Totally lossless projections

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Abstract

In this brief note, we discuss projective morphisms of perfect categories which are fully faithful, i.e., totally lossless.

Contents

1 Prologue

In some sense, the notion of an object, say perhaps a space, being totally adherent to the object from which it is the projection, is in effect a statement of the invertibility. That is to say, for the map

$$\mathscr{A} \xrightarrow{Perf} \mathbb{C}^\infty_\bullet \ni \mathcal{J}^\infty_n \overset{n}{\otimes} \mathbb{R}^n$$

where $\mathcal{J}^\infty_n$ is the nth jet of an infinitary bundle, and $\overset{n}{\otimes}$ is the n-fold symmetric product of a topological space homeomorphic to $\mathbb{R}^n$, the op-map

$$\mathscr{A} \xleftarrow{Perf^{op}} \mathbb{C}^\infty_\bullet$$

is also perfect.

We shall refer the reader to [1] for information about $\mathscr{A}$ in the case where it is a stack. For the infinite jet bundle, we refer the reader to [2]. One of the

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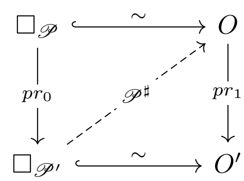
aims of the paper at hand is to extend the map (of a perfect subspace of) $\mathscr{A}$ to the more general notion of a *perfect map*, say, of rings, or perhaps more general objects yet.

So, let $\square_{\mathscr{P}}$ be a perfect subspace relative to the classifying stack $\mathscr{S}$. Let there be a flat embedding $\square_{\mathscr{P}} \overset{\sim}{\hookrightarrow} O$ for some O in an arbitrary category C. Further, let the following diagram



be commutative, where $\mathscr{P}^\sharp$ right-lifts against $\sim$. As is seen here, $\square_{\mathscr{P}} \xleftarrow{pr_0^{op}} \square_{\mathscr{P}'}$ is a retract of $O \xleftarrow{pr_1^{op}} O'$. Let us suppose for concreteness that O is a measurable metric space. Then the map $\mathscr{P}^\sharp$ induces either a measure or a metric onto $\mathscr{P}'$. Recall that a measurable metric space consists of the following data:

1. A space, X
2. A notion of distance, $d(x, y)$, between any two points x, y in X.
3. A measure, μ, over the metric space (X, d) such that $\mu(d(x, y)) = |d(x, y)$.

Therefore, a measurable metric space is a triple $X_d^\mu = (X, d, \mu)$, but we shall simply write X for the space when all other information is understood from context.

For the strongest sort of equivalence, warranting the "$=$" sign, i.e. $X = X$, we mean a perfectly lossless, bijective immersion $X \hookrightarrow X$, where for any point $x \in X$, $d(x, pr_n(x)) = 0$.

Let us assume that every path $P : x \to y \in Aut(X)$ is infinitesimally generated by an Abelian group object $\mathscr{G}$. Then, the map

$$\mathscr{G} \times \mathscr{G} \longrightarrow Aut(X)$$

is a bijection on fibers. Thus, we may safely write

$$\mathscr{G} \times \mathscr{G} = Aut(X)$$

as an equivalence of the strongest kind. Sketching a proof of this lemma may involve attaching a sub-object identifier, ρ, to every map $\mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$, and constructing a commutative diagram such that an element $x \in X$ is the push-out of all of these maps.

2 Differentiability

2.1 C^1 spaces

We begin our inquiry into the differentiability of spaces when they are subject to totally lossless maps, which we have yet to define, by considering spaces of first-order differentials. Without yet generalizing, consider the following:

Lemma 1. *If the map $X \in C^1 \to Y$ is totally lossless, then Y is a $C^{>0}$ space.*

Proof. Consider a map $X \in C^1 \to Y$, where Y is a C^0-space. Then, the map is not totally lossless, because the map is not a monomorphism. To show this, for every path $P : x \to y \in X$, consider the derivative dP. Thus, the cardinality of the set X is twice as large as the cardinality of the set Y, and the elements of the form dP fail to biject.

Thus, by contradiction, Y must be $C^{>0}$. □

Even stronger, it suffices to show that if we require bijectivity for all, and only all, of the elements of X, then Y *must* be *exactly* C^1.

