Levelwise Accessible Equivalence Classes

Ryan J. Buchanan*

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Abstract

For a space of directed currents, geometric data may be accessible by means of a certain $\frac{1}{n}$-type functor on a sheaf of germs. We investigate pointwise periodic homeomorphisms and their connections to foliations.

1 Background

Let there be an open cover $\{U_\alpha\}_{\alpha\in A}$ of a set of metrizable point patches:

$$\amalg_{\alpha_i}^{\alpha_\omega} fib(P)$$

where P is a set of paths in the path groupoid $M^{<1>}$ over a manifold M.

$$P = Hom(M, \mathcal{G})$$

where $\mathcal{G}$ is a Lie groupoid endowed with its set of self-maps, $\mathcal{G}^{\mathcal{G}}$.

Remark 1. *The path groupoid need not generate a manifold; for instance, the immersion*

$$M^{<1>} \hookrightarrow G \times_{G_n} G$$

may only be an orbifold.

Recall that, an object $x \in \mathscr{C}$ is said to be *Cauchy*, if:

Definition 1.

$$(\lim_{\longleftarrow} \sum_{i=0}^{\infty} \tilde{x}_i) \in \mathcal{C}$$

where $\mathscr{C}$ is an essentially small category, is a Cauchy point*, $\tilde{x}_\lambda \in \mathscr{C}$, where λ is the co-dimension of the bounded, metrizable pre-space sheaf $\mathcal{S}(x)$. The map $i \to \lambda$ factors through the proper Cartesian square:*

*Roseburg, OR

$$\begin{array}{ccccccccc} & & \tilde{x}_i & \xrightarrow{\frac{1}{n}} & \tilde{x}_\bullet & \xrightarrow{\frac{n-1}{n}} & \tilde{x}_\lambda & & \\ & & {\scriptstyle s\to t}\downarrow & & \downarrow{\scriptstyle s\to t} & & \downarrow{\scriptstyle s\to t} & & \\ 0 & \longrightarrow & i & \longrightarrow & \bullet & \longrightarrow & \lambda & \longrightarrow & 0 \end{array}$$

The above exact sequence is produced by lifting a class of germs from the pre-sheaf $Pr_\bullet(x)$ to an automorphism of monodromy.

$$Pr_\bullet(x) \ltimes \mathscr{M}$$

The left-action on $\mathscr{M}$ equips the pre-geometric operad data with a groupoid structure. In specific, retracts of every $m \in \mathscr{M}$ exist, and right-lift against the identity action on a sub-object identifier,

$$\rho_\bullet \sim \tilde{x} \in x$$

.

Definition 2. *A fat geometric realization, $||x||_\bullet$, is a level-wise discrete automorphism from the isotropy group i_x of a point x to itself.*

Here, the variable x has a representation $\hat{x} \in \mathbb{N}$, and the isotropy group action

$$i_x \circ_{Aut(x)} i_{\hat{x}}$$

lies in the essential image of a certain kind of "untangling" of isofibrations, $x \to_n x$, where every proper functor occurs $\frac{1}{n}$ of the time. The linking of probabilities with geometric data, i.e., having all probabilities equally spread across a manifold, is in some sense intrinsically tied to the mean curvature of a manifold.

Here, n is the *level* of the map

$$\mathscr{C} \longrightarrow Set_{\mathscr{C}}$$

where for every $c_i \in \mathscr{C}$, there are orthogonal projections for every $i \pm 1 \in \mathbb{N}$ That is,

$$c_{i-1} \vdash c_i \vdash c_{i+1} \vdash ... \vdash c_n$$

factors through itself *at most* n times, and so it has an upper bound

$$sup(\mathscr{C}) = c_n$$

If, at every path $P \in Aut(\mathscr{C})$, there is a sub-object identifier ρ, then we can track the current-directed directional derivative, for at least all étale stacks.

$$\partial P_\rho = \int_0^\infty \rho_\alpha(\mathfrak{S}\mathfrak{p}^{\mathfrak{et}})$$

This gives us Segal's classical $\Gamma_{2_p}^{s_n} = \mathcal{M} \ltimes E\mathbb{X}$ classifying space, where $\mathbb{X}$ is a locally privileged, accessible frame with respect to the canonical coherency condition

$$\omega_{can} = \omega_{\delta_i} \circ ... \circ \omega_{\emptyset}$$

where ω_{can} is a non-degenerate lift of the automorphism class $Aut(x)$ of a point $\{x, x^{-1}\}$ of an isotropy groupoid $\mathcal{G}_x$.

