$3n+1$ conjecture: a proof or almost

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The Collatz algorithm is rewritten to remove divisions by two and to transform it from a hailstone to a steadily growing value. In contrast with the original problem this new sequence becomes reversible and it is reverted in combinatorial way to find all integers leading to the sequence end. Computer programs are available for demonstrations and experimenting.

1 Collatz differently

1.1 Notations

In the original formulation for any integer number $X_i > 0$ to obtain X_{i+1} we either multiply X_i by 3 and add 1 if it is odd, or divide it by 2 until the result remains even. Such an algorithm leads to a, so called, hailstone behaviour of X_i.

For any integer number $X_i > 0$ represented in base-2 we will use H_i (Head) to designate the most significant bit position and T_i(Tail) the least significant bit position (number of trailing zeros). For example for a binary number

```
10001010101000
H000000000T000
```

$H = 13$ and $T = 3$.

1.2 Key statement

The new sequence will be

$$X_{i+1} = 3X_i + 2^{T_i} \tag{1.1}$$

and Collatz states it will eventually lead to $H_n = T_n$

$$X_n = 2^{H_n} = 2^{T_n} \tag{1.2}$$

In other words to a single 1 shifted left by T_n bits.

Remark 1. T_n is exactly the number of divisions by 2 we would accomplish with the regular Collatz algorithm.

Any additional step for $i > n$ will merely multiply X_i by 4

$$X_{i+1} = 3X_i + 2^{T_i} = 3 \cdot 2^{T_i} + 2^{T_i} = 4 \cdot 2^{T_i} = 2^{T_i+2}$$

or shift it left by two positions.

Example for 49:

i	binary X_i	decimal X_i	original X_i
0	110001	49	49
1	10010100	148	37
2	111000000	448	7
3	10110000000	1408	11
4	1000100000000	4352	17
5	11010000000000	13312	13
6	1010000000000000	40960	5
7	100000000000000000	131072	(end)1
8	10000000000000000000	524288	(useless)1

Let us demonstrate in details a step for $X_2 = 448 = 111000000$. After multiplication by 3 instead of dividing the result by 2 we add $2^6 = 1000000$:

i	binary X_i	decimal X_i	original X_i
2	111000000	448	7
	10101000000	448*3	
	+1000000	$+2^6$	
3	10110000000	1408	11

With this new formulation the recursion will be:

$$X_1 = 3X_0 + 2^{T_0}$$

$$X_2 = 3X_1 + 2^{T_1} = 3\left(3X_0 + 2^{T_0}\right) + 2^{T_1} = 3^2 \cdot X_0 + 3^1 \cdot 2^{T_0} + 3^0 \cdot 2^{T_1}$$

$$X_n = 3^n \cdot X_0 + 3^{n-1} \cdot 2^{T_0} + \cdots + 3^1 \cdot 2^{T_{n-2}} + 3^0 \cdot 2^{T_{n-1}} \tag{1.3}$$

1.3 Sequence properties

Fact 2. *The number of steps to complete the sequence is exactly the number of odd values in the original Collatz.*

Fact 3. *For a step i the new value is exactly the original Collatz value multiplied by T_i.*

Fact 4. *For any $j > i$:*

$$T_j > T_i \tag{1.4}$$

One could say: the problem is no longer a hailstone.

Fact 5. *On a side note, between two neighbour steps:*

$$H_{i+1} - H_i = 1\,or\,2$$

and in average the head speed before it meats the tail is $S_H = Av\,(H_{i+1} - H_i) = \log_2 3$. Meanwhile the tail moves with average speed $S_T = 2$ (for $H_i - T_i > 2$).

So, intuitively, we would expect the tail to catch and substitute the head (this is exactly what Collatz is about).

2 Proof or not ?

For a given sequence end $X_n = 2^{T_n}$ there are generally many starting points X_0 leading to X_n. For instance, both $X_0 = 26$ and $X_0 = 85$ end with $X_n = 256 = 2^8$.

2.1 Key moment

Instead of generating values according to Eq.1.2 we will look for all possible paths back from $2^{T_n} = 1 \ll T_n$.

Reverting Eq.1.2 yields

$$X_i = \left(X_{i+1} - 2^{T_i}\right)/3 \tag{2.1}$$

where

$$0 \leq T_i < T_{i+1} \tag{2.2}$$

and

$$\bmod\ \left(X_{i+1} - 2^{T_i}, 3\right) = 0 \tag{2.3}$$

This means that starting from a $X_n = 2^{T_n}$ we can find all possible values for X_{n-1} by testing T_{n-1} against Eq.2.2 and Eq.2.3. Then repeat for each T_{n-1}. And so on we will discover all values leading to X_n.

For example, observing closer a value $2^{T_n} = 1000000...0000$ one can see that the number of suitable values for T_{n-1} is $T_n/2$ (number of zero pairs). Moreover, the lowest acceptable $T_{n-1} = 1$ if T_n is even otherwise $T_{n-1} = 2$. While the highest is always $T_{n-1} = T_n - 2$.

All child values of 2^{T_n} with even T_n and odd T_n never overlap (see Example). Thus picking up two large starting points 2^{T_n-1} and 2^{T_n} will seed uniquely values situated below $2^{T_n}/3$. Tending T_n to infinity then will fill the integers from 1 to ∞.

Proof. If for any integer X_0 there is always a way to reach it from a 2^{T_n} according to Eq.2.1 the same path can be followed back by means of Eq.1.2. □

3 Example

Values reverted from 2^7 and 2^8 with Eq.2.1:

```
128        10000000  odd  Tn=7
  42         101010 = (10000000-10)/11         = 1111110/11
  40         101000 = (10000000-1000)/11       = 1111000/11
    13         1101 = (101000-1)/11            = 100111/11
    12         1100 = (101000-100)/11          = 100100/11
  32         100000 = (10000000-100000)/11     = 1100000/11
    10         1010 = (100000-10)/11           = 11110/11
      3          11 = (1010-1)/11              = 1001/11
    8          1000 = (100000-1000)/11         = 11000/11
      2          10 = (1000-10)/11             = 110/11

256       100000000  even  Tn=8
  85        1010101 = (100000000-1)/11         = 11111111/11
  84        1010100 = (100000000-100)/11       = 11111100/11
  80        1010000 = (100000000-10000)/11     = 11110000/11
    26        11010 = (1010000-10)/11          = 1001110/11
    24        11000 = (1010000-1000)/11        = 1001000/11
  64        1000000 = (100000000-1000000)/11   = 11000000/11
    21        10101 = (1000000-1)/11           = 111111/11
    20        10100 = (1000000-100)/11         = 111100/11
      6         110 = (10100-10)/11            = 10010/11
    16        10000 = (1000000-10000)/11       = 110000/11
      5         101 = (10000-1)/11             = 1111/11
      4         100 = (10000-100)/11           = 1100/11
        1         1 = (100-1)/11               = 11/11
```

4 Source code

This document and computer programs may be found here:

https://github.com/sashamakarenko/collatz