Optimal fractional $PI^{\beta(t)}D^{\alpha(t)}$ controllers and numerical simulation for DC motor speed control

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Abstract We model the rotation process of the motor for variable-order fractional control, which has been active in recent research, and perform numerical simulation of its optimal control and automatic control process. In this paper, we verify numerical method and error estimation of variable order fractional linear dynamic system with time-varying coefficients, a variable-order fractional PID controller design method where the integral of the absolute error with time weight is minimized is proposed using particle swarm optimization algorithm and demonstrate its effectiveness through numerical simulation for DC motor speed control. Numerical experiments show that the performance of the VFPID controller is superior to PID and FPID, especially $VFPID_B$ (B-type variable order FPID) controller has the best performance. Finally, when the differential order varies, the subtypes of variable-order fractional derivatives are analyzed for the effects on the control objective, its effectiveness is newly clarified, and their research and practice is highlighted.

Keywords: Variable-Order fractional PID Control, Variable-Order fractional Dynamic System, Variable Order fractional calculus, Particle swarm optimization

1. Introduction

In the 1940s, the practical significance of fractional differential operators began to emerge, revealing the fact that it was effective to use fractional differential operators in mathematically modeling the mechanical constitutive relations of viscoelastic media.

Since fractional derivatives have non-localities, past dependencies, and so on, unlike classical derivatives, fractional calculus has features that better model natural phenomena including complex phenomena in various fields such as materials science, chemistry, physics, biology, economic [36].

One of the main applications of fractional calculus is fractional PID control that has received considerable attention in its scientific research and industrial applications [1-3].

So far, the results show that in many cases the fractional PID controller outperforms the classical integer-order PID controller much [4,11].The attractiveness of fractional-order PID controller lies in their potential to increase control performance and robustness of closed-loop system due to the extra control parameters available compared to conventional controller [15, 16].

For FPID(often denoted as $PI^{\beta}D^{\alpha}$), integral and derivative is replaced by β-order fractional integral and α-order fractional derivative, respectively, and thus the $PI^{\beta}D^{\alpha}$, which contains two additional parameters, provides additional flexibility in designing controllers of better performance [7].

In particular, fractional-order PID controller is widely used in industrial automation and manufacturing engineering due to its advantages of high torque per weight, long lifetime, lower noise and high reliability while controlling motors [8-10].

The authors of [5,6,12] designed a fractional PID controller (FOPID) for speed control of DC motor. They used particle swarm optimization algorithm (PSO) to optimize FOPID controller parameters and

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showed that it has more flexible and higher robust performance than classical PID.

In [13], the robustness of fractional order controllers with varying load of DC motor is studied and in [14] a comparative experiment with the results of PID and fuzzy logic controllers is performed to show the control performance and robustness of parameter tuning method with adaptive network fuzzy inference system (ANFIS) structure. Recent studies have shown that fractional differential equations with constant order cannot adequately model some real natural phenomena, such as complex diffusion occurring in highly fine inhomogeneous or disordered porous media [17-20].

To overcome this limitation, variable order (VO) differential operators have been proposed and various definitions and subtypes have been formulated [21-23].The generalization from constant orders to VO operators gives new insight for more accurate modeling of phenomena and mathematical description of complex systems. In recent years, with the increasing number of mathematical studies on calculus, differential equations, optimal control, and automatic control with variable order, these theoretical studies have been applied in various fields such as motor speed control and considering variable order fractional properties in practice is important and practical [24-26].Thus, the mathematical properties and approximate calculation methods of VO fractional operators have been studied and their applications and computational algorithms have been developed [27- 33]. In particular, in [34, 35], authors propose a VFPID controller design method using variable order fractional operators and demonstrate its effectiveness.

Since PID controller is part of FPID and FPID controller is part of VFPID, VFPID controller provides more degrees of freedom and it is more flexible than PID or FPID. Although VFPID controller offers more degrees of freedom, they have the drawback of increasing the amount of computation and additional control complexity. In particular, there is no stability criterion for dynamic system and feedback controllers with VO.

Moreover, although it is always possible to model linear time-varying fractional systems with fixed order in the frequency domain, it is not always possible for linear time-varying fractional systems with variable order. Thus, designing a feedback controller for dynamic systems with variable order remains an open issue with today's technology.

