Proof of the Collatz Conjecture

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Abstract

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2……The inverted Collatz sequences can be represented as a tree with 1 as its root node. In order to prove the Collatz conjecture, one must demonstrate that the tree covers all positive integers. In this paper, we construct a Collatz tree with 1 as its root node by connecting infinite number of basic trees. Each basic tree relates to each positive integers. We prove that a Collatz tree is a connected tree and covers all positive integers.

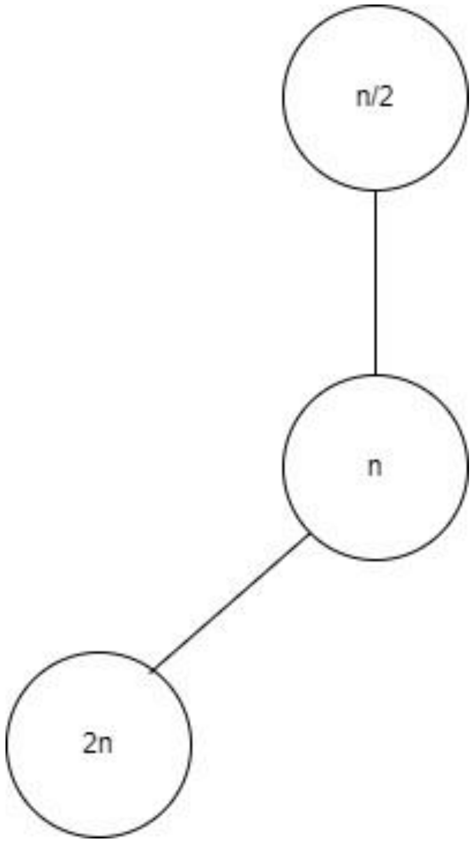
1. Introduction

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches a trivial cycle 1, 2, 1, 2……The inverted Collatz sequences can be represented as a tree with 1 as its root node. In order to prove the Collatz conjecture, one must demonstrate that this tree covers all positive integers.[1].

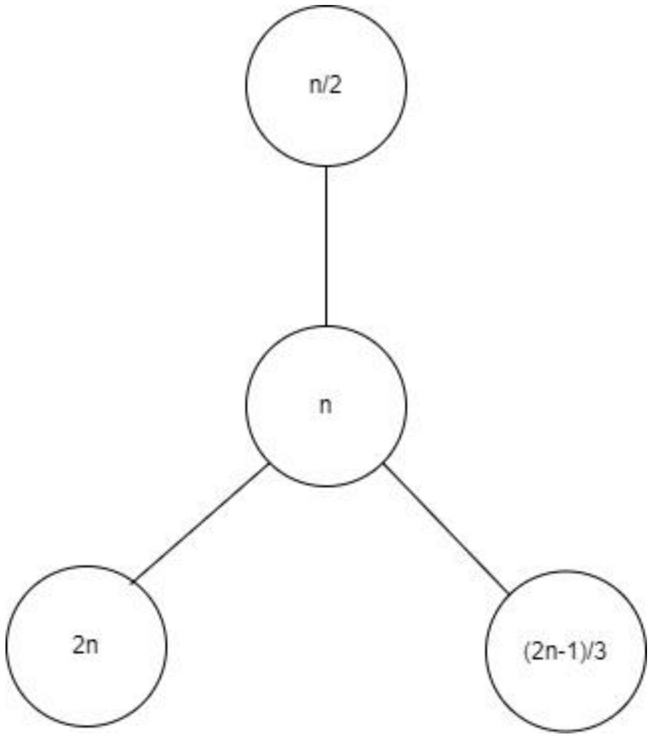
2. A basic tree

The basic tree is constructed for each positive integer as follows:

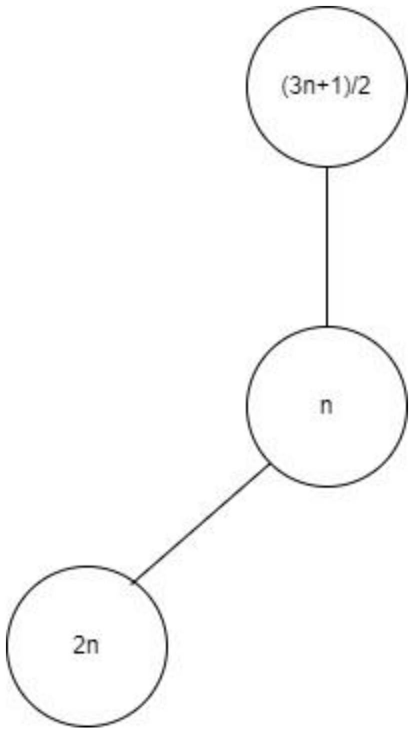
Let n be a positive integer node. Its parent node is $\frac{n}{2}$ if n is even, or $\frac{3n+1}{2}$ if n is odd. Its left child is 2n. Its right child is $\frac{2n-1}{3}$ if n $\equiv$ 2mod3, or no right child if n $\neq$ 2mod3. Thus there are four types of basis tree as shown in Figure 1.



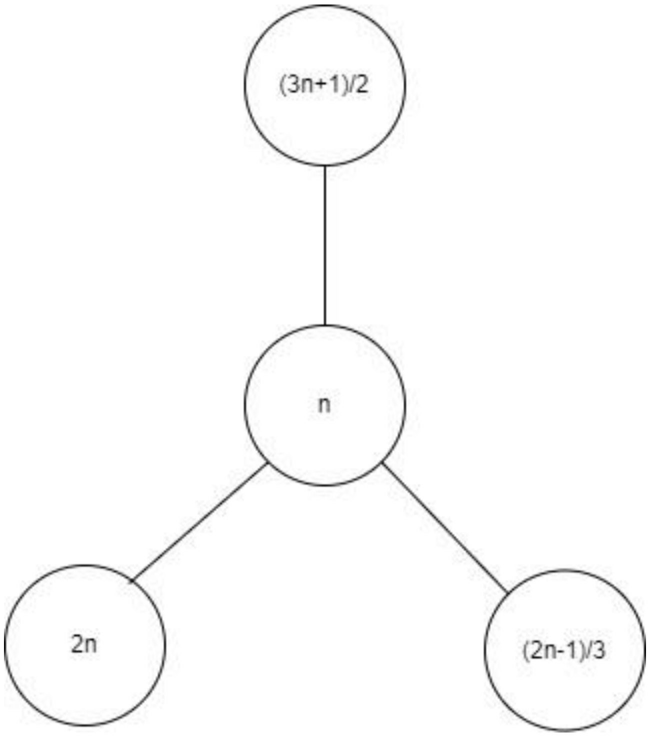
(a) n is even and not equal to 2mod3



(b) n is even and equals to 2mod3



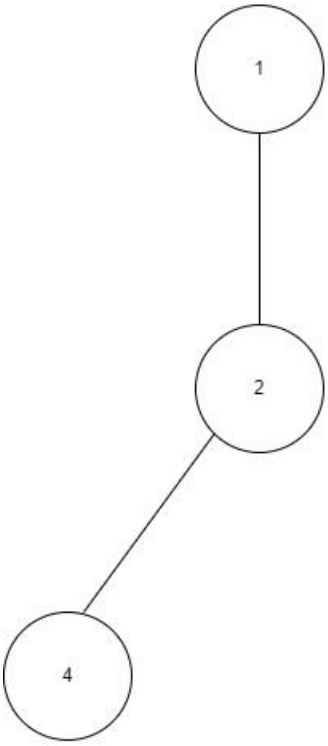
(c) n is odd and not equal to 2mod3

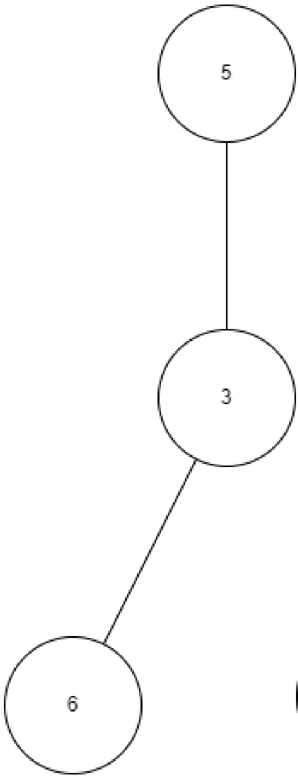


(d) n is odd and equals to 2mod3

Figure 1, four types of basic trees

Examples of basic trees shown in Figure 2.


