hypothesis may be approached to arrive at a solution. In the paper, instead of merely using estimates of integrals and sums (which are imprecise and may therefore be of little or no reliability) in the support of arguments, where feasible actual computations and precise numerical facts are used to support arguments, for precision, for more sharpness in the arguments, and for "checkability" or ascertaining of the conclusions. This paper is the revised and expanded version of a paper [5] published in 2022.

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Introduction

The Riemann hypothesis is an important mathematical problem as its validity would affirm the manner of distribution of the prime numbers. The hypothesis asserts that all the non-trivial zeros of the zeta function ζ lie on the critical strip between Re(s) = 0 and Re(s) = 1 at the critical line Re(s) = 1/2 only. The moot point is whether there would be zeros appearing at other locations on this critical strip, e.g., at Re(s) = 1/4, 1/3, 3/4, 4/5, etc., which would disprove the hypothesis. All this would be looked into in this paper.

Shown below is the Riemann zeta function ζ with its terms:-

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots \qquad (1.1)$$

where s is the complex number $1/2 + bi$

For the term $1/2^{1/2\,+\,bi}$ above, e.g., whether it would be positive or negative in value would depend on which part of the complex plane this term $1/2^{1/2\,+\,bi}$ would be found in, which depends on 2 (n) and b (it does not depend on 1/2 - 1/2 and 2 (n) only determine how far the term is from zero in the complex plane). This term could be in the positive half of the complex plane whereby the term would have a positive value or the negative half of the complex plane whereby the term would have a negative value. Therefore some of the terms in the Riemann zeta function ζ would have positive values while the rest would have negative values (which depend on the values of n and b). The sum of the series in the Riemann zeta function ζ is found with a formula, e.g., the Riemann-Siegel formula, or, the Euler-Maclaurin summation formula. The Riemann zeta function ζ would turn out a non-trivial zero at the critical line Re(s) = 1/2, as more and more terms are added, when it reaches a point at the critical line Re(s) = 1/2 where the positive terms (in the positive half of the complex plane, as explained above) cancel out the negative terms (in the negative half of the complex plane), i.e., a non-trivial zero indicates the point in the Riemann zeta function ζ whereby the total value of the positive terms equals the total value of the negative terms. There would be an infinitude of such non-trivial zeros at the critical line Re(s) = 1/2, which had been proved by G. H. Hardy. Whether there would be zeros off this critical line Re(s) = 1/2 on the critical strip bounded by Re(s) = 0 and Re(s) = 1 as more and more terms are added to the Riemann zeta function ζ is still an open question, which Riemann himself had thought highly unlikely though he had not been able to provide a proof.

Riemann evidently anticipated that there would be an equal, or, almost equal number of primes among the terms in the positive half and the negative half of the complex plane when there is a zero, whereby the distribution of the primes would be statistically fair - the more terms are added to the Riemann zeta function ζ, the fairer or “more equal” would be the distribution of the primes in the positive half and the negative half of the complex plane when there is a zero. This is like the tossing of a coin whereby the more tosses there are the “more equal” would be the number of heads and the number of tails. In other words, in the longer term, with more and more terms added to the Riemann zeta function ζ, more or less 50% of the primes should be found in the positive half of the complex plane and the balance 50% should be found in the negative half of the complex plane, the more terms there are the fairer or “more equal” would be this distribution, when there is a zero, when the positive terms cancel out the negative terms in the Riemann zeta function ζ.

It is evident that through the non-trivial zeros the order or pattern of the distribution of the primes could be discerned. [1-6]

Proof of Riemann Hypothesis

We bring up some notable points about the non-trivial zeros of the Riemann zeta function ζ(s) defined by a power series shown below:-

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots$$

At the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1 all the non-trivial zeros would be found on an oscillatory sine-like wave which oscillates in spirals, there being an infinitude of these spirals, which represent the complex plane. All the properties of the prime counting function π(n) are in some way encoded in the properties of the Riemann zeta function ζ, evidently resulting in the primes and the non-trivial zeros being some sort of mirror images of one another - the regularity in the way the primes progressively thin out and the progressively better approximation of the quantity of primes towards infinity by the prime counting function $\pi(n)$, as is described by the following equation:-

$$\lim_{n\to\infty} \pi(n)/(n/\log n) = 1 \qquad (2.1)$$

(The prime number theorem states that the limit of the quotient of the 2 functions $\pi(n)$ and $n/\log n$ as n approaches infinity is 1, the larger the number n is, the better is the approximation of the quantity of primes by the prime counting function $\pi(n)$, as is implied by Equation (2.1) above; all this is in spite of the fact that the primes are scarcer and scarcer as n is larger and larger.),

mirror or reflect the regularity in the way the non-trivial zeros of the Riemann zeta function ζ line up at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, the non-trivial zeros becoming progressively closer together there, with no zeros appearing anywhere else on the critical strip, and, all this has been found to be true for the first 10^{13} non-trivial zeros.