Let $\mathcal{V}_\phi$ be a Grothendieck universe with the universal property ϕ. That is to say, for every member $v \in \mathcal{V}$, the set of local diffeomorphisms

$$Diff(v)$$

is ϕ-small.Let X be a topological space and $\mathscr{X}$ an étale stack. Then, the series

$$X \to \mathcal{V} \to \mathscr{X}$$

factors uniquely and canonically through the set of equivalences on orbits of the isotropy group $\mathcal{G}_x$, giving us a map

$$Eff_{\tilde{x}} : X \xrightarrow{\omega_{can}\mathcal{G}_x} \mathscr{X}$$

Example 1. *Let Sym_x be the following category: objects are orbits of an isotropy group $\mathcal{G}_x$ defined by fixing sub-objects of a Lie groupoid $\mathcal{G}$ point-wise, and morphisms are local diffeomorphisms fixing x. Then, we have:*

$$Eff_{\tilde{x}} \in Fun(Sym_x) = \omega \longrightarrow \omega$$

acting on simplicial k-forms $\omega_k(x)$.

Effective (parts of étale) maps are fully faithful by definition, as was shown in [2].

2 Definitions

Let $\vec{\partial}x$ denote the directional derivative at a point x with respect to an ambient space $\mathcal{A}$. Let

$$x \hookrightarrow \mathscr{X}$$

be a submersion, and let their be a weak equivalence

$$x \xrightarrow{w.e.} im(x)$$

.

Let there be a pro-discrete covering $\mathcal{U}(im(x))$, called the *foliation* $(\mathcal{F}_x)$ of x into leaves. We define each leaf to be a collection of sections over the fibers $f_* : im(x) \to x'$, such that the union

$$\bigcup_{i=0}^{n} fib_{pro}(im(x_i))$$

is contained within the space $\mathcal{A}' \in \mathscr{X}$, such that the sequence

$$0 \to x \to im(x) \to x'$$

is surjective at each step.

Proposition 1. *Let $\mathscr{G}$ be an almost-free group, and fix an isotropy group $\mathscr{G}_x$. Then, any map sending $\tilde{x} \in \mathscr{G}_x$ to an étale stack,*

$$St : \tilde{x} \longrightarrow \mathfrak{St}^{\text{ét}}$$

induces a folation on the essential image of x.

Proof. To sketch a proof of this proposition would require us to find some collection of objects (say, germs of a structure sheaf) which representatively biject to (correspond to) leaves of the foliation $\mathcal{F}_x$. To see that this is possible, let $\mathcal{U}(x)$ be stratified and equipped with a metric γ. Then, we have

$$\tilde{x} \xrightarrow{\text{ét}} \mathcal{U}(x) \xrightarrow{\gamma} x \sqcap y$$

and for every distinct point y, i.e., such that $x \cap y$ is empty, but $x \sqcap y$ is full, we have that the source and target maps

$$(s,t) \in \mathscr{G}_x \times_{\tilde{x}} \mathscr{G}_x$$

are discrete, and so there is a natural stratification $(x \vee y)_\gamma \to \mathcal{F}_x$.

□

We can now proceed to define the "Carchedi current":

Definition 3. *Fix a map $\mathscr{G}_x \longrightarrow \mathcal{F}_x$ from an isotropy group of a point to a foliation which fixes the point leafwise. The Carchedi current, C_{cur}, is defined as follows.*

Fix a leaf $\ell \in \mathcal{F}_x$. Say that a map $x \longrightarrow \ell$ is "levelwise accessible" if it is

1. *leafwise disjoint from all other ℓ'.*
2. *in the essential image of $\vec{\partial}x$*

and if there exists some open cover, $\mathcal{U}_\gamma(x)$ such that all of the Cauchy points for x exist and are leafwise exact. Then, $C_{cur}(x)$ is defined to be the levelwise accessible limit of all of the paths $[P]$ over the tangent bundle of x.