This paper proposes a new method for designing VFPID controller for control of fractional time-varying linear dynamic system with variable order. To do this, the Grünwald-Letnikov definition and backward difference approximation are applied to discretization of the variable order fractional differential equation. Then, PSO algorithm is used to search the optimal parameter of VFPID controller that minimize the integral absolute error (IAE) of the closed-loop system.

The rest of the paper is organized as follows.

In Section 2, a brief overview of VO fractional operators and subtypes is presented. In Section 3, the modeling of linear fractional order systems for speed control of DC motors and the identification method for designing optimal VFPID are described. In this section, the optimization algorithm PSO to find the discretization method and control parameters for FDEs with VO fractional order operators is briefly described. In Section 4, the advantages of using subtypes of VFPID in controlling variable-order fractional linear dynamic system are illustrated for DC motor speed control. Finally, a conclusion is drawn

2. Preliminaries

Definition ([1]): x(.):[a, b]→R and $\alpha(t) \in C^n[a, b]$. For all t ∈ [a, b]

(a) The left variable order Fractional Integral (VOFIs) of type A and type B in the sense of Riemann-Liouville is defined by

$$ {}_a^A I_t^{\alpha(t)} x(t) = {}_a I_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(\alpha(t))} \int_a^t (t-\tau)^{\alpha(t)-1} x(\tau) d\tau \qquad {}_a^B I_t^{\alpha(t)} x(t) = \int_a^t \frac{1}{\Gamma(\alpha(\tau))} (t-\tau)^{\alpha(\tau)-1} x(\tau) d\tau \tag{1} $$

(b) The right variable order Fractional Derivative (VOFDs) of type A and type B in the sense of Riemann-Liouville is defined by

$$ {}_a D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha(t)-1} x(\tau) d\tau, \qquad {}_a^B D_t^{\alpha(t)} x(t) = \frac{d^n}{dt^n} \left(\int_a^t \frac{1}{\Gamma(n-\alpha(\tau))} (t-\tau)^{n-\alpha(\tau)-1} x(\tau) d\tau \right) \tag{2} $$

(c) The left variable order Fractional Derivative (VOFDs) of type A and type B in the sense of Caputo is defined by

$$ {}_a^C D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_a^t (t-\tau)^{n-\alpha(t)-1} x^{(n)}(\tau) d\tau, \ {}_a^{B,C} D_t^{\alpha(t)} x(t) = \int_a^t \frac{1}{\Gamma(n-\alpha(\tau))} (t-\tau)^{n-\alpha(\tau)-1} x^{(n)}(\tau) d\tau \tag{3} $$

(d) The right variable order Fractional Derivative Definition (VOFDs) of type A and type B in the sense of Grünwald-Letnikov is defined by

$$ {}_a^{GL} D_t^{\alpha(t)} x(t) = \lim_{\substack{h\to 0 \\ Nh=t-a}} \sum_{k=0}^{N} \frac{(-1)^k}{h^{\alpha(t)}} \binom{\alpha(t)}{k} x(t-kh), \ {}_a^{B,GL} D_t^{\alpha(t)} x(t) = \lim_{\substack{h\to 0 \\ Nh=t-a}} \sum_{k=0}^{N} \frac{(-1)^k}{h^{\alpha(kh)}} \binom{\alpha(kh)}{k} x(t-kh) \tag{4} $$

where $n-1 < \alpha(t) \le n, n \in N$ and in special case, $\alpha(t) = \alpha \in R$.

3. Design of optimal fractional $PI^{\beta(t)}D^{\alpha(t)}$ controller for time-varying fractional dynamic sy stem with variable order

In this section, we consider the optimal fractional order controller design method for multi-term time-varying fractional order dynamical systems if the differential order is given as a function.

Since the uniqueness of the solution of multi-term fractional differential equation with variable order has not yet been studied, we assume the uniqueness of the solution and study the optimal fractional order controller design method using the Grünwald-Letnikov fractional derivative approximation formula .The multi-term variable order fractional order linear nonhomogeneous differential equation with time-varying coefficients is as follows:

$$ {}^c D_{0+}^{\alpha(t)} y(t) + \sum_{i=1}^{L} a_i(t) {}^c D_{0+}^{\alpha_i(t)} y(t) = g(t), \quad 0 < t < T \tag{5} $$

with the initial conditions

$$ D^k y(t)\big|_{t=+0} = b_k \in R, \ k = 0,1,\cdots,n-1 \tag{6} $$

Where, T>0 is any real number and $\alpha(t), \alpha_i(t) \in C_{[0,T]}$, $i = 1,\cdots,L$, $\alpha(t) > \alpha_1(t) > \cdots > \alpha_m(t) \ge 0$, $n-1 < \alpha(t) \le n$, $n_i - 1 < \alpha_i(t) \le n_i$, $n, n_i \in \mathrm{N}$.