Riemann had posited that the margin of error in the estimate of the quantity of primes less than a given number with the prime counting function $\pi(n)$ could be eliminated by utilizing the following J function which is a step function involving the non-trivial zeros expressed in terms of the zeta function ζ, which has been shown to be effective (2 steps are involved here - first, the prime counting function $\pi(n)$ is expressed in terms of the $J(n)$ function, then the $J(n)$ function is expressed in terms of the zeta function ζ, with the $J(n)$ function forming the link between the counting of the prime counting function $\pi(n)$ and the measuring (involving analysis and calculus) of the zeta function ζ, which would result in the properties of the prime counting function $\pi(n)$ somehow encoded in the properties of the zeta function ζ):-

$$J(n) = Li(n) - \sum_p Li(n^p) - \log 2 + \int_n^\infty dt/(t(t^2 - 1) \log t) \qquad (2.2)$$

where the first term $Li(n)$ is generally referred to as the "principal term" and the second term $\sum_p Li(n^p)$ had been called the "periodic terms" by Riemann, Li being the logarithmic integral

The above formula might look fearsome but is actually not. The third term log 2 is a number which is 0.69314718055994… while the fourth term $\int_n^\infty dt/(t(t^2 - 1) \log t)$ which is an integral representing the area under the curve of a certain function from the argument all the way out to infinity can only have a maximum value of 0.1400101011432869…. Since these 2 terms taken together (and minding the signs) are limited to the range from -0.6931… to -0.5531…, and since the prime counting function $\pi(n)$ deals with really large quantities up to millions and trillions they are much inconsequential and could be safely ignored. The first term or principal term $Li(n)$, where n is a real number, should also be not much of a problem as its value could be obtained from a book of mathematical tables or computed by some math software package such as *Mathematica* or *Maple*. However, special attention should be given to the second term $\sum_p Li(n^p)$ which concerns the sum of the non-trivial zeros of the zeta function ζ (p in this second term is a "rho", which is the seventeenth letter of the Greek alphabet, and it means "root" - a root is a non-trivial zero of the Riemann zeta function ζ (a root is technically speaking a solution or value of an unknown of an equation that could be factorized)). Riemann had evidently called the second term "periodic terms" as the components there vary irregularly.

The prime number theorem asserts that $\pi(n) \sim Li(n)$ (technically $Li(n) = \int_2^n dx/\log(x)$) which also implies the weaker result that $\pi(n) \sim n/\log n$. However, with $Li(n)$ the prime count estimate would have a margin of error. The Riemann hypothesis asserts that the difference between the true number of primes $p(n)$ and the estimated number of primes $q(n)$ would be not much larger than $\sqrt{n}$. With the above $J(n)$ function we could eliminate this margin of error and obtain an exact estimate of the quantity of primes less than a given number as follows:-

$$J(n) = \text{exact quantity of primes less than a given number}$$

Since the third and fourth terms of the $J(n)$ function are inconsequential and could be safely ignored, as is described above, deducting the second term from the first term should be sufficient, as is shown below:-

$$J(n) = Li(n) - \sum_p Li(n^p) = \text{exact quantity of primes less than a given number}$$

The above in brief shows the close connection between the primes and the non-trivial zeros of the Riemann zeta function ζ, the primes and the non-trivial zeros being some sort of mirror images of one another as is described above, with the distribution of the non-trivial zeros being regarded as the music of the primes by mathematicians.

An important point to note is that though the non-trivial zeros at the critical line $\text{Re}(s) = 1/2$ become more and more closely packed together the farther along we move up this critical line while the primes occur farther and farther along the number line, the density of the one is approximately the reciprocal of the density of the other whereby the complementariness, regularity, symmetry is evident, this regularity of the distribution of the non-trivial zeros mirroring the regularity of the distribution of the primes.

Thus the regularity in the way the primes progressively thin out and the progressively better approximation of the quantity of primes towards infinity by the prime counting function $\pi(n)$ as is described by Equation (2.1) above and stipulated by the prime number theorem, which all implies the regularity of the distribution of the primes, also somehow implies the regularity of the distribution of the non-trivial zeros of the Riemann zeta function ζ (the non-trivial zeros are found only at the critical line $\text{Re}(s) = 1/2$, lined up there in an orderly manner, as is described above) which is in accordance with the Riemann hypothesis, since the properties of the prime counting function $\pi(n)$ are in some way encoded in the properties of the Riemann zeta function ζ, with the distribution of the primes (which shows regularity) and the distribution of the non-trivial zeros (which also shows regularity) mirroring one another, as is explained above. Refer to Appendix 1 below to see further this close connection between the primes and the Riemann zeta function ζ whereby it is shown in the appendix that the Riemann zeta function ζ has the property of prime sieving encoded within it (compare: sieve of Eratosthenes), which further supports the reasoning here. The input of a function determines its output. The input of the Riemann zeta function ζ are the terms in its series while its output are the non-trivial zeros, and, these terms would determine the status of the non-trivial zeros, e.g., whether the non-trivial zeros would be found or would not be found after the terms are added together and whether the non-trivial zeros would be orderly or disorderly if found. In the case here, the regularity of the distribution of the primes at the terms which also have the property of prime sieving encoded within as is explained above and in Appendix 1 below, on the input side of the zeta function ζ evidently leads to the regularity of the distribution of the non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ only on its output side (i.e., the non-trivial zeros are found grouped in an orderly manner on the output side only when the terms on the input side of the zeta function ζ have the power $a + bi$ (which is a complex number) with a (the real axis) = 1/2), as is described above.

