Real Numbers: a new (quantum) look

... with a hierarchical structure

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Abstract

• The motivation for re-designing R, as a Number System with "quantum structure", is briefly provided, including hints to applications.

• Rational numbers Q have much more structure beyond the ordered field structure which leads to Real Numbers as a metric completion, essential in Analysis and Classic Geometry (continuum / Limits / Calculus etc.):

A) Farey sequence filtration;

B) One point compactification, part of the *integral projective line* P^1Z:, i.e. the *rational unit circle*: $Q \to S^1_Q$ (including Pythagorean triples!).

• A *topological completion* of Q provides a new construction of the real numbers, compatible with the above structure, adequate for applications in Number Theory and Quantum Physics.
It is based on *continued fractions representation of real numbers*, and their "universal" binary representations using *the modular group*.

• Background on continued fractions, modular group.

Genesis of this project ...

Quantization of Classical Physics is difficult because of the "Continuum" (Real Numbers) ... Idea: Why not re-designing Physics as a "Discrete Theory" ...
2005: Quantum Computing, (Quantum) Digital World Theory" etc.; left to do: "Quantize The Qubit"!
2020: Solution: Finite groups for Standard Model etc. proton and neutron states (baryons) as *modular curves with finite structure* (Platonic, Archimedian solids etc.) [6] (... almost there!).

... but how to **Dispense of Real Numbers!?** see [1].

2023: (Stumbled upon) Real Numbers → Continued Fractions → *Integral Mobius transformations*! (Modular group, again!), so *DISCRETE Lorentz Transformations*: FINITE SPACE-TIME and STATES!!

Solution : *Redesign REAL NUMBERS*!

What is "wrong" with the Reals Numbers?

Top 10 reasons (for now) are:
1) They do not describe real quantities (Quantum Physics required quantization etc.);
2) Resulted from an "over-reaction" to extending Number Systems:

$$N \to Z \to Q \to R \to C \to H \to O...$$

the only analytic step in the sequence; in parallel algebraic numbers were developed, AG-periods etc. and R phased out gradually;
3) In Sciences we need 2D, to include periodicity (electric circuits etc.) so we need C; but do we really need $C = R[i]$!?
4) Physics is conformal; Q are ratios, "conformal" ($Q \to P^1 Z \to P^1 Q$, but then R is metric. We need "new reals" that are *fractional transformations*, to preserve the "field theory" tradition when extending algebraic numbers!
5) Nature is conformal and discrete: we always have *conjugate variables* (2D-conformal transformations. including rotations) and a *quantum unit* (compactification) [2].

(cont.)

6) Numbers are "shadows" of Math-Objects: almost all transcendental numbers don't have such associated shadows (they are ghosts of Cauchy completion: no Cauchy sequences to justify we need them!).
7) We need to move on from "ratios and fields" (e.g. Q etc.) to homogeneous structures (e.g. theory of ideas) and equations, as "true relations" (and look at their symmetries); i.e. a geometric picture: projective spaces / Hopf algebras (parallel and serial addition; renormalization via R-H Problem and Birkoff decomposition etc.) [2].
8) R separates Math into Real Analysis and friends etc., and Number Theory and "friends" (Alg. NT, AG). The theory of p-adics is a theory in $char = p$ only at "tangent level" (F_p); otherwise it is a mixture of AG and Analysis ("char 0").
9) Complete Q "to the left" and get p-adic numbers; "to the right" yields R; there should be a way to join them: by relating the prime at infinity and finite primes.

...3.14...

10) We don't really use R anyways! (excepting School, of course!).

The Plan

Another "clue" is that we encounter the *modular group* everywhere! which is the symmetry group of the rationals Q ($SL_2(Z)$ preserves latices, are "symplectic transformations" and conformal in 2D etc.) ... So, we should probably *find a way to "extend" the field Q with its symmetries $Aut(Q)$*, to the "groupoid" of fractions and relations between them P^1Q: Farey graph / map; this opens a Geometrization Program ...
So, since we can't just get rid of real numbers, design a "bridge" between R and a "new" representation of the reals: CF and canonical Modular Group representation.

$$SL_2(Z) =< S, T > \stackrel{CF}{\to} R, \quad and\ complete\ it : R_{CF} = \overline{SL_2(Z)}.$$

$$N \to Z \to Q \to \overline{SL_2(Z)}_{AG} \to R_{CF}[i] \to H \to O...$$

Where Alg.-Geom. really means "as needed by the Theory"[1] ... and don't forget the extra structure of a groupoid, to use for modular curves with tessellation, SM etc.