The feedback control signal by $PI^{\beta(t)}D^{\alpha_0(t)}$ control for this dynamic system is as follows:

$$ u_b(t) = k_p e(t) + k_i \, {}_0 I_t^{\beta(t)} e(t) + k_d \, {}_0 D_t^{\alpha_0(t)} e(t) \tag{7} $$

The target control signal

$$u_f(t)={}^cD_{0+}^{\alpha(t)}x_d(t)+\sum_{i=1}^{L}a_i(t){}^cD_{0+}^{\alpha_i(t)}x_d(t) \tag{8}$$

and the control input signal is $u(t)=u_f(t)+u_b(t)$, then the dynamic error system for the cl osed-loop control system of the control object is as follows:

$$\begin{cases} e(t)=x(t)-x_d(t) \\ {}^cD_{0+}^{\alpha(t)}e(t)+\sum_{i=1}^{L}a_i(t){}^cD_{0+}^{\alpha_i(t)}e(t)=k_pe(t)+k_{i\,0}I_t^{\beta(t)}e(t)+k_{d\,0}D_t^{\alpha_0(t)}e(t) \end{cases} \tag{9}$$

If [0,T] is divided by uniform mesh, for t_i=mh, m=0,...N with $0=t_0<t_1<t_2<\ldots<t_N=T$, the d ynamic error equation is written as the backward finite difference equation of the following left type A or left type B of (3).

The left A- type backward finite difference equation

$$\begin{aligned}&\sum_{k=0}^{m}\omega_k^{\alpha(t_m)}e(t_m-kh)+\sum_{i=1}^{L}a_i(t_m)(\sum_{k=0}^{m}\omega_k^{\alpha_i(t_m)}e(t_m-kh))=\\ &=k_p\,e(t_m)+k_i\sum_{k=0}^{m}\omega_k^{-\beta(t_m)}e(t_m-kh)+k_d\sum_{k=0}^{m}\omega_k^{\alpha_0(t_m)}e(t_m-kh),m=0,1,2,\cdots,N\end{aligned} \tag{10}$$

Or the left type B backward finite difference equation

$$\begin{aligned}&\sum_{k=0}^{m}\varpi_k^{\alpha(t_m)}e(t_m-kh)+\sum_{i=1}^{L}a_i(t_m)(\sum_{k=0}^{m}\varpi_k^{\alpha_i(t)}e(t_m-kh))= \qquad m=0,1,2,\cdots,N\\ &=k_p\,e(t_m)+k_i\sum_{k=0}^{m}\overline{\omega}_k^{-\beta(t_m)}e(t_m-kh)+k_d\sum_{k=0}^{m}\omega_k^{\alpha_0(t_m)}e(t_m-kh),\end{aligned} \tag{11}$$

By Podlubny's matrix method, we rewrite the upper equation in matrix form as follows.

$$\begin{aligned}&\begin{pmatrix}\omega_0^{\alpha(t_0)}&&&\\ \omega_1^{\alpha(t_1)}&\omega_0^{\alpha(t_1)}&&\\ \vdots&\vdots&\ddots&\\ \omega_N^{\alpha(t_N)}&\omega_{N-1}^{\alpha(t_N)}&\cdots&\omega_0^{\alpha(t_N)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}+\sum_{i=1}^{L}\begin{pmatrix}a_i(t_0)\omega_0^{\alpha_i(t_0)}&&&\\ a_i(t_1)\omega_1^{\alpha_i(t_1)}&a_i(t_1)\omega_0^{\alpha_i(t_1)}&&\\ \vdots&\vdots&\ddots&\\ a_i(t_N)\omega_N^{\alpha_i(t_N)}&a_i(t_N)\omega_{N-1}^{\alpha_i(t_N)}&\cdots&a_i(t_N)\omega_0^{\alpha_i(t_N)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}=\\ &=k_p\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}+k_i\begin{pmatrix}\omega_0^{-\beta(t_0)}&&&\\ \omega_1^{-\beta(t_1)}&\omega_0^{-\beta(t_1)}&&\\ \vdots&\vdots&\ddots&\\ \omega_N^{-\beta(t_N)}&\omega_{N-1}^{-\beta(t_N)}&\cdots&\omega_0^{-\beta(t_N)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}+k_d\begin{pmatrix}\omega_0^{\alpha_0(t_0)}&&&\\ \omega_1^{\alpha_0(t_1)}&\omega_0^{\alpha_0(t_1)}&&\\ \vdots&\vdots&\ddots&\\ \omega_N^{\alpha_0(t_N)}&\omega_{N-1}^{\alpha_0(t_N)}&\cdots&\omega_0^{\alpha_0(t_N)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}\end{aligned} \tag{12}$$

