DIVISIBLE CYCLIC NUMBERS

Julian Beauchamp

Abstract. There are known to exist a number of (multiplicative) cyclic numbers (CNs), but in this paper I introduce what appears to be a new kind of number, which we call *divisible cyclic numbers* (DCNs) and wonder what properties they may possess. Strangely, I can find no reference to them anywhere. Given that they are simple to understand and quite commonplace, it would be remarkable if they were hitherto unknown to the mathematical world.

0.1. **Definition.** I define a divisible cyclic number (DCN) as an integer for which all cyclic permutations of its digits are divisible by a smaller integer without remainder.

Since DCNs appear to be a new kind of number, I have created the following notations. Let $\delta_{(n)}$ be a divisible cyclic number that is divisible by some integer, n, and let $\Delta_{(n)}$ be the set of all $\delta_{(n)}$. An arrow, $\rightarrow$, shifts the front digit to the back.

For example, $485695 \in \Delta_{(7)}$ such that:

$$485695 \rightarrow 856954 \rightarrow 569548 \rightarrow 695485 \rightarrow 954856 \rightarrow 548569 = \delta_{(7)}.$$

All these permutations are divisible by 7. Here are some more examples:

$\delta_{(7)}$: 871856984236 $\rightarrow$ 718569842368 $\rightarrow$ 185698423687 $\rightarrow$ 856984236871...

$\delta_{(11)}$: 1265 $\rightarrow$ 2651 $\rightarrow$ 6512 $\rightarrow$ 5126

$\delta_{(11)}$: 123589716459 $\rightarrow$ 235897164591 $\rightarrow$ 358971645912 $\rightarrow$ ...

$\delta_{(13)}$: 786448 $\rightarrow$ 864487 $\rightarrow$ 644878 $\rightarrow$ 448786 $\rightarrow$ 487864 $\rightarrow$ 878644

$\delta_{(17)}$: 7994325875621569 $\rightarrow$ 9943258756215697 $\rightarrow$ 9432587562156979 $\rightarrow$ ...

$\delta_{(37)}$: 518 $\rightarrow$ 185 $\rightarrow$ 851.

$\delta_{(41)}$: 58917 $\rightarrow$ 89175 $\rightarrow$ 91758 $\rightarrow$ 17589 $\rightarrow$ 75891.

If this has not been previously discovered, it would be very surprising. And why does it work? And what determines the digit-length? We will return shortly to what determines the digit-length and the minimum digit-length of any $\delta_{(n)}$.

0.2. **Multiplicative Cyclic Numbers (CNs).** How do DCNs differ from the more familiar cyclic number? A CN is an integer for which cyclic permutations of the digits are successive integer multiples of the number. Unlike *divisible* cyclic

Date: Sept 2023.
Key words and phrases. Number Theory, Cyclic Numbers, Prime Numbers.

numbers, CNs are *multiplicative* cyclic numbers. The most widely known is the six-digit number 142857, whose first six integer multiples are:

$$142857 \times 1 = 142857$$

$$142857 \times 2 = 285714$$

$$142857 \times 3 = 428571$$

$$142857 \times 4 = 571428$$

$$142857 \times 5 = 714285$$

$$142857 \times 6 = 857142$$

Cyclic numbers are generated from the reciprocals of the following primes, p, listed in the OEIS (A001913) and always possess $p-1$ digits:

7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, 167, 179, 181, 193, ... (A001913 in OEIS).

After 142857 (6 digits, generated by 1/7), the next few CNs are:
0588235294117647 (16 digits, generated by 1/17)
052631578947368421 (18 digits, generated by 1/19)
0434782608695652173913 (22 digits, generated by 1/23)
0344827586206896551724137931 (28 digits, generated by 1/29)
0212765957446808510638297872340425531914893617 (46 digits, generated by 1/47)

DCNs, it seems, are closely related (though much more common) and raise several questions.