We proceed to consider whether the Riemann hypothesis could be false. Let us here assume that the Riemann hypothesis is false. If the Riemann hypothesis is false, there would be non-trivial zeros appearing at other locations on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4, 4/5$, etc., i.e., the distribution of the non-trivial zeros would be irregular, which would contradict the reasoning above, and also would imply that the distribution of the primes would also be irregular (since the primes and the non-trivial zeros are mirror images of each other, having similar "distribution" characteristics, as is explained above), which means that the prime number theorem described above, whose effectiveness is evidently due to the regularity and orderliness of the distribution of the primes, as is described above (wherein it is stated that there is regularity in the way the primes progressively thin out), would hence be false. But this would be an absurdity as the prime number theorem had been proved through non-elementary methods by Hadamard and de la Vallee Poussin who had in 1896 independently proved that none of the non-trivial zeros lie on the very edge of the critical strip on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$, which was enough for deducing the prime number theorem; the prime number theorem had also been proved later by Erdos and Selberg using elementary methods. Therefore, since the assumption of the falsity of the Riemann hypothesis would lead to the absurdity that the prime number theorem is false, the Riemann hypothesis cannot be false, and, since it cannot be false it has to be true. The reasoning here is reasoning by contradiction or *reductio ad absurdum* which is commonly used in mathematics.

Furthermore, Riemann's $J(n)$ function with its error term as is described above has been found to be able to estimate the quantity of primes less than a given number with **accuracy**; this somehow implies that the non-trivial zeros, many thousands of which are used to compute the error term $\sum_p Li(n^p)$ (with thousands of zeros with positive values (in the positive half of the complex plane) cancelling out thousands of zeros with negative values (in the negative half of the complex plane) when added together, the difference between these positive values and these negative values after adding together being the quantity in the error term, which is deducted from $Li(n)$ the principal term to give the exact quantity of primes less than a given number), are orderly and well-behaved as per the Riemann hypothesis, for if the non-trivial zeros were disorderly and poorly behaved the $J(n)$ function would in all probability not be able to estimate the quantity of primes less than a given number with accuracy - this could be regarded as further though perhaps subtle proof that the Riemann hypothesis is true. [6]

All of this somehow implies the truth of the Riemann hypothesis with the first 10^{13} non-trivial zeros all nicely, neatly and orderly lined up at the critical line $\mathrm{Re}(s) = 1/2$ exactly mid-way on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$.

So far the role of the non-trivial zeros of the Riemann zeta function ζ and the close connection between the primes and the Riemann zeta function ζ and how this is in some way responsible for the regularity of the distribution of the non-trivial zeros on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ have been explained, with some other deductions about the distribution of the non-trivial zeros having been presented as well. Next would be the explanation of the actual reason why all the non-trivial zeros of the Riemann zeta function ζ would lie on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ at the critical line $\mathrm{Re}(s) = 1/2$ only.

The locations of these non-trivial zeros on the critical strip are described by a complex number $s = 1/2 + bi$ where the real part is 1/2 and i stands for the square root of -1. It should be noted that the mathematical operations and logic of the complex numbers $a + bi$, where a and b are real numbers and i is the imaginary number square root of -1, are practically the same as for the real numbers and are even more versatile. For the Riemann zeta function ζ to be zero, its series would have to have both the positive terms and negative terms cancelling each other out, though the positive or "+" signs in the series may indicate positive values only which is misleading. We would here consider the possibility of any non-trivial zeros being off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4, 4/5$, etc.

It had been proven that there would not be zeros at $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. As is stated above, the first 10^{13} non-trivial zeros are found only at the critical line $\mathrm{Re}(s) = 1/2$. Nature appears to demand that these zeros must appear only at $\mathrm{Re}(s) = 1/2$, exactly mid-way on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ whereby the symmetry is perfect. "1/2" in the complex number $1/2 + bi$, which is "square root", also appears to be compatible with and work fine with "i", which is "square root of -1" - both of them are square roots. $1/2 + bi$ has what is called a complex conjugate $1/2 - bi$ so that when $1/2 + bi$ and $1/2 - bi$ are added together the terms bi in both $1/2 + bi$ and $1/2 - bi$ would cancel out one another leaving behind their respective real parts only - in this way the troublesome i which does not actually make mathematical sense would be exterminated. 1/2 is also the reciprocal of the smallest prime and the smallest even number 2, which is significant.

