Gravitational redshift and curvature of space-time according to the Schwarzschild model

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Abstracs

We are working on concrete applications of the relativistic Schwarzschild model to the cosmos. In this report we relate the Gaussian curvature of space-time according to the Schwarzschild model with the gravitational redshift. The relativistic Schwarzschild model solves Einstein's equations exactly assuming a point gravitational mass and empty space in its vicinity. This model leads to a static and symmetric solution of the mathematical equation of space-time that allows its Gaussian curvature to be calculated at each point. We have calculated some curvature values and found an equation to calculate them which allows us to extend the results to a wider range of distances. We have quantitatively related this Gaussian curvature equation to the gravitational redshift equation by establishing a new equation between these two magnitudes. Finally, we carefully studied this found equation and applied it to the assumption of the supermassive black hole in our galaxy SAGITTARIUS A*

Keywords: Gravitational redshift, Schwarzschild model, curvature of space-time, cosmology.

1.Introduction

First, we are concerned with the problem of calculating the curvature of space-time caused by a spherical and static black hole at a point located at a distance "r" from the center of the black hole. This point will always be further away from the event horizon or Schwarzschild radius, "Rs". Schwarzschild solves the equations of the general theory of relativity [1] for an assumption of a point gravitational mass and a surrounding empty space, establishing a metric and an equation for space-time that turns out to be stationary in time and with spherical symmetry, resulting in a 2D surface, (the Flamm paraboloid), which is represented in Fig. 1.

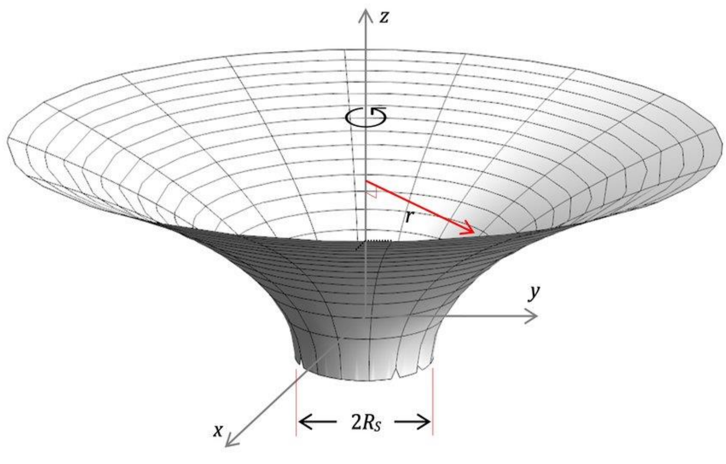

Fig. 1 Space-time in the Schwarzschild model. Flamm paraboloid

2. Resolution of the mathematical problem

Flamm's paraboloid, mathematical solution to the Schwarzschild model, is a 2D surface inserted in a space R^3. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the the point mass "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [2], and with it we are going to calculate values of Gaussian Curvature.

Surface parameters (r, φ)

$0 \leq r < \infty, \;\; 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r \cos\varphi$

$y = r\, \mathrm{sen}\varphi$

$z = 2(Rs(r- Rs))^{1/2}$

and by vector equation:

$f(x,y,z) = (r \cos\varphi,\; r\, \mathrm{sen}\varphi,\; 2(Rs(r- Rs))^{1/2})$

Determination of velocity, acceleration, and normal vectors to the surface

$\eth f/\eth\varphi = (-r\, \mathrm{sen}\varphi, r\cos\varphi, 0)$ $\qquad \eth^2 f/\eth\varphi^2 = (-r \cos\varphi,\; -r\, \mathrm{sen}\varphi,\; 0)$

$\eth f/\eth r = (\cos\varphi, \mathrm{sen}\varphi, (r/Rs\; -1)^{-1/2})$ $\qquad \eth^2 f/\eth r^2 = (0,\; 0,\; (-1/(2R_s)).\,(r/Rs\; -1)^{-3/2})$

$\eth f/\eth\varphi\eth r = (-\mathrm{sen}\varphi,\; \cos\varphi,\; 0)$

$n = (\eth f/\eth\varphi \times \eth f/\eth r) = (r\cos\varphi/(r/Rs\; -1)^{1/2},\; r\,\mathrm{sen}\varphi/(r/Rs\; -1)^{1/2},\; -r)$