0.3. **Are there any trivial sets of DCNs?** We immediately notice some trivial sets (in base 10). By trivial I mean DCNs which have digits of any length or none, or contain, say, only even digits. These include:

a) $\Delta_{(2)}$. Every $\delta_{(2)}$ contains only even digits, since the divisor is 2 for every permutation. From this, it also follows that, for all odd primes, p, the set $\Delta_{(2p)}$ is an empty set (since an odd number does not divide an even number);

b) $\Delta_{(3)}$ is a complete set since *every* multiple of 3 is in this set (regardless of its permutation). There is no restriction on digit-length;

c) $\Delta_{(5)}$ is trivial because every $\delta_{(5)}$ contains only digits 5 and 0. Any other digit would yield a remainder after division;

d) $\Delta_{(10)}$ is an empty set since any $\delta_{(10)}$ contains only zeros.

For this reason, from here on, we consider only non-trivial $\delta_{(n=p>5)}$ (although it would be possible to consider $\delta_{(3)}$ or $\delta_{(5)}$ when 3 or 5 are factors of composite $n > 5$).

0.4. **How many Digits can a DCN have?** I first found DCNs by accident while playing with the multiplicative CN, 142857. I subtracted 1 and divided by 7. And then rotated the digits and to my surprise they were all divisible by 7. I then found the following 6-digit numbers by trial and error believing that cyclic numbers in $\Delta_{(7)}$ required *exactly* digits 1-6:

124635; 125643; 134526: 134652; 136542; 142653; 145236; 145362.

But further tests revealed that they are not limited to digits 1-6, and that *all* 6-digit numbers divisible by 7 have the same divisibly-cyclic property. Indeed, if we take, say, the last in the list, 145362, and keep adding 7 (i.e. 145369; 145376; 145383; 145390; etc), each new number belongs to the set $\Delta_{(7)}$ but only up to $999999 = 10^6 - 1$. As soon as we reach 7 digits, we find that $1000006 \notin \Delta_{(7)}$ (since 6100000 is clearly not divisible by 7).

The same was true for $\Delta_{(11)}$. For example, the number $9977441375 \in \Delta_{(11)}$ has 10 digits. If we keep adding 11, we generate more elements in the set (i.e. 7599774413; 7599774424; 7599774435; 7599774446; and so on, up to $9999999999 = 10^{10}-1$). But again, as soon as the DCN reaches 11 digits, we find that $10000000010 \notin \Delta_{(11)}$. All of which strongly suggests that DCNs are defined (somehow) by the number of digits they possess.

0.5. **What determines the Digit-Length?** To answer this question, we start with observations before making some conjectures. I define $L_{(p)}$ as the base digit-length of $\delta_{(p)}$.

After various tests I discovered that the set $\Delta_{(7)}$ seems to have $L_{(7)} = 6$ (eg 124635). I have not yet found a $\delta_{(7)}$ with less than 6 digits. But I also discovered that $\Delta_{(7)}$ includes multiples of the base-length. For example, 923556987458 has 12 digits $= 2L_{(7)}$.

The set $\Delta_{(11)}$ seems to have base digit-length $L_{(11)} = 2$ (eg 22, 33, etc); $2L_{(11)} = 4$ (eg 1265); $3L_{(11)} = 6$ (eg 123475); $4L_{(11)} = 8$ (eg 29235569); $5L_{(11)} = 10$ (eg 2923556955); $6L_{(11)} = 12$ (eg 123589716459); and so on.

The set $\Delta_{(13)}$ seems to have $L_{(13)} = 6$ (eg 852969); $2L_{(11)} = 12$ (eg 597697654489); $3L_{(11)} = 18$, and so on.

The set $\Delta_{(17)}$ seems to have $L_{(13)} = 16$ (eg 7994325875621569); $2L_{(13)} = 32$ (eg 55585296587412536985223654789632); and so on.

The set $\Delta_{(37)}$ seems to have $L_{(37)} = 3$ (eg 518); $2L_{(37)} = 6$ (eg 516890); $3L_{(37)} = 9$ (eg 855516886), and so on.

Is there an underlying pattern? What exactly determines the value of $L_{(p)}$? How many digits $\delta_{(p)}$ might have?

0.6. **Conjecture 1.** I conjecture that only for the primes listed in OEIS A001913 (mentioned above) the following it true:

$$L_{(p)} = (p-1) \text{ (i.e. the base digit-length)}$$

This certainly appears to be the case for $p = 7, 17, 19, 23...$, while simultaneously not for $p = 11, 13, 31, 37...$ and so on, which all have smaller base-digits .