The following list of the first 10 terms of the series of the Riemann zeta function ζ with consecutive fractional powers $s \leq 1/2$ shows that the sums with smaller powers increase progressively, i.e., the smaller the power s is the larger the percentage of increase in the quantity is:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + 1/6^{1/2} + 1/7^{1/2} + 1/8^{1/2} + 1/9^{1/2} + 1/10^{1/2} + \ldots = 5.03$
(The Riemann hypothesis asserts that all zeros would be found in this series only.)

[2] $\zeta(1/3) = 1 + 1/2^{1/3} + 1/3^{1/3} + 1/4^{1/3} + 1/5^{1/3} + 1/6^{1/3} + 1/7^{1/3} + 1/8^{1/3} + 1/9^{1/3} + 1/10^{1/3} + \ldots = 6.20$
(The sum 6.20 here represents an increase of 23.26% compared to the sum 5.03 in [1] above while the percentage of decrease in power from $s = 1/2$ to $s = 1/3$ is 33.33%.)

[3] $\zeta(1/4) = 1 + 1/2^{1/4} + 1/3^{1/4} + 1/4^{1/4} + 1/5^{1/4} + 1/6^{1/4} + 1/7^{1/4} + 1/8^{1/4} + 1/9^{1/4} + 1/10^{1/4} + \ldots = 6.97$
(The sum 6.97 here represents an increase of 38.57% compared to the sum 5.03 in [1] above while the percentage of decrease in power from $s = 1/2$ to $s = 1/4$ is 50%.)

[4] $\zeta(1/5) = 1 + 1/2^{1/5} + 1/3^{1/5} + 1/4^{1/5} + 1/5^{1/5} + 1/6^{1/5} + 1/7^{1/5} + 1/8^{1/5} + 1/9^{1/5} + 1/10^{1/5} + \ldots = 7.46$

(The sum 7.46 here represents an increase of 48.31% compared to the sum 5.03 in [1] above while the percentage of decrease in power from $s = 1/2$ to $s = 1/5$ is 60%.)

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<u>Note</u>: Though the respective percentages of increase in quantity above, namely, 23.26%, 38.57% & 48.31%, are disproportionate with and lower than the respective percentages of decrease in power, namely, 33.33%, 50% & 60%, at a later stage when there are more and more terms in the series, there being an infinitude of terms, when the sums get larger and larger, the percentages of increase in quantity would all be infinitely higher than the percentages of decrease in power, as is evident from **Table 1** below. The same would apply for the quantities when the powers $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., as could be extrapolated from the above list (and is evident from Appendix 3 below).

(The series of the Riemann zeta function ζ with powers $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., would have sums which are all smaller than the sums shown in the above list for powers $s \leq 1/2$ as could be extrapolated from the above list. For the largest power on the critical strip $s = 1$, which has no zeros, the sum of the first 10 terms is a mere 2.93. Refer to Appendix 2 below for an analogous example.)

It is evident from all the above that when the sum of the series in the Riemann zeta function ζ increases too quickly as is the case when the powers $s < 1/2$, when disproportionateness between the increases and decreases in the respective quantities and powers sets in as is described above, or, too slowly as is the case when the powers $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., as could be extrapolated from the above list, the equilibrium, balance or regularity would be lost and there would not be zeros. (Refer to Appendix 2 below for an analogous example.) All the non-trivial zeros of the Riemann zeta function ζ would be at the optimum or equilibrium power $s = 1/2$ only. (The analogue of this optimum or equilibrium power could be that of a shirt or pants which exactly fits a person, e.g., size A could be too small for the person, size C too large, while size B fits just fine.) At least 10^{13} zeros have been found at $s = 1/2$ while none has been found for $s < 1/2$ and $s > 1/2$. Also, there is the easier solubility of equations with fractional powers $s = 1/2$ as compared to equations with fractional powers $s < 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc., and $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., which is explained below:-

$s = 1/2$ is the largest root among the roots with 1 as the numerator. As such $s = 1/2$ as a fractional power with 1 as the numerator gives the largest result as compared to the fractional powers with 1 as the numerator $s < 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc. (but this largest result brings the *smallest increase in quantity* as compared to the results of the fractional powers with 1 as the numerator $s < 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc., when divided by 1, e.g., $1/2^{1/2} < 1/2^{1/3} < 1/2^{1/4} < 1/2^{1/5}$, etc. - *this is an important similarity to the case for n = 2 for Fermat's last theorem described in Appendix 4 below*) - equations with fractional powers $s = 1/2$ would evidently be easier to solve than equations with fractional powers $s < 1/2$ (e.g., in a computation $s = 1/2$ needs only 1 rooting step while $s = 1/5$ needs 4 rooting steps) and $s > 1/2$, e.g., s = 2/3, 3/4, 4/5, etc. (e.g., in a computation s = 1/2 needs only 1 rooting step, while s = 4/5 needs 7 steps - 3 squaring steps for s = 4 & 4 rooting steps for s = 1/5).