or

$$\begin{aligned}&\begin{pmatrix}\overline{\omega}_0^{\alpha(t_0)}&&&\\ \overline{\omega}_1^{\alpha(t_1)}&\overline{\omega}_0^{\alpha(t_0)}&&\\ \vdots&\vdots&\ddots&\\ \overline{\omega}_N^{\alpha(t_N)}&\overline{\omega}_{N-1}^{\alpha(t_{N-1})}&\cdots&\overline{\omega}_0^{\alpha(t_0)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}+\sum_{i=1}^{L}\begin{pmatrix}a_i(t_0)\overline{\omega}_0^{\alpha_i(t_0)}&&&\\ a_i(t_1)\overline{\omega}_1^{\alpha_i(t_1)}&a_i(t_1)\overline{\omega}_0^{\alpha_i(t_0)}&&\\ \vdots&\vdots&\ddots&\\ a_i(t_N)\overline{\omega}_N^{\alpha_i(t_N)}&a_i(t_N)\overline{\omega}_{N-1}^{\alpha_i(t_{N-1})}&\cdots&a_i(t_N)\overline{\omega}_0^{\alpha_i(t_0)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}=\\ &=k_p\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}+k_i\begin{pmatrix}\overline{\omega}_0^{-\beta(t_0)}&&&\\ \overline{\omega}_1^{-\beta(t_1)}&\overline{\omega}_0^{-\beta(t_0)}&&\\ \vdots&\vdots&\ddots&\\ \overline{\omega}_N^{-\beta(t_N)}&\overline{\omega}_{N-1}^{-\beta(t_{N-1})}&\cdots&\overline{\omega}_0^{-\beta(t_0)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}+k_d\begin{pmatrix}\overline{\omega}_0^{\alpha_0(t_0)}&&&\\ \overline{\omega}_1^{\alpha_0(t_1)}&\overline{\omega}_0^{\alpha_0(t_0)}&&\\ \vdots&\vdots&\ddots&\\ \overline{\omega}_N^{\alpha_0(t_N)}&\overline{\omega}_{N-1}^{\alpha_0(t_{N-1})}&\cdots&\overline{\omega}_0^{\alpha_0(t_0)}\end{pmatrix}\begin{Bmatrix}e_0\\e_1\\\vdots\\e_N\end{Bmatrix}\end{aligned} \tag{13}$$

where

$$\omega_k^{\alpha(t_m)}=(-1)^k h^{-\alpha(t_m)}\binom{\alpha(t_m)}{k}\quad,\quad\varpi_k^{\alpha(t_m)}=(-1)^k h^{-\alpha(kh)}\binom{\alpha(kh)}{k}\quad m=0,1,2,\cdots,N \tag{14}$$

$$\binom{\alpha(t)}{k} = \frac{\Gamma(\alpha(\mathrm{t})+1)}{\Gamma(\mathrm{k}+1)\Gamma(\alpha(\mathrm{t})-\mathrm{k}+1)} = \frac{(-1)^k \Gamma(k-\alpha(t))}{\Gamma(-\alpha(t))\Gamma(k+1)}\ ,\quad \binom{\alpha(t)}{0} = 1$$

Solving the above system of equations(10-13) yields a solution of the Riemann-Liouville se nse with zero initial conditions and the solution of the Caputo sense is obtained by adding th e following terms with respect to the initial condition to the solution of the Riemann-Liouvill e sense with zero initial condition.