[1]... and push i inside!

Modality and a few Benefits

The idea is to use CF representation to represent real numbers as ST-sequences of the modular group, including duality in Haar Wavelet Analysis.

It allows to investigate periods in the context of the modular group, in a geometric setting (projective space etc.). In fact think that it is a re-thinking of what C is ...
The geometric setting involves ant-podal map, "bifield structure" (Hopf algebra with duality) etc. A relation with modular forms is expected, and from the algebraic side, with $L - functions$ and galois groups.
Galois groups (π_1, algebraic fundamental groups, abstract GG etc.) need "upgraded" to Hopf algebras of symmetries with duality (Hopf objects in a category). Then a Pontryagin duality may even turn into a Laglands duality etc.

... and A little History of Real Analysis

• *Fourier Analysis* 1800s (used also by Babylonians!?) [2] is the study of functions (e.g. real), in terms of periodic functions: *Signal analysis* etc. is based on an *additional structure of the Reals*: *Translations* (i.e. $Z \to R$) and *dilations modulo Z* $D_n(x) = nx \mod Z$.

• The next step is *Wavelet Analysis*, where a hierarchy of details is introduced via *translations* $T(x) = x - 1$ and *scaling* by a powers of 2, $S(x) = 2^k x$: "zooming in or out" on the details of a function (signal); this is heavily used in transmission of pictures on the Web (see how a large image is loaded by your browser - on a slow connection!).

• ... but 2 is a choice (like 10 in decimal representations)! *"Modular Group Analysis"* of functions (new area) is "Universal", *just adding inversion* $S(z) = 1/z$. And, it is Quantum Physics and Number Theory "friendly" (allows to understand the atomic world etc.).

[2] Wiki: "An early form of harmonic series dates back to ancient Babylonian mathematics, where they were used to compute ephemerides (tables of astronomical positions)!!

Main idea and implications

- *Fourier Analysis*: $T(x) = x + 1$ inv., $D_a(x) = ax$ affine group;
- *Wavelet Analysis* introduces "zooming in/out" *grading* by powers of 2.

Limitations: fixes the *prime at infinity*,
separating Real Analysis from Number Theory. It is not NT friendly ($D_n(x) = nx \ mod \ Z$ "screambles" irrationals).

- *Modular Group Analysis* "adds" inversion $S(x) = 1/x$; advantages:
  1) Maps $\infty \to p$ *prime*: *unifies Real Analysis and Number Theory*;
  2) Adds a *scale structure* to the Reals (Farey & CF filtration etc.);
  3) From (1) we expect to "bring Riemann Hypothesis Home" (to NT; not just a *bridge to Weil's RH* / Deligne Th. - see [11]).
  4) Provides a *natural Algebraic-Geometric framework for Quantization* [6] and beyond (finite structures for *Standard Model of Elem. Part. Phys.*).

Goals and Designing Plans

• Introduce a new construction of the Reals, generalizing the idea of Haar wavelets, and compatible with the modular group action on fractions;

• Study the "meaningfull real numbers": $Q \subset Alg.Numbers \subset Periods$.

• Present background material on *continued fractions* and *modular group* (2D-congruence Arithmetic);

• Benefits: bring Analysis "home", to Algebraic Number Theory and Geometry.

The Rational Numbers with The Farey Filtration

Farey Fractions

• Fractions in the interval $[0,1]$ can be grouped by the size of their denominators (in reduced form):
$F_1 = \{0/1, 1/1\}$,
$F_2 = \{0/2, 1/2, 1/1\} = F_1 \cup \{1/2\}$,
$F_3 = \{0/3, 1/3, 1/2, 2/3, 1/1\}$, $F_3 = F_2 \cup \{1/3, 2/3\}$ etc.

Definition

The Farey sequences of fractions F_n are defined inductivly:
a) $F_1 = \{0/1, 1/1\}$;
b) F_n is defined by adjoining to F_{n-1} the "new" (irreducible) fractions with denominator n, which are of the form k/n with $gcd(k,n) = 1$ (irreducible fractions of denominator n).

Ex. $F_6 = F_5 \cup \{1/6, 5/6\}$ (see [2] for more details.