1
2
4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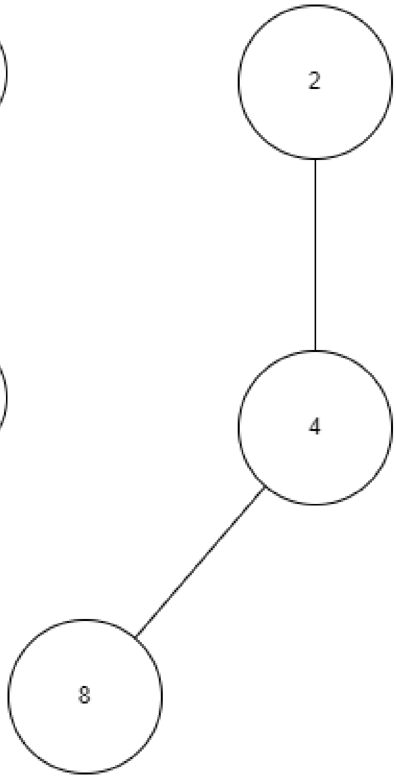
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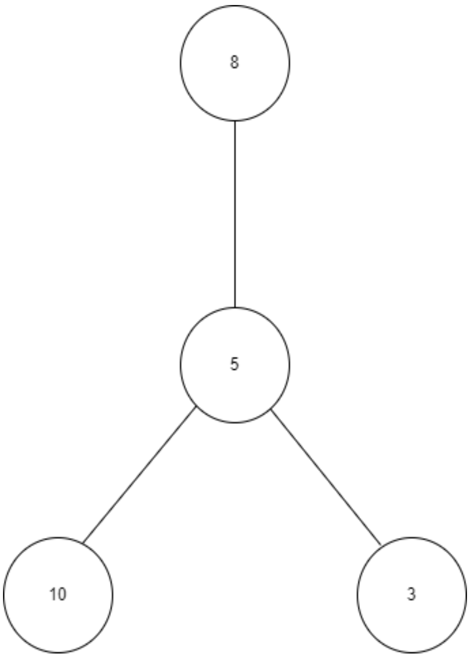
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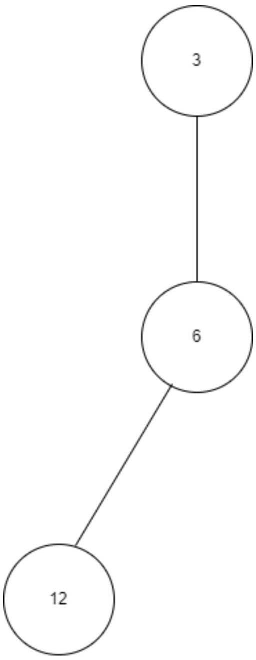
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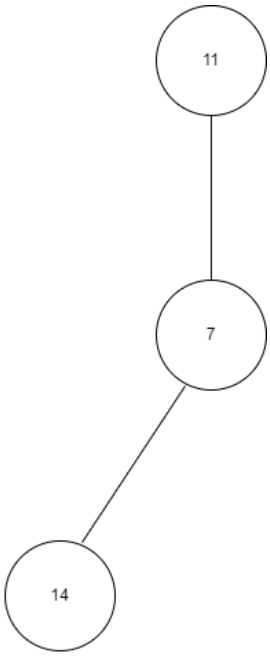