$$h^{\alpha(t_m)} \sum_{k=0}^{n-1} \frac{b_k\, t_m^{k-\alpha(t_m)}}{\Gamma(k+1-\alpha(t_m))} \quad m = 1,2,\cdots,N \tag{15}$$

Thus, the solution of the dynamic error equation is expressed as follows:

- For type A

$$e_m = -\sum_{k=1}^{m} \omega_k^{\alpha(t_m)}\, e(t_m - kh) + h^{\alpha(t_m)} \sum_{k=0}^{n-1} \frac{b_k\, t_m^{k-\alpha(t_m)}}{\Gamma(k+1-\alpha(t_m))} \quad m = 1,2,\cdots,N \tag{16}$$

- For type B

$$e_m = -\sum_{k=1}^{m} \overline{\omega}_k^{\alpha(t_k)}\, e(t_m - kh) + h^{\alpha(t_m)} \sum_{k=0}^{n-1} \frac{b_k\, t_m^{k-\alpha(t_m)}}{\Gamma(k+1-\alpha(t_m))} \quad m = 1,2,\cdots,N \tag{17}$$

Theorem[25]. By variable order fractional calculus definitions (2) and (4), the error estimates and discrete formulas of left variable order fractional derivatives of type A and type B are expressed as follows:

$${}_a D_{t_m}^{\alpha(t)} x(t) = \sum_{k=0}^{m} \omega_k^{\alpha(t_m)} e(t_m - kh) + \mathrm{O}(\mathrm{n}+1+\alpha_{\mathrm{m}})\ , \qquad {}_a^{\mathrm{B}} D_{t_m}^{\alpha(t)} x(t) = \sum_{k=0}^{m} \overline{\omega}_k^{\alpha(t_m)} e(t_m - kh) + \mathrm{O}(\mathrm{n}+1+\alpha^*) \tag{18}$$

here

$$\alpha^* = \begin{cases} \sup\{\alpha(t) | n-1 < \alpha(t) < n,\ t \in [0,T]\},\ t_m \le 1 \\ \inf\{\alpha(t) | n-1 < \alpha(t) < n,\ t \in [0,T]\},\ t_m > 1 \end{cases}$$

Remark1. It can get the approximate solution of variable order fractional nonlinear differen tial equation with time-varying coefficients by transforming it into a system of nonlinear equa tion by using the Grünwald-Letnikov definition.

Remark2. It can apply this approximate calculation method not only for the dynamical system of the left Caputo (Riemann-Liouville) sense but also the right Caputo (Riemann-Liouville) sense

It can be seen that the fractional $PI^{\beta}D^{\alpha}$ controller requires not only three parameters K_p, K_i, K_d, but also the real parameters β, α of integral and differential operators.

In addition, if β, α are given by the functions $\beta(t), \alpha(t)$, in other words, if $\alpha(t) = a + be^{-ct}, \beta(t) = e + fe^{-gt}$, constants a,b,c,e,f,g must also be added as parameters.

Fig.1 shows the flexibility of VFPID control and the necessity of variable order fractional calculus.

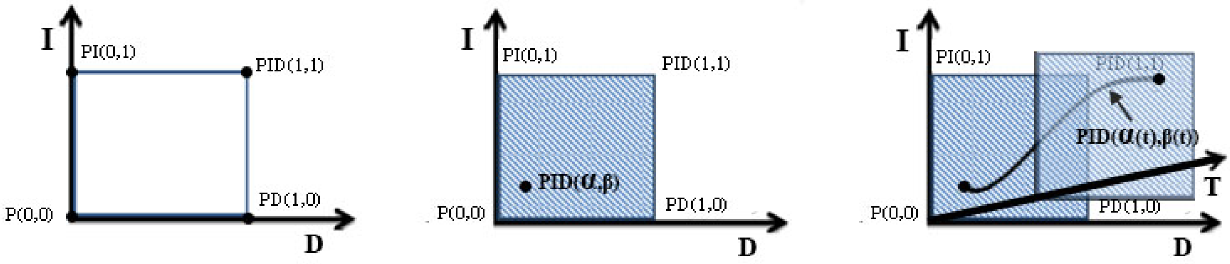

Fig1. permissible domain of differential order

Next, we take the integral of time-weighted absolute error (ITAE) as the objective function.

$$J(u)=\int_0^{\infty} t\left|e(t,u)\right|dt,\ u\in R^n \tag{19}$$

To adjust the VFPID controller by the objective function means solving a system of nonlinear equation. Thus, nine free parameters can be identified optimally in a search space by using PSO.