0.7. **Conjecture 2.** I also conjecture that for *all other* primes, $p = 11, 13, 31, 37...$ whose base digit-lengths are smaller that $(p-1)$, DCNs exist for the following equation (for all positive integers $x > 1$):

$$L_{(p)}x = (p-1).$$

0.8. **Conjecture 3.** Bringing these two together, I also conjecture that for *all primes*, the number of digits is closely related to the numbers of OEIS A002371 (itself related to the sequence above). This sequence is the period length of the decimal expansion of the prime numbers. It begins as follows:

0, 1, 0, 6, 2, 6, 16, 18, 22, 28, 15, 3, 5, 21, 46, 13, 58, 60, 33, 35, 8, 13, 41, 44, 96, 4, 34, 53, 108, 112, 42, 130, 8, 46,...

For example, the 4th prime is 7, so $\frac{1}{7} = 0.\overline{142857}$, whose period length is 6, so A002371(4)=6. Or the 5th prime number is 11, so $\frac{1}{11} = 0.\overline{09}$, whose period length is 2, so A002371(5)=2.

The sequence also gives the smallest solution for m for the modular equation:

$$10^m - 1 \equiv 0 (\text{mod } p).$$

For example, when $p = 7$, 999999 is the smallest value for which this equation holds (where $m = 6$). Or when $p = 11$, 99 is the smallest value for which this equation holds (where $m = 2$).

So with a degree of confidence we can conjecture that the number of digits of any given $\delta_{(p)}$ is equal to (or a multiple of) A002371(n), where n is the nth prime number, such that:

$$L_{(7)} = 6, \text{ or multiple of } 6$$
$$L_{(11)} = 2, \text{ or multiple of } 2$$
$$L_{(13)} = 6, \text{ or multiple of } 6$$
$$L_{(17)} = 16, \text{ or multiple of } 16$$
$$L_{(19)} = 18, \text{ or multiple of } 18$$
$$L_{(23)} = 22, \text{ or multiple of } 22$$
$$L_{(29)} = 28, \text{ or multiple of } 28$$
$$L_{(31)} = 15, \text{ or multiple of } 15$$
$$L_{(37)} = 3, \text{ or multiple of } 3$$
$$...$$

This correlates nicely with what we have observed so far about digit length. But can it be proven? Unfortunately, as yet, there is no known method for generating sequence A002371.

0.9. **Do DCNs exist for composite numbers?** Finally, we ask whether DCNs exist for composite numbers, when $n = pq$, where q is another prime (n not having prime factors 2 or 5). The answer is yes. For example, when $p = 7$, $q = 11$:

$$806113 = \delta_{(7\times 11)} = \delta_{(77)}.$$

The digit-length of $\delta_{(77)}$, $L_{(77)} = 6$, seems to be determined by the longest base digit-length, that is $L_{(7)} = 6$, so $L_{(7\times 11)} = 6$.

This seems to be supported if we consider the otherwise trivial case of $p = 3$. When $p = 3$, $q = 7$ we can find the following DCN:

$$617589 = \delta_{(21)}.$$

This digit-length, $L_{(21)} = 6$, is determined not by $L_{(3)} = 1$, but by the longer base digit-length, $L_{(7)} = 6$. Therefore, $L_{(3\times 7)} = 6$.

Thus we can conjecture that, where $L_{(p)} < L_{(q)}$, $\delta_{(pq)}$ has base digit-length:

$$L_{(pq)} = L_{(q)}.$$

0.10. **Conclusion.** By way of conclusion, we conjecture that a DCN, $\delta_{(n)}$, divisible in all its cyclic permutations by prime p (if $n = p$) or by composite n (if n does not contain prime factors 2 or 5), will always exist whenever its base digit-length is equal to the smallest value for m that satisfies the modular equation $10^m - 1 \equiv 0(\text{mod } p)$ as represented in OEIS A002371 (as seen above), and always exists when $L_{(p)} = (p - 1)$ when $n = p$.

However, as well as the question of how to determine whether a cyclic number is divisibly-cyclic, we are left with the more fundamental question, Why does it work at all?

Chester, UK
Email address: julianbeauchamp47@gmail.com