$[n] = r\,((1/(r/Rs\; -1)) +1)^{1/2}$

$\mathbf{n} = n/[n]$

Curvature and curvature parameters

Gauss curvature $\quad$ **$K = LN\text{-}M^2/EG\text{-}F^2$**

$L = \eth^2 f/\eth\varphi^2.\,\mathbf{n}$ $\qquad E = \eth f/\eth\varphi.\; \eth f/\eth\varphi$

$N = \eth^2 f/\eth r^2.\,\mathbf{n}$ $\qquad G = \eth f/\eth r.\; \eth f/\eth r$

$M = (\eth f/\eth\varphi\eth r).\,\mathbf{n}$ $\qquad F = \eth f/\eth\varphi.\; \eth f/\eth r$

Completing a previous work of ours, [3], we have particularized the equations in 20 points between 1 and 1400 Schwarzschild radii, Rs, calculating the corresponding curvatures as shown in the results table 1.

Thus, although in the metric there is a singularity at the point 1Rs, the value of the Gaussian curvature for the singularity is resolved mathematically calculating a limit. We have calculated that limit for Gauss curvature.

3.- Results of curvature values

Table 1. Gaussian curvature values according to the Schwarzschild model

Distance to the point mass	**Value of Gauss Curvature k**	Distance to the point mass	**Value of Gauss Curvature k**
1Rs	**-0,5000 x Rs^{-2}**	60Rs	**-2,325.10^{-6} x Rs^{-2}**
1,2Rs	**-0,2873 x Rs^{-2}**	80Rs	**-9,596.10^{-7} x Rs^{-2}**
1,4Rs	**-0,1821 x Rs^{-2}**	100Rs	**-4,925.10^{-7} x Rs^{-2}**
1,6Rs	**-0,1220 x Rs^{-2}**	200Rs	**-5,963.10^{-8} x Rs^{-2}**
1,8 Rs	**-0,0790 x Rs^{-2}**	400Rs	**-4,800.10^{-9} x Rs^{-2}**
2Rs	**-0,0625 x Rs^{-2}**	600Rs	**-2,376.10^{-9} x Rs^{-2}**
3Rs	**-0,0186 x Rs^{-2}**	800Rs	**-9,710.10^{-10} x Rs^{-2}**
4Rs	**-0,0078 x Rs^{-2}**	1000Rs	**-5,059.10^{-10} x Rs^{-2}**
5Rs	**-0,0030 x Rs^{-2}**	1200Rs	**-2,883.10^{-10}x Rs^{-2}**
6Rs	**-0,0023 x Rs^{-2}**	1400Rs	**-1,810.10^{-10} x Rs^{-2}**