Particle swarm optimization (PSO) algorithm is as follows:

Step1. Initialize a swarm of particles with random position, velocity and acceleration.

Step2. Evaluate the fitness of each particle.

Step3. Compare the individual fitness of each particle with its previous pbest (a particle with the best position). If the fitness is better, update the fitness to pbest.

Step4. Compare the individual fitness of each particle with its previous gbest (the best particle of its previous swarm), if the fitness is better, update the fitness to gbest.

Step5. Update the velocity and position of each particle.

Step6. Return to step 2 and repeat until stopping condition is satisfied.

4. Numerical simulation analysis of $PI^{\beta(t)}D^{\alpha(t)}$ controller for DC motor speed control

In this section, the performance evaluation of fractional order controllers is considered through fractional order PID controller design for speed control of armature controlled DC motor.

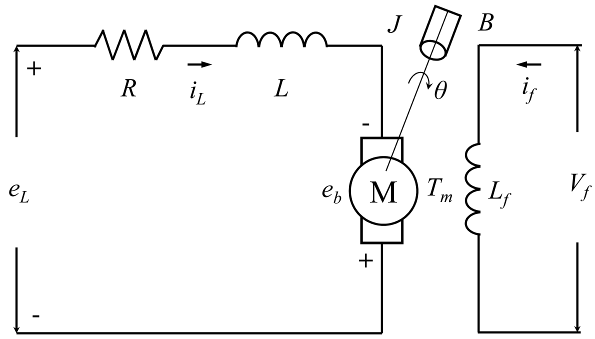

Fig.2. Schematic of armature controlled DC motor

The DC motor operating variables and physical constants used in [4] are used in our simul ation for performance comparison.

Then, $T_m = K_1 K_f i_f i_a$, where k1 is a constant. In the armature controlled DC motor, the excitation current is kept constant. Then, $T_m = K_T\, i_a$, where K_T is the motor torque constant.

The differential equation and moment equation of armature circuit are as follows:

$$L\frac{d\,i_a}{dt}+R\,i_a+e_b-e_a=0\ ,\qquad J\frac{d^2\theta}{dt^2}+B\frac{d\theta}{dt}=T_m=K_T i_a$$

Table1. Motor operating variables and constants [4].

NO	Symbol	Unit	Description	Symbol	Unit	Value	Description
1	R	Ω	Armature Resistance	P	HP	5	Rated Power
2	L	H	Inductance of armature winding	V	V	240	Rated Armature Voltage
3	i_a	A	Armature current	R_a	Ω	2.518	Armature Resistance
4	i_f	A	Field current	L_a	H	0.028	Armature Inductance
5	e_a	V	applied armature voltage	R_f	Ω	281.3	Field Resistance
6	e_b	V	back EMF	L_f	H	156	Field Inductance
7	T_m	Nm	torque developed by motor	K_b		0.0924	Back EMF constant
8	θ	Rad	angular displacement of motor shaft	K_T		0.0924	Motor constant
9	ω	rad/ sec	angular speed of motor shaft	B	Nm*s/ rad	0.0005	Friction coefficient of motor
10	J	Kg-m^2	equivalent moment of inertia of motor and load referred to motor shaft	J	Kg-m^2	0.003	Moment of inertia of motor
11	B	Nm*s/r ad	equivalent friction coefficient of motor and load referred to motor shaft	v	RPM	1750	Rated Speed
12				V_f	V	300	Rated Field Voltage

The motor rotation equation by controlling armature voltage is ordered as follows:

$$\mathrm{LJ}\frac{\mathrm{d}^2\omega}{dt^2}+(BL+RJ)\frac{d\omega}{dt}+(RB+K_bK_T)\omega=K_Te_a \tag{20}$$

The existence of the solution is guaranteed and numerical simulation calculations can be performed for various PID control with different orders using the definitions of Grünwald-Letnikov derivative (type A, B). In the simulation, let the variable order fractional functions of the controller and coefficients have the following forms:

$$\alpha_0(t)=a+b\exp(-ct), a>1, b>0, c>0 \quad ,\beta(t)=d+f\exp(-gt), d,f,g>0 \tag{21}$$

$\alpha(t)=2, \alpha_1(t)=1,\ \alpha_2(t)=0,\ a(t)=9.0909*10^{-4},\ a_1(t)=0.08190476, a_2(t)=0.1060255$

The upper limit of time for optimization is t=5s and the step size of discretization is h=0.1s. Optimization is tested for dynamic systems of type A or type B and PID, FPID, VFPID$_A$ (VFPID by A-type variable order fractional operators) and VFPID$_B$ controllers (VFPID by B-type variable order fractional operators).