All of this is similar to the case for Fermat's last theorem, which had been proved by Andrew Wiles. (Refer to Appendix 4 below for explanation on Fermat's last theorem to see this important similarity with Fermat's last theorem - it could be seen that the Riemann hypothesis is analogous to Fermat's last theorem which means what is true for Fermat's last theorem would also be true for the Riemann hypothesis.)

We bring up an important point here. If more and more terms are added to the series in the list of the sums of the Riemann zeta function ζ above where the consecutive fractional powers $s \leq 1/2$, which presently have 10 terms each, the differences in the sums between that for power s = 1/2 and that for powers s < 1/2, e.g., s = 1/3, 1/4, 1/5, etc., and, that for power s = 1/2 and that for powers s > 1/2, e.g., s = 3/4, 4/5, 5/6, etc., would be greater and greater, i.e., the differences between these sums would be more pronounced the more terms are added to the series. We could see this point by comparing, e.g., the sums of the first 5 terms of the Riemann zeta function ζ for consecutive fractional powers $s \leq 1/2$ and the sums of the first 10 terms of the Riemann zeta function ζ for consecutive fractional powers $s \leq 1/2$, which is as follows, and extrapolating from there:-

For the comparison, we here compute the sums for the first 5 terms of the series of the Riemann zeta function ζ with consecutive fractional powers $s \leq 1/2$ as follows, after which the results of this computation are incorporated in **Table 1** (shown in bold) below:

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + \ldots = 3.24$
(The Riemann hypothesis asserts that all zeros would be found in this series only.)

[2] $\zeta(1/3) = 1 + 1/2^{1/3} + 1/3^{1/3} + 1/4^{1/3} + 1/5^{1/3} + \ldots = 3.69$
(The sum 3.69 here represents an increase of **13.89%** (the increase here is **23.26%** for the 1^{st}. 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

[3] $\zeta(1/4) = 1 + 1/2^{1/4} + 1/3^{1/4} + 1/4^{1/4} + 1/5^{1/4} + \ldots = 3.98$
(The sum 3.98 here represents an increase of **22.84%** (the increase here is **38.57%** for the 1^{st}. 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

[4] $\zeta(1/5) = 1 + 1/2^{1/5} + 1/3^{1/5} + 1/4^{1/5} + 1/5^{1/5} + \ldots = 4.15$
(The sum 4.15 here represents an increase of **28.09%** (the increase here is **48.31%** for the 1^{st}. 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

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Table 1 below of the above-mentioned **percentage *increases*** for the sums for the first 2 terms to the first 10 terms for ζ(1/3), ζ(1/4) & ζ(1/5) would give a clearer picture:-

	1^{st}. 2 Terms	1^{st}. 3 Terms	1^{st}. 4 Terms	1^{st}. 5 Terms	1^{st}. 6 Terms	1^{st}. 7 Terms	1^{st}. 8 Terms	1^{st}. 9 Terms	1^{st}. 10 Terms	1^{st}. 11 Terms	…
[1] ζ(1/2)	-	-	-	-	-	-	-	-	-	-	
[2] ζ(1/3)	4.68%	8.30%	11.47%	**13.89%**	16.16%	18.11%	20.09%	21.87%	**23.26%**	To Be Extrapolated	
[3] ζ(1/4)	7.60%	13.54%	18.28%	**22.84%**	26.30%	29.78%	32.88%	35.88%	**38.57%**	To Be Extrapolated	
[4] ζ(1/5)	9.36%	16.59%	22.94%	**28.09%**	32.88%	37.22%	41.32%	45.01%	**48.31%**	To Be Extrapolated	

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It is evident that the percentage increases shown above would go up in value continuously to infinity with the infinitude of the terms of the Riemann zeta function ζ. All this indicates more and more bad news for the solubility of the Riemann zeta function ζ for powers $s < 1/2$, and, $s > 1/2$ (as could be extrapolated from the above; refer to Appendix 2 and Appendix 3 (which provides an example) below) when there are more and more terms in the Riemann zeta function ζ, i.e., for powers $s < 1/2$ and $s > 1/2$, the more terms there are in the Riemann zeta function ζ the less soluble it would be. This is a serious irregularity and is the reason why there are no zeros for the Riemann zeta function ζ for powers $s < 1/2$ and $s > 1/2$.

For the Riemann zeta function ζ, $s = 1/2$ is evidently the optimum or equilibrium power whereby there would be zeros. No zeros would be found on the critical strip bounded by Re(s) = 0 and Re(s) = 1 for $s < 1/2$ and $s > 1/2$ because if $s < 1/2$ the sum of the series in the zeta function ζ increases too fast when more and more terms are added to the series and if $s > 1/2$ the sum of the series in the zeta function ζ increases too slowly when more and more terms are added to the series; $s = 1/2$ is evidently optimum, just fits - evidently the only power which is conducive for the production of zeros, i.e., solutions for the Riemann zeta function ζ - all of this is similar to the case for Fermat's last theorem which is explained in Appendix 4 below, as is stated earlier. We clarify here what solubility of an equation such as the Riemann zeta function ζ means - a non-trivial zero of the Riemann zeta function ζ is a root - a root here is a solution or value of an unknown of an equation which could be factorized.