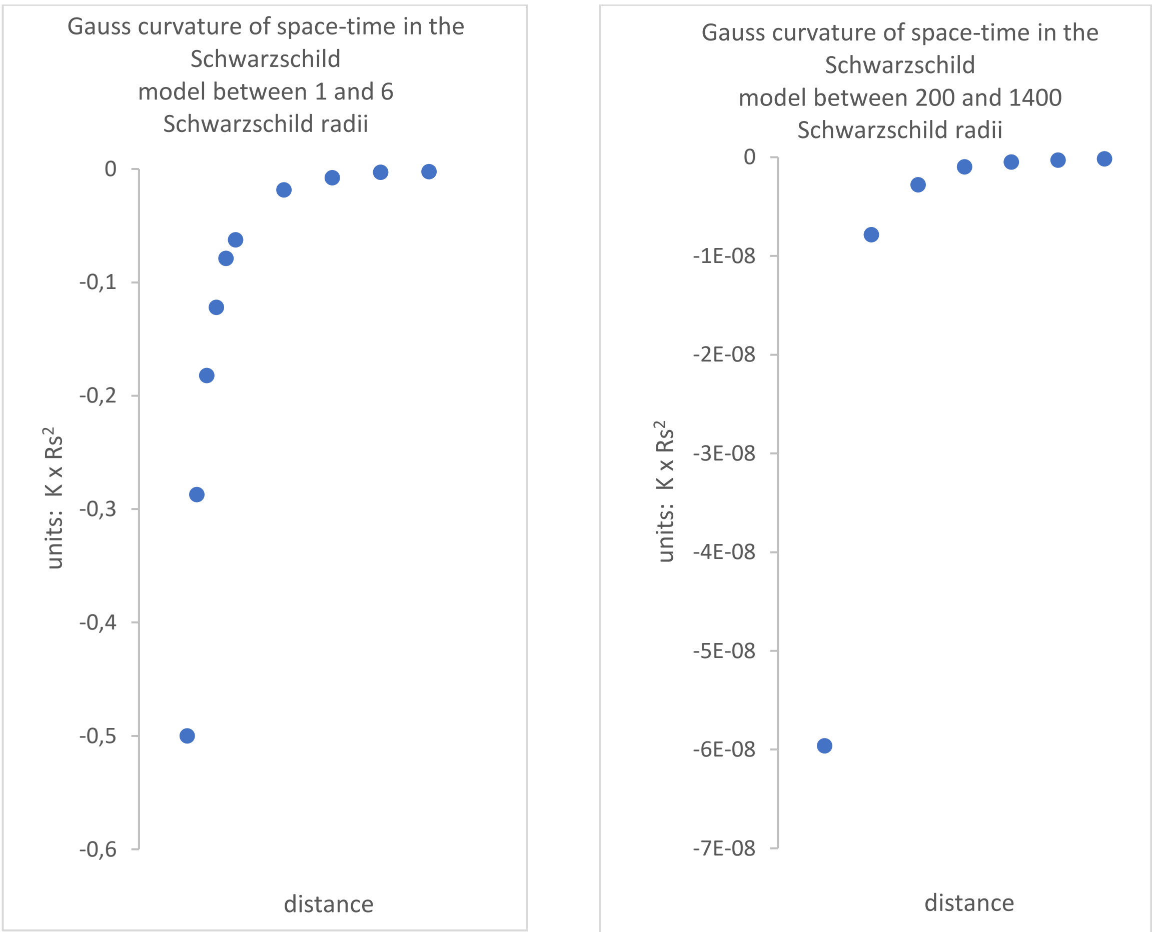

Fig 2 Gauss curvature of space-time in the Schwarzschild model

4.- An equation to calculate the curvature of space-time according to the Schwarzschild model

An adjustment equation has been obtained using an Excel program by regression methods throughout this wide range of distances. The degree of quality of the fit obtained by calculating the parameter R^2 is very high, 0.9999. For this reason, it is to be expected that this equation allows interpolate the calculation of Gaussian curvature values, in this wide range of distances, with high accuracy without the need to carry out the laborious calculations that would otherwise have to be done.

Fit equation between 1 and 1400 Schwarzschild radii

Gaussian curvature: $k = -0{,}5268\ (r/Rs)^{-3{,}054} \times Rs^{-2}$

Fit quality $R^2 = 0{,}9999$

Rounding decimals and according to definition of Schwarzschild radium, Rs

$Rs = 2GM/c^2$

where G is the universal gravitation constant, and M is the mass of the black hole, we can express the adjustment equation we have found as the following approximate equation:

$$\mathbf{k = -GM/c^2r^3}$$

where k is the Gaussian curvature of space-time according to the Schwarzschild model.

5.- Gaussian curvature of spacetime and gravitational redshift

We are going to relate our gauss curvature k with the redshift due to the relativistic effect when photons pass through a gravitational field [4]. The redshift "z" is expressed, according to the Schwarzschild metric:

$$1 + z = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{1}{\sqrt{1-\frac{Rs}{r}}}$$

where G is the universal gravitational constant.

M is the mass that causes the gravitational field, which is assumed to be a point mass.

r is the distance from the mass to the point to calculate the curvature.

Rs is the Schwarzschild radius, $Rs = 2GM/c^2$.

According to the formula found by us for the calculation of Gaussian curvature according to the Schwarzschild spacetime model:

$$k = \frac{-GM}{r^3c^2} = \frac{-Rs}{2r^3}$$

result:

$$1+ z = \frac{1}{\sqrt{1+2r^2k}}$$

$$(1+z)^2 = 1/(1+2r^2k)$$

$$(1+z)^2\ (1+2r^2k) = 1$$

$$\mathbf{k = \frac{1}{2r^2}\left(\frac{1}{(1+z)^2} - 1\right)}$$

Equation found that relates the Gaussian curvature, k, of space-time to the gravitational redshift.