The upper and lower bounds of the parameters for optimization are

$k_p,\ k_d, k_i\in[0,2]$, $a\in[0,2]$, $b\in[0,1]$, $c\in[0,1000]$, $d\in[0,2]$, $f\in[0,1]$, $g\in[0,1000]$

In the PSO algorithm, the number of particles is set to $N=40$ and the maximum number of iterations is $I_{max}=20$.

Table 2 shows the control parameters and performances of non-optimization according to th e alternative FPID coefficients of [4], and table 3 shows the optimal parameter identification values and control performances of PID, FPID, VFPID controllers.

Fig. 3 shows the transient process for error tracking of the controllers when initial conditio ns are $e(0) = -0.1$, $\dot{e}(0) = 0.001$.

Numerical simulation results show comparison between the PID, FPID and VFPID controlle rs where parameters are optimized with the fractional controllers where parameters are not opt imized.

In the previous works without optimization (K_p= 0.05, K_d= 0.0525, K_i= 0.98) the control performance was much lower than the integer-order PID controller in our case (optimal parameter identification) in spite of the fractional controller, as seen in Fig. 3. Moreover, in partial cases when the fractional orders are a real number greater than 1.3, instability was observed.

We can see that FPID is better than PID and VFPID is better than FPID in their performances, and affirm VFPID to be of the best performance, especially $VFPID_B$ is very good.

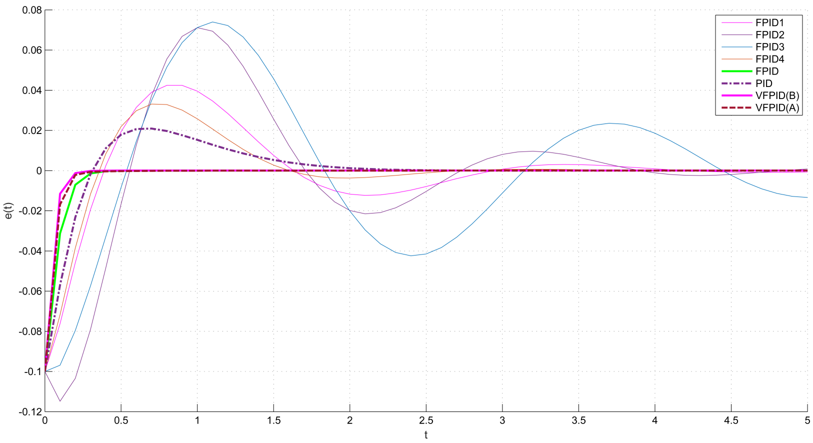

Fig.3. Comparison of Transient curves of Different Controllers

Table2. Performance indexes and FPID control characteristics ((K_p=0.05, K_d=0.0525, K_i=0.98; non-optimization case)

Index	β	A	ITAE	peak time (s)	settling time (s)	Overshoot (%)	Color
FPID1	1.1	0.5	0.6520982093	0.8	4.673	4.244	pink
FPID2	0.7	1.5	1.2179365150	1.0	3.865	7.119	purple
FPID3	1.1	1.2	2.6134138604	1.1	5이 상	7.395	blue
FPID4	0.9	0.5	0.2916051309	0.7	2.483	3.309	orange

Table3. Performance indexes and control characteristics by our method ($\alpha_1(t) = 1$; optimization case)

Controller		ITAE	K_p	K_i	K_d	a	B	c	d	f	g	Overshoot (%)	Peak time(s)	settling time (s)	Color
PID		0.181	2.0	0.94	0.1							2.091	0.697	1.869	black
FPID		0.005	0.1	2.0	0.1	0.132			0.1			0	0	0.289	pale green
VFPID	$VFPID_A$	0.004	1.5	2.0	2.0	0.1	0.1	590.5	0.1	0.1	256.4	0	0	0.236	Paint orange
	$VFPID_B$	0.001	0.1	0.1	1.0	1.0	0.972	253.9	0.1	0.1	470.5	0	0	0.198	pink