We here elaborate more on the apparently subtle points in the above paragraph which might be difficult to comprehend. To comprehend the point that if $s < 1/2$ the sum of the series in the Riemann zeta function ζ increases too fast when more and more terms are added to the series we need to make a close study of and understand **Table 1** above referring also to the computations above this table, and, to comprehend the point that if $s > 1/2$ the sum of the series in the Riemann zeta function ζ increases too slowly when more and more terms are added to the series we need to study closely and understand **Table 2** in **Appendix 3** below referring also to the computations above this table. A careful study of **Table 1** above would reveal that for $s < 1/2$ all the sums for these series, e.g., for s = 1/3, 1/4, 1/5, 1/6, etc., would diverge more and more from the sum for $s = 1/2$ when more and more terms are added to all these series including $s = 1/2$. As evidently only the series for s = 1/2 are conducive for the production of zeros, what this implies is that as more and more terms are added to the series for $s < 1/2$ such as s = 1/3, 1/4, 1/5 and 1/6, it would be less and less likely for these series to be able to produce zeros (i.e., these series would be less and less soluble) due to the rate of ***increase*** of their sums (relative to the sum for $s = 1/2$) becoming greater and greater (in fact too greatly) with more and more terms added to these series including $s = 1/2$. Likewise, a careful study of **Table 2** in **Appendix 3** below would also reveal that for $s > 1/2$ all the sums for these series, e.g., for s = 2/3, 3/4, 4/5, 5/6, etc., would diverge more and more from the sum for $s = 1/2$ when more and more terms are added to all these series including $s = 1/2$. Since evidently only the series for s = 1/2 are conducive for the production of zeros, what this also implies is that as more and more terms are added to the series for $s > 1/2$ such as s = 2/3, 3/4, 4/5 and 5/6, it would be less and less likely for these series to be able to produce zeros (i.e., these series would be less and less soluble) due to the rate of ***decrease*** of their sums (relative to the sum for $s = 1/2$) becoming greater and greater (in fact too greatly) with more and more terms added to these series including $s = 1/2$. In other words, for the Riemann zeta function ζ for powers $s < 1/2$ and $s > 1/2$, the more terms there are in the Riemann zeta function ζ the less soluble it would be, which is a serious irregularity. Extrapolating from **Table 1** and **Table 2** it is evident that the non-trivial zeros would not be found on the critical strip bounded by Re(s) = 0 and Re(s) = 1 for s < 1/2 and s > 1/2.

There is the feeling that for s < 1/2 and s > 1/2 the Riemann zeta function ζ might yield some non-trivial zero or zeros after innumerable terms, e.g., after many billions, trillions or more terms, have been added to the series, as past experience has shown this could happen. However, extrapolations with **Table 1** above and **Table 2** in **Appendix 3** below would show that this is not possible. It may happen only when the following occur: (a) For s < 1/2 all the sums for these series, e.g., for s = 1/3, 1/4, 1/5, 1/6, etc., would diverge less and less (instead of more and more), even gradually so, from the sum for s = 1/2 when more and more terms are added to all these series including s = 1/2. (b) For s > 1/2 all the sums for these series, e.g., for s = 2/3, 3/4, 4/5, 5/6, etc., would diverge less and less (instead of more and more), even gradually so, from the sum for s = 1/2 when more and more terms are added to all these series including s = 1/2. As the Riemann hypothesis is shown to be analogous to Fermat's last theorem in Appendix 4 below (which means what is true for Fermat's last theorem is also true for the Riemann hypothesis) and Fermat's last theorem posits that there are solutions only for n = 2 and none for n > 2 and n < 2, by the same principle there should not be solutions for s < 1/2 and s > 1/2 and the feeling that for s < 1/2 and s > 1/2 the Riemann zeta function ζ might yield some non-trivial zero or zeros after innumerable terms have been added to the series appears misplaced.

As per the explanations in the paper, we conclude that all the non-trivial zeros of the Riemann zeta function ζ could be expected to be found at the critical line Re(s) = 1/2 only and not anywhere else on the critical strip bounded by Re(s) = 0 and Re(s) = 1. □

Appendix 1

The Riemann zeta function ζ(s), shown below, is the sum over all natural numbers n:-

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots$$

The function could also be written in the following way (using Euler's product formula) showing its connection with the primes:-

$$\zeta(s) = \prod_{p \text{ prime}} p^s/p^s - 1 = 2^s/2^s - 1 \text{ x } 3^s/3^s - 1 \text{ x } 5^s/5^s - 1 \text{ x } 7^s/7^s - 1 \text{ x } \ldots \qquad (3.1)$$

where the product is over the consecutive primes p, providing the first hint that the Riemann zeta function ζ(s) is closely connected to the primes; it could be seen above that the Riemann zeta function ζ(s) has the property of prime sieving encoded within it (compare: sieve of Eratosthenes).