2.1.1 Curvature

Let there be two curves $P : x \to y$ and $P' : x' \to y'$ in X, such that the metric $d(x, y)$ varies at a point x_0 from the metric $d(x', y')$ at the point x'_0. Then, we have the inequality

$$dP \neq dP'$$

and we can therefore obtain the *curvature* of each path by calculating the distance

$$|P|_{c_i} = d((dP, dP'), avg(dP, dP'))$$

Sections of the aforementioned paths form connections, which are isometries under all totally lossless maps. Here, $|P|$ is the class of all curves in X, and $\lim_{i \to \infty} |P|_{c_i} = 0$ for sufficiently many samples.

In other words, curvature is calculated by comparing the distance travelled along a fiber with the average distance across the equivalence class of all fibers after an equal amount of time. For a space X, we denote the comparison space by X_{comp}, and when

$$X_{comp} = X$$

then zero average sectional curvature will be detected, and thus the space is said to have curvature 0.

Proposition 1. *For a totally lossless projective map $X \to Y$,*

$$|Y|_c = \pm |X|_c$$

If the chirality of the space is reversed under the map, then the orientation of the fibers are reversed as well. So, if ho_X is the homology of X, then

$$coho_Y = ho_X$$

means the spaces are of opposite sign. This is semiotically meaningful, as we can consider a superposition

$$\{X, Y | coho_Y = ho_X\} \;\; X + Y = 0$$

to be the meaningless state, or the neutral and trivial state under the consideration of the total fuzzy truth space of a particular "logical world," say W_∞.

Warning 1. *While the superposition of two opposing spaces, with perfectly lossless maps between the two, is zero, each is to be observed individually as physically meaningful; that is, they will both present with their inherent sectional curvature relative to some comparison manifold*

$$Z = (X \wedge Y)_{comp}$$

Remark 1. *Any group of n spaces may be compared to a common index Z by way of the formula*

$$Z = (X \wedge ... \wedge \omega)_{comp}$$

with n-2 spaces omitted by the ellipses. This allows us to write a configuration space as $(Z, |p|, C, \mathcal{U})$, where $|p|$ is a class of point-like objects, C is the category of p, and $\mathcal{U}()$ is the neighborhood operator.

Definition 1. *Let p_0 be a point object in a given category C. The neighborhood operator, $\mathcal{U}(p_0)$ is a restriction of the set of outward morphisms $p_0 \to p_{>0}$, $|p||_r$, where r is the radius of the neighborhood.*

Therefore, any configuration space C has a built-in notion of sectional curvature, as well as "connections," which are defined as curved sections of fibrations.

3 Invariance

What sort of property are we imagining when we think of "invariance?" A couple of possibilities may spring to mind: Lorentz invariance, gauge invariance, etc. Here, invariance is defined as follows:

Definition 2. *Let there be some object p_0 traveling along a path P over time. If the maps $id : p_0 \to p_0 @ t = 0$ and $id' : p_0 \to p_0 @ t = n$, where t is the elapsed time, are identical in form, then p_0 is an invariant object with respect to identity.*

To say that an object, in the real world, is invariant, is to say that it is “the same thing” as it was earlier, and so mathematically this would mean that the identity function behaves exactly the same across time. This would mean that the map $t \longrightarrow t + n$ is totally lossless for said object.

To say that a map is totally lossless is *stronger* than to say it is invertible. It is, also, to say that it is *perfect*, in the most blatant sense. Thus, consider the class

$$|p|$$

of possible $|p|_n$. We have, for all n, $triv_n = triv_{n \pm k}$. That is to say, the function which *kills* the data under consideration, i.e. *trivializes* the object, is stable over time. Thus, the class of objects then represents a meta-identity which determines, at each moment, the invariant identity on p.

Proposition 2. *If an object is invariant, it will be of the same type T at the times $t = 0$ and $t = n$.*

A proof of the above proposition would essentially be trivial, but we shall provide one nonetheless. It follows from the very definition of invariance; if an object i has type T, then i_{id} must obviously also have type T. Thus, if there is a sequence

$$i \to i_{id} \to j \to j_{id}$$

,

then $ker(j_{id}) = im(i_{id})$, and so the type of every object is transitive.

4 References

[1] R.J. Buchanan, P. Emmerson, O. Hancock
On the Mechanics of Quasi-quanta Realization, (2023)
[2] H. Nishimura *Synthetic Differential Geometry of Jet Bundles*, (2001)