Next, assuming that it is affected by strong external electromagnetic fields or rotates in a fluid field, the motor rotation equation by controlling the armature voltage can be modeled as the following fractional order dynamic equation:

$$\mathrm{LJ}\frac{d^2\omega}{dt^2}+(BL+RJ)D_{0+}^{1.5}\omega+(RB+K_bK_T)\omega=K_Te_a \tag{22}$$

Set $\alpha_1(t)=1.5$ and the other parameters to be set as above and numerical simulation are performed.

Numerical experimental results of the obtained optimal parameters and performance indexes are shown in Fig. 4 and Table 4.

The error transient curves of the various controllers (PID, FPID, $\mathrm{VFPID_A}$, and $\mathrm{VFPID_B}$) are shown in Fig. 4, and Table 4 shows the optimized parameter values and performance indexes by PSO algorithm.

It can be seen that FPID controller outperforms PID controller and VFPID controller is better than FPID controller, especially $\mathrm{VFPID_B}$ controller has the best control performance than other types of controllers.

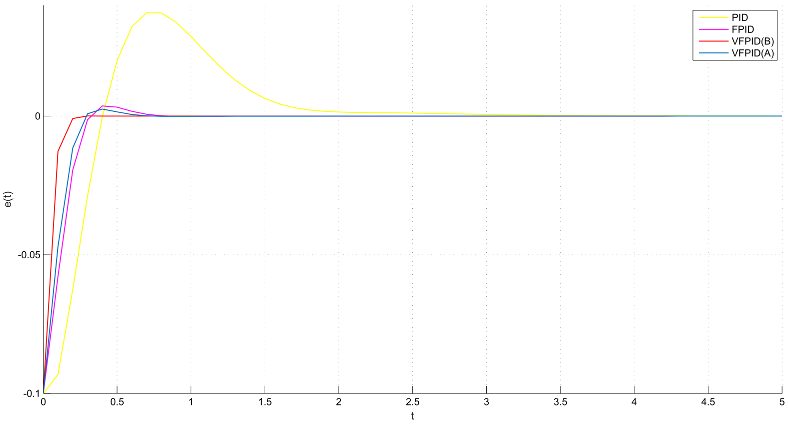

Fig.4. Comparative experiments of different controllers for fractional dynamic system ($\alpha_1(t)=1.5$)

Table4. Performance indexes and control characteristics of different controllers ($\alpha_1(t)=1.5$)

Controller		**ITAE**	**K_p**	**K_i**	**K_d**	**a**	**b**	**c**	**d**	**f**	**g**	**Overshoot (%)**	**Peak time(s)**	**settling time (s)**	**Color**
PID		0.32	2.0	1.23	0.1							3.726	0.71	1.899	Yellow
FPID		0.016	0.89	2.0	1.34	0.11			0.1			0.364	0.45	0.810	Pink
VFPID	$\mathrm{VFPID_A}$	0.011	2.0	2.0	2.0	0.1	0.1	1.39	0.1	0.1	3.35	0.255	0.39	0.689	Blue
	$\mathrm{VFPID_B}$	0.001	0.2	2.0	1.18	0.93	0.99	589.1	0.1	0.8	81.5	0	0.3	0.3	Red

5.Conclusion

In this paper, in order to investigate the relationship and validity of variations of the variable-order fractional Operator, we proposed a numerical method of variable-order linear fractional dynamic system using the approximate calculation formulas of the variable-order fractional calculus operators and introduced error estimation.

Based on this, a variable-order fractional PID controller design method for fractional dynamic system was proposed using particle swarm optimization algorithm and applied to DC motor speed control to perform performance evaluation.

Finally, numerical experiments have shown that the VFPID controller is superior to PID controller and FPID controller, especially the $VFPID_B$ controller has the best performance.

Discussion

Since different types of variation definitions are proposed for variable order fractional calculus, the correlation of variations and control characteristics in the frequency domain as well as solution properties of variable order fractional differential equations should be studied.

It is also important to determine the analog circuits, hardware implementation and efficiency of controllers according to variations.

It is noteworthy that the performance of $VFPID_B$ is particularly superior.

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