We are going to check the accuracy of the equation by comparing the curvature values obtained from it with the values obtained by us exactly using the Schwarzschild equations, Table 1. The results are in Table 2

Table 2. Gauss curvature values and related corresponding redshift values

distance r	**1+z**	**k Schwarzschild model**	$\mathbf{k = \frac{1}{2r^2}\left(\frac{1}{(1+z)^2} - 1\right)}$
1Rs	∞	**-0,5000 x Rs^{-2}**	**- 0,5000 x Rs^{-2}**
1,2Rs	2,4495	**-0,2873 x Rs^{-2}**	**-0,2893 x Rs^{-2}**
1,4Rs	1,8708	**-0,1821 x Rs^{-2}**	**-0,1822 x Rs^{-2}**
1,6Rs	1,6330	**-0,1220 x Rs^{-2}**	**-0,1221 x Rs^{-2}**
1,8Rs	1,5000	**-0,0790 x Rs^{-2}**	**-0,0857 x Rs^{-2}**
2Rs	1,4142	**-0,0625 x Rs^{-2}**	**-0,0625 x Rs^{-2}**
3Rs	1,2247	**-0,0186 x Rs^{-2}**	**-0,0185 x Rs^{-2}**
4Rs	1,1547	**-0,0078 x Rs^{-2}**	**-0,0078 x Rs^{-2}**
5Rs	1,1180	**-0,0030 x Rs^{-2}**	**-0,0040 x Rs^{-2}**
6Rs	1,0954	**-0,0023 x Rs^{-2}**	**-0,0023 x Rs^{-2}**
60Rs	1,0084	**-2,325.10^{-6} x Rs^{-2}**	**-2,304.10^{-6} x Rs^{-2}**
80Rs	1,0063	**-9,596.10^{-7} x Rs^{-2}**	**-9,751.10^{-7} x Rs^{-2}**
100Rs	1,0050	**-4,925.10^{-7} x Rs^{-2}**	**-4,963.10^{-7} x Rs^{-2}**
200Rs	1,0025	**-5,963.10^{-8} x Rs^{-2}**	**-6,227.10^{-8} x Rs^{-2}**
400Rs	1,0012	**-4,800.10^{-9} x Rs^{-2}**	**-7,486.10^{-9} x Rs^{-2}**
600Rs	1,0008	**-2,376.10^{-9} x Rs^{-2}**	**-2,219.10^{-9} x Rs^{-2}**
800Rs	1,0006	**-9,710.10^{-10} x Rs^{-2}**	**-9,377.10^{-10} x Rs^{-2}**
1000Rs	1,0005	**-5,059.10^{-10} x Rs^{-2}**	**-5,001.10^{-10} x Rs^{-2}**
1200Rs	1,0004	**-2,883.10^{-10}x Rs^{-2}**	**-2,776.10^{-10} x Rs^{-2}**
1400Rs	1,0003	**-1,810.10^{-10} x Rs^{-2}**	**-1,530.10^{-10} x Rs^{-2}**

We see, comparing both columns and for the calculated distances, that the precision of the results is notable, which seems to indicate that our equation is valid relating Gaussian curvatures of space-time according to the Schwarzschild model and the gravitational redshift and that, therefore, it can be used over a wider range of distances.

Figure 3 compares point by point between 1 and 6 Schwarzschild radii the redshift with the gauss curvature, we can observe a great correlation between both.