11
7
14

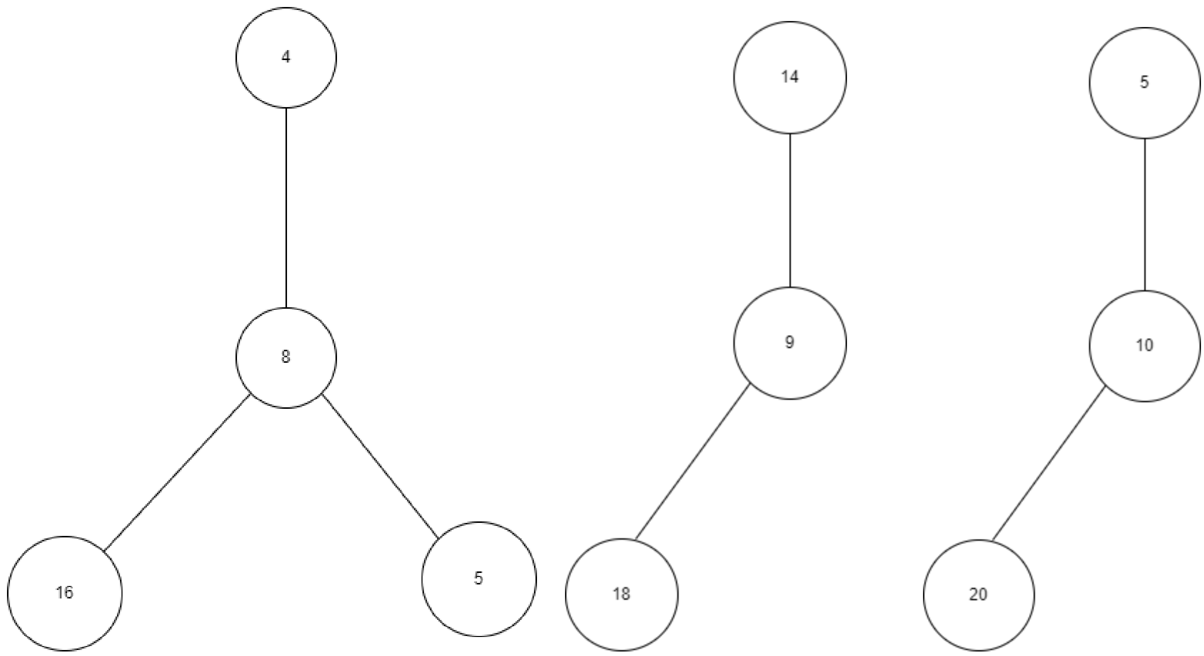
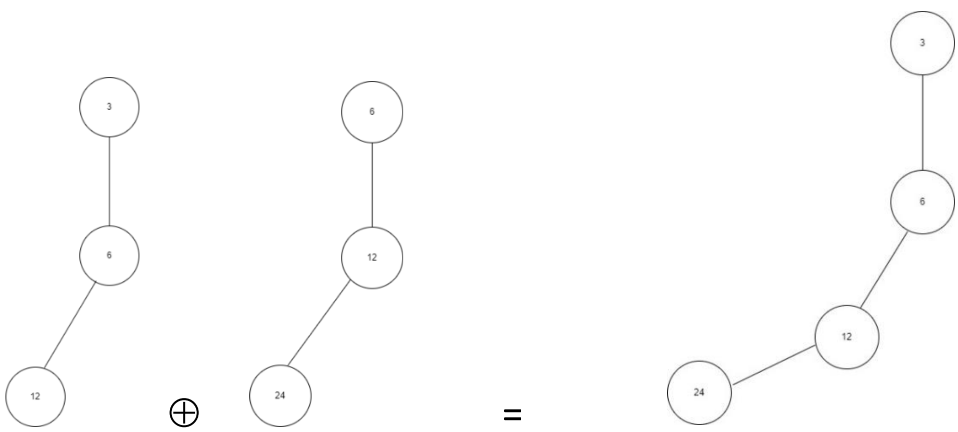


Figure 2. Basic trees of positive integers 2, 3, 4, 5, 6, 7, 8, 9, 10

3. **How to connect two basic trees**

   A simple rule to connect two basic trees is that these two basic trees must have a common branch as three cases shown in Figure 3.