Appendix 2

Below are the values of the reciprocals of, say, 100, with consecutive fractional powers $s \leq 4/5$, these reciprocals being representative of the terms of the Riemann zeta function ζ:-

[1] $1/100^{4/5}$ = 1/39.8107171 = 0.025 (This quantity represents a decrease of 75% compared to [4] below while the increase in power from $s = 1/2$ to $s = 4/5$ is only 60%.)

[2] $1/100^{3/4}$ = 1/31.62278 = 0.032 (This quantity represents a decrease of 68% compared to [4] below while the increase in power from $s = 1/2$ to $s = 3/4$ is only 50%.)

[3] $1/100^{2/3}$ = 1/21.5444 = 0.046 (This quantity represents a decrease of 54% compared to [4] below while the increase in power from $s = 1/2$ to $s = 2/3$ is only 33.33%.)

[4] $1/100^{1/2}$ = 1/10 = 0.100 (The terms of the series of the Riemann zeta function ζ as per the Riemann hypothesis fall under this category. 10^{13} zeros have been found under this category only.)

[5] $1/100^{1/3}$ = 1/4.6416 = 0.215 (This quantity represents an increase of 115% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/3$ is only 33.33%.)

[6] $1/100^{1/4}$ = 1/3.1623 = 0.316 (This quantity represents an increase of 216% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/4$ is only 50%.)

[7] $1/100^{1/5}$ = 1/2.5119 = 0.398 (This quantity represents an increase of 298% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/5$ is only 60%.)

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Note the disproportionateness between the respective percentages of decrease in quantity and the respective percentages of increase in power for the reciprocals with powers $s > 1/2$, and, between the respective percentages of increase in quantity and the respective percentages of decrease in power for the reciprocals with powers $s < 1/2$.

Appendix 3

The following list of the first 5 terms of the series of the Riemann zeta function ζ with consecutive fractional powers $s \geq 1/2$ shows that the sums with larger powers decrease progressively, i.e., the larger the power s is the larger the percentage of decrease in the quantity is:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + \ldots = 3.24$
(The Riemann hypothesis asserts that all zeros would be found in this series only.)

[2] $\zeta(2/3) = 1 + 1/2^{2/3} + 1/3^{2/3} + 1/4^{2/3} + 1/5^{2/3} + \ldots = 2.85$
(The sum 2.85 here represents a decrease of **12.04%** compared to the sum 3.24 in [1] above.)

[3] $\zeta(3/4) = 1 + 1/2^{3/4} + 1/3^{3/4} + 1/4^{3/4} + 1/5^{3/4} + \ldots = 2.68$
(The sum 2.68 here represents a decrease of **17.28%** compared to the sum 3.24 in [1] above.)

[4] $\zeta(4/5) = 1 + 1/2^{4/5} + 1/3^{4/5} + 1/4^{4/5} + 1/5^{4/5} + \ldots = 2.59$
(The sum 2.59 here represents a decrease of **20.06%** compared to the sum 3.24 in [1] above.)

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Table 2 below is a tabulation of the above-mentioned **percentage *decreases*** for the sums for the first 2 terms to the first 5 terms for ζ(2/3), ζ(3/4) & ζ(4/5):-

	1st. 2 Terms	1st. 3 Terms	1st. 4 Terms	1st. 5 Terms	1st. 6 Terms	…
[1] ζ(1/2)	-	-	-	-	-	
[2] ζ(2/3)	4.52%	7.63%	9.98%	**12.04%**	To Be Extrapolated	
[3] ζ(3/4)	6.65%	11.08%	14.37%	**17.28%**	To Be Extrapolated	
[4] ζ(4/5)	7.86%	12.98%	16.78%	**20.06%**	To Be Extrapolated	

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Appendix 4

For the case for $x^n + y^n = z^n$ for Fermat's last theorem which asserts that there are no solutions for $n > 2$, we here explain why there are no solutions for $n > 2$. We begin by selecting a Diophantine equation which has the smallest odd prime number 3 and the smallest composite number 4 (which is the square of the smallest prime number 2), i.e., the smallest Diophantine equation which has 2 as the power, for illustration which is presented below:-

$$3^2 + 4^2 = 5^2 \qquad (3.2)$$

If the power of 2 in the series on the left above were increased to 3, which is the next, consecutive whole number, e.g., the sum on the right would not be a whole number anymore, which is in accordance with Fermat's last theorem, as is shown below:-

$$3^3 + 4^3 = 4.49795^3$$

The regularity of the power of 2 is now lost, which is for the smallest Diophantine equation which initially had 2 as the power. For the larger Diophantine equations with initial powers of 2 the irregularity after increasing their powers to 3, which is the next, consecutive whole number, or, higher powers, could be expected to be worse.