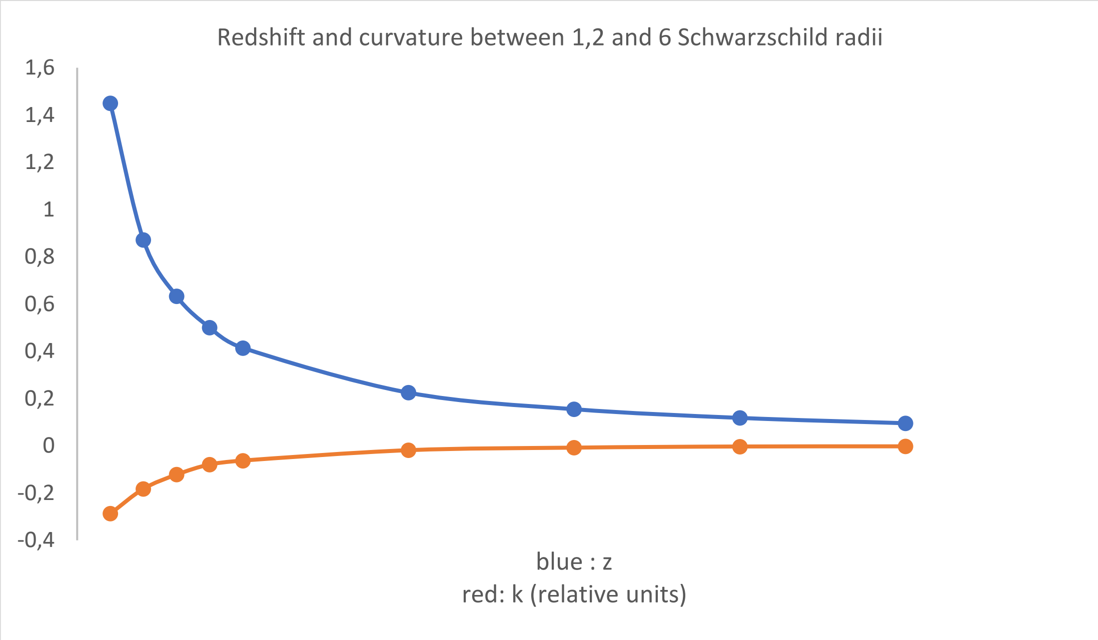

Fig. 3 Redshift and curvature between 1,2 and 6 Schwarzschild radii

6.- Application to SAGITTARIUS A*

In a previous study of ours [5] we had calculated the values of the Gaussian curvature of space-time from the Schwarzschild model on the assumption of the supermassive black hole of our galaxy SAGITTARIUS A*. We assume that Sagittarius A* behaves like a symmetric and static black hole, that is, the Schwarzschild model is assumable for the calculation of curvatures.

Sagittarius A*, [6], [7], [8], is the supermassive black hole at the galactic center of the Milky Way. Like the nuclei of most spiral and elliptical galaxies, the Milky Way contains a black hole at its center.

The mass of SAGITTARIUS A* is estimated at 3,6 million suns if the mass of the sun [9] is $1.989.10^{30}$ Kg, the mass of Sagittarius A* is estimated at $7.16.10^{36}$ Kg, its Schwarzschild radius, Rs, thus turns out to be $10,61.10^{9}$ meters.

With the same assumptions we elaborate Table 3 where this case is contemplated. We relate the gravitational redshift values to the curvature values corresponding to the SAGITTARIUS A* black hole, also giving the distance in meters.

Table 3. Redshift values and gauss curvature values for SAGITTARIUS A*

Distance in Schwarzschild radii	Distance in meters	1+z	Gauss curvature, (m^{-2})
1Rs	10.10^{9}	∞	-4, 44160E-21
1,2Rs	12.10^{9}	2,4495	-2,89352E-21
1,4Rs	14.10^{9}	1,8708	-1,82214E-21
1,6Rs	16.10^{9}	1,6330	-1,22071E-21
1,8 Rs	18.10^{9}	1,5000	-8,57339E-22
2Rs	20.10^{9}	1,4142	-6,24988E-22
3Rs	30.10^{9}	1,2247	-1,85158E-22
4Rs	40.10^{9}	1,1547	-7,81248E-23

5Rs	50.10^9	1,1180	-3,99903E-23
6Rs	60.10^9	1,0954	-2,31386E-23
60Rs	600.10^9	1,0084	-2,30426E-26
80Rs	800.10^9	1,0063	-9,7515E-27
100Rs	1000.10^9	1,0050	-4,96275E-27
200Rs	2000.10^9	1,0025	-6,22664E-28
400Rs	4000.10^9	1,0012	-7,48652E-29
600Rs	6000.10^9	1,0008	-2,21956E-29
800Rs	8000.10^9	1,0006	-9,36657E-30
1000Rs	10000.10^9	1,0005	-4,99625E-30
1200Rs	12000.10^9	1,0004	-2,77611E-30
1400Rs	14000.10^9	1,0003	-1,52992E-30

Conclusions

First, we have calculated some values of the Gaussian curvature of space-time according to the Schwarzschild model. As the calculations are laborious, from the results obtained we have obtained an equation that allows us to obtain them in a simpler way. We have studied this equation and we have seen that it reproduces the curvature values quite accurately and allows them to be interpolated and extrapolated to a wider range of distances.

We have then studied the relationship between the gravitational redshift and the Gaussian curvature of space-time in the Schwarzschild model. Thus, we have found an equation that relates both magnitudes. We have verified the validity of this equation by comparing its results with the exact curvature values obtained by us in this report studying the space-time equation of the model. of Schwarzschild in a range of distances between 1 Schwarzschild radius and 1400 Schwarzschild radii. This comparison demonstrates the validity of the equation even over a wider range of distances. Thus, the equation relates the Gaussian curvature of space-time to the gravitational redshift in the vicinity of Schwarzschild black holes.

Finally, we have applied the equation found to a real case of our galaxy, its supermassive black hole SAGITTARIUS A*, we have related the curvature values in its vicinity with the corresponding gravitational redshift. Obviously, the results are valid if in a first approximation it is correct to consider, at the level of curvature calculation, SAGITTARIUS A* a static and symmetric black hole, that is, one that can be treated according to the Schwarzschild model.

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