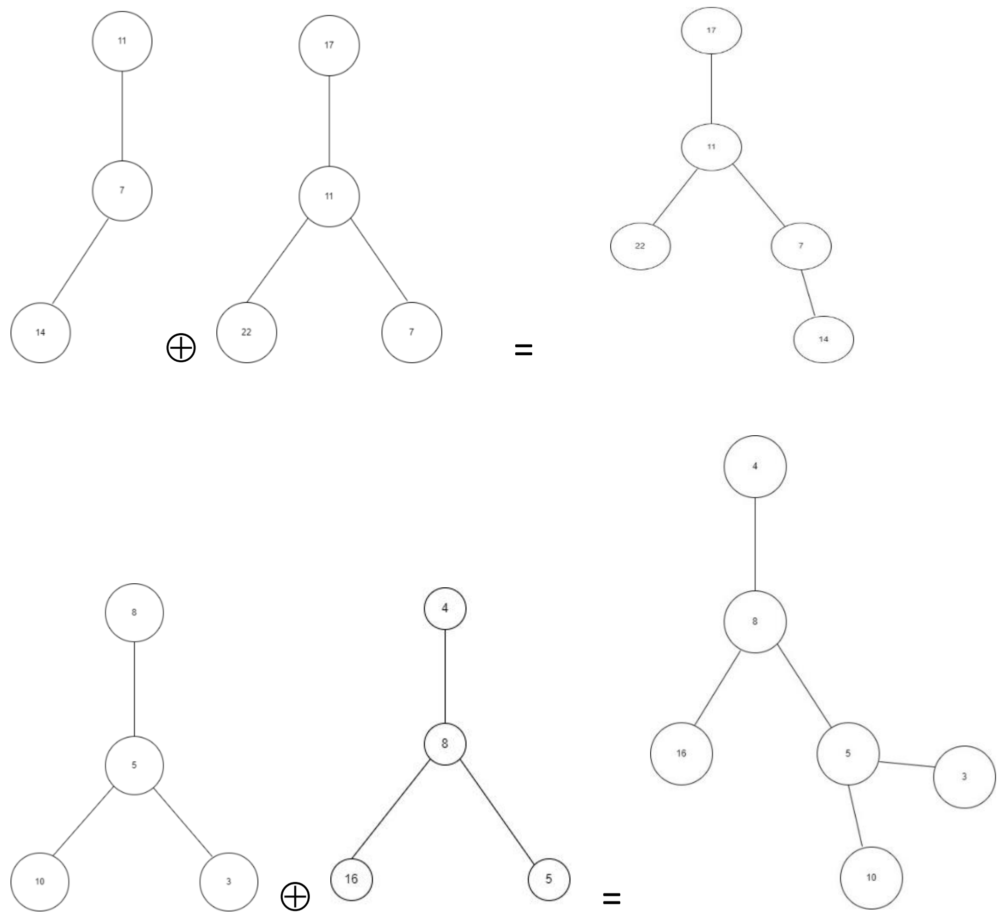


Figure 3. Various cases of basic trees connection

4. **Collatz tree with node 1 as its root node**

A Collatz tree is formed by connecting basic trees as shown in Figure 4. This tree is arranged in levels 0 to ∞. There is only node 1 in level 0. For i $\geq$3, number of nodes in level i is less than number of nodes in level i+1. There is no nontrivial cycle or divergence sequence in this tree. But in order to prove the Collatz conjecture, this tree must cover all positive integers