In the next step we bring up the values of, say, 100, of consecutive whole number powers n, say, 2 to 5, this quantity 100 being representative of the terms of the equation $x^n + y^n = z^n$ as per Fermat's last theorem, to explain the reason for this irregularity, which is as follows:-

[1] $100^2 = 10{,}000$ (The terms of the series of Fermat's last theorem fall under this category. All zeros would be found under this category only.)

[2] $100^3 = 1{,}000{,}000$ (This quantity represents an increase of 9,900% compared to [1] above while the increase in power from $n = 2$ to $n = 3$ is only 50%.)

[3] $100^4 = 100{,}000{,}000$ (This quantity represents an increase of 999,900% compared to [1] above while the increase in power from $n = 2$ to $n = 4$ is only 100%.)

[4] $100^5 = 10{,}000{,}000{,}000$ (This quantity represents an increase of 99,999,900% compared to [1] above while the increase in power from $n = 2$ to $n = 5$ is only 150%.)

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The quantities from the consecutive whole number powers $n > 2$ above increase progressively compared to [1], the larger the power n is the larger the percentage of increase in the quantity is. The increases in the respective quantities and powers are also disproportionate when compared to one another, with the increases in the respective quantities being evidently much too quick. All this shows that the equilibrium, balance or regularity of $x^n + y^n = z^n$ when $n = 2$ as per Fermat's last theorem cannot be maintained when $n > 2$, when disproportionateness between the increases in the respective quantities and powers sets in as is described above, as the increase in quantity is too quick, and, when $n < 2$, e.g., $n = 5/4$, 3/2, 7/4, etc., as the increase in quantity is too slow as could be extrapolated from the above example. (Refer to Appendix 2 above for an analogous example.) For Fermat's last theorem, $n = 2$ could be regarded as the optimum or equilibrium power, the only power whereby $x^n + y^n = z^n$ is possible. There is also the easier solubility of equations with whole number powers $n = 2$ as compared to equations with powers $n > 2$, e.g., $n = 3$, 4, 5, etc., and $n < 2$, e.g., $n = 5/4$, 3/2, 7/4, etc., which is explained below:-

$n = 2$ is the smallest whole number power which brings an increase in quantity. As such $n = 2$ is the whole number power which brings the *smallest increase in quantity* as compared to the whole number powers $n > 2$, e.g., $n = 3$, 4, 5, etc., for instance $2^2 < 2^3 < 2^4 < 2^5$, etc. - equations with whole number powers $n = 2$ would evidently be easier to solve than equations with powers $n > 2$ (with general equations with powers $n = 5$ having been proven unsolvable - $n = 2$ needs only 1 squaring step while $n = 5$ needs 4 squaring steps) and $n < 2$, e.g., $n = 5/4$, 3/2, 7/4, etc. (e.g., in a computation $n = 2$ needs only 1 squaring step, while $n = 7/4$ needs 9 steps - 6 squaring steps for $n = 7$ & 3 rooting steps for $n = 1/4$).

Like the Riemann zeta function ζ (which comprises of a series) that is able to turn out zeros, the series in Fermat's last theorem all have their own zeros, e.g., for the series $3^2 + 4^2 = 5^2$, $13^2 + 84^2 = 85^2$ and $65^2 + 72^2 = 97^2$, their zeros are respectively as follows:-

(1) $3^2 + 4^2 - 5^2 = 0$
(2) $13^2 + 84^2 - 85^2 = 0$
(3) $65^2 + 72^2 - 97^2 = 0$
Etc.

Like the case of the Riemann zeta function ζ wherein its longer series of positive and negative terms would cancel out each other when added up together to produce a zero, the various shorter series of Fermat's last theorem could also be added up together with their positive terms and negative terms cancelling out each other to produce a zero. For instance, the above series (1), (2) and (3) with their zeros for Fermat's last theorem could be added together as a longer series to produce a zero as follows:-

$3^2 + 4^2 - 5^2 + 13^2 + 84^2 - 85^2 + 65^2 + 72^2 - 97^2 = 0$

or, as follows with the terms in ascending order of magnitude:-

$3^2 + 4^2 - 5^2 + 13^2 + 65^2 + 72^2 + 84^2 - 85^2 - 97^2 = 0$

Hence, the uncanny similarity or resemblance between Fermat's last theorem and the Riemann hypothesis, as is evident above.

The optimum or equilibrium power whereby zeros are possible $n = 2$ (square) for Fermat's last theorem is the reciprocal and opposite of the optimum or equilibrium power $s = 1/2$ (square root) for the Riemann hypothesis. Also, both these powers have the number "2" which is the smallest prime number and the smallest even number. There also appears to be complementariness and symmetry between these 2 powers. $n = 2$ and its reciprocal $s = 1/2$ are evidently important quantities which may be comparable to π (3.14159265) or e (2.71828). All this is significant.

There are evidently great similarities between Fermat's last theorem and the Riemann hypothesis, both being analogues of one another, which implies that if Fermat's last theorem is true, as it indeed is true as Andrew Wiles had proved it in 1994, then the Riemann hypothesis is also true.

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