Proposition 2. *If $\mathcal{M}$ is a smooth manifold, and for some point $x \in \mathcal{M}$, $C_{cur}(x)$ exists, then $\mathcal{M}$ is a $C^{>0}$ manifold.*

Proof. Trivial. □

3 Periodicity

Denote by f^n the nth iterate of f, such that $f^0 = id_X$ and $f^n = f \circ f^{n-1}$ for $n \in \mathbb{N}$. For any $x \in X$, the subset $O_f(x) = \{f^n(x) | n \in \mathbb{Z}_+\}$ is called the *f-orbit* of x.

Definition 4. *A homeomorphism $f : X \longrightarrow X$ will be called pointwise periodic if, for each $x \in X$, there exists an integer $n(x)$ such that $f^{n(x)}(x)$ is the identity on x.*

Remark 2. *In other words, if a homeomorphism is pointwise periodic, then the orbit of every point is finite. However, we do not postulate an upper bound for $n(x)$.*

We establish an important proposition here:

Proposition 3. *If a homeomorphism $f^\sharp : \mathscr{G} \times \mathscr{X} \longrightarrow \mathscr{G} \times \mathscr{X}$ is levelwise accessible with level $n(x)$, then it is pointwise periodic with period n.*

That is to say, for a foliation $\mathcal{F}_x$, with x fixed, there exists a section of a fiber $\mathscr{F} \in \mathcal{O}_X$ with periodic retracts to the centralizer of the underlying groupoid $\mathscr{G}$. This is because, as was shown in [3], fixing a point x is a stronger action than inducing a pointwise periodic homeomorphism over a stack $\mathscr{X}$, such that the orbits of a given point are finite.

So, for a simply connected arc-space $\mathfrak{X}$, a foliation $\mathcal{F}_{x \in \mathfrak{X}}$ which fixes a point x must necessarily be pointwise periodic under the map $f^\flat : \mathfrak{X} \longrightarrow \mathfrak{X}$. This means that an open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ for an index space A (assumed to be ambient), the map

$$g : \partial X \to \partial X$$

is recurrent, by definition, when

$$X = \sum_{i=0}^{k<\infty} \alpha_i$$

. Assuming that the Carchedi current exists for a continuously lift

$$g^\sharp : \bigcup_{i=0}^{k} \partial X \to \mathfrak{Y}$$

where $\mathfrak{Y}$ has a base-space with determinant n, then the map is pointwise periodic with period $k - n$. Further, in such case, the isotropy group of each point is *portable*, in the sense of Emmerson.

Let $\mathfrak{M}$ be a commutative monoid, and let $Rep(\mathfrak{M})$ be full. If, and only if,

a.) The Carchedi current exists on $Rep(\mathfrak{M})$

and

b.) $Aut(\mathfrak{M}) \circ_k \mathfrak{M} = Id_m$ for some $m \in \mathfrak{M}$

then

$Rep(\mathfrak{M})$ is pointwise periodic.

That is to say, if there exists some k such that $f^k(m) = Id_m$, then there is an operad, $\circ_k$, whose right action on commutative monoids is periodic of period k. The underlying assumption, is, of course, that $\mathfrak{M} \times * \in \mathscr{C}$, and that $\mathscr{C}$ is essentially small. This allows us to write

$$Eff_*(\mathfrak{M}) \subset Per$$

,

where Per is the category whose objects are integrated isotropy groupoids, and whose morphisms are all pointwise periodic.

4 References

[1] D. Carchedi, *On the homotopy type of higher orbifolds and Haefliger classifying spaces*, (2016)
[2] D. Carchedi, *Etale stacks as prolongations*, (2019)
[3] I. Naghmouchi, *Pointwise-recurrent Dendrite Maps*, (2011)