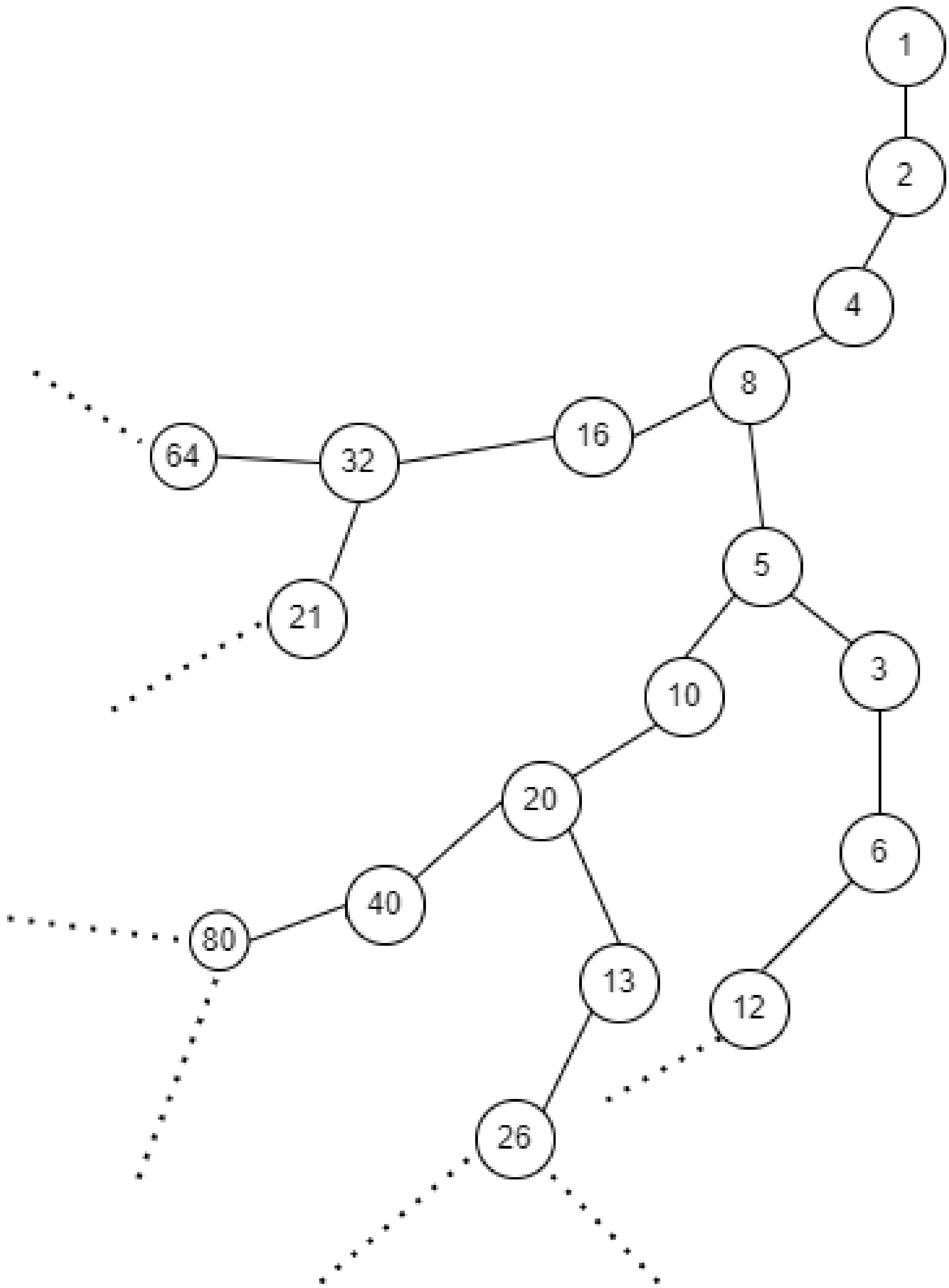


Figure 4. A Collatz tree

We can show that there is only one Colltaz tree for all positive integers. Let assume there is another Collatz tree that is completely different from the Collatz tree shown in Figure 4. This is impossible because node 1 must be in the level 0 of any Collatz tree. Thus there is only one Collatz tree covers all positive integers.

5. **Conclusion**

A connected Collatz tree shown in Figure 4 covers all positive integers. By starting at any node in a tree, there is a unique path from that node to a node 1.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.