• Abstract Algebra interpretation (briefly): a fraction $3/5$ can be mapped to $3 \in Z/5Z$; then $F_n \cong F_{n-1} \cup (Z/n, \cdot)*$, i.e. a disjoint union of the units $U(Z/n)$ of the rings $(Z/n, +, x)$ ...

Filtrations vs. Grading ...

This provides a *filtration of the rationals*:

$$Q = \cup_{n \in N} F_n, \quad F_1 \subset F_2 \cdots F_{n-1} \subset F_n \cdots$$

• A *grading structure* on a vector space/ ring / field etc. is a much richer, but also rigid structure. Ex.:
A) Vector spaces: $V = V_1 \oplus V_2 \oplus V_2 ...;$
B) Polynomials: $R[x] = R \oplus R \oplus R ...,\quad P(x) = c_0 + c_1 x + c_2 x^2 ...$

• A *grading structure* on a ring (e.g. polynomials) is equivalent to a derivation rule, i.e. DERIVATIVE, via the *Power Rule*!!

$$(x^3)' = 3x^2,$$

comes from the grading, no limits or Calculus needed!!

Topology from Filtration

A *topological structure* (what is "near", limits etc.) can be defined using *open sets*, *topology* for short, or *sequences*: *Sequencial Space / Sequencial Topology* [2].

$$\textit{Main idea}: \quad \textit{Convergent Sequences} \leftrightarrow \textit{Topology}.$$

Hence, instead of using a metric to define *Cauchy sequences* of rational numbers, we define the class of *sequences cofinal with the Farey filtration*, as "convergent" by definition.

The real number $x \in R$ can be represented as continued fractions $CF(x)$, which in turn, can be represented as a sequence W of the standard generators $U(z) = 1/z$ and $T(z) = z + 1$ of the modular group $SL_2(Z)$.
1) In this way the filtration structure of Q can be transferred to the Reals, and a natural *depth of approximation* defined, instead of using a metric.
2) The relation with *p-adic numbers* will be studied elsewhere.

(Details / proofs, will appear in the joint article with Anurag Kurumbail)

The Modular Group (MG)

A group everybody should know ... Why?

• Congruence arithmetic is about Z and Z/n: 1D ...

• 2D-Congruence arithmetic is a "complexification" of the above: $SL_2(Z)$ conformal transformations on the *rational circle* $S^1_Q = Q/Z$, including *Pythagorean triples* ... (relating nice elementary topics).

Definition

The *modular group* $MG = PSL_2(Z)$ is the group of 2D-matrices with integer coefficients $SL_2(Z)$, modulo ± 1.

To a modular transformation, we associate a (complex) *fractional transformation* (*integral Mobius transformation*):

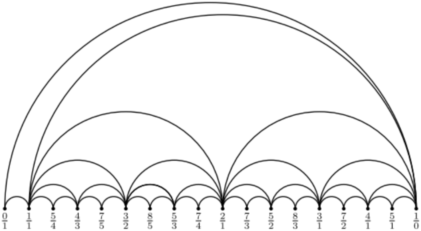
$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in Z \mapsto T(z) = \frac{az+b}{cz+d}, \ z \in C.$$

Generators of MG & Farey Fractions

Ex. *Unit translation* $T(z) = z + 1$ (addition of 1 as a generator of $(Z, +)$), and *geometric inversion* $S(z) = -1/z$ are generators of the modular group. On fractions: $T(m/n) = (m+n)/n, S(m/n) = -n/m$.

There are many interesting properties of MG and its action on Farey fractions, but not enough time now ...

$$T(2/1) = \frac{2}{1} + 1 = 3/2$$



Each arc denotes an action of a MT

Continued Fractions (CF)

Briefly, to get to our goal: $\bar{Q} = R$ ...

Real numbers have a *continued fraction* representation:

a) CF of a rational function is finite:

$$7/5 = 1 + 1/CF(5/2) = 1 + 1/[2 + 1/2] = [1; 2, 2];$$

b) CF of a quadratic number is periodic, e.g.

$$\sqrt{2} = [1; 2, 2, 2...] = [1; (2)];$$

c) CF of algebraic numbers?;

d) ... of (Algebraic-Geometric) Periods? e.g.:

$$\int_0^1 \frac{4}{x^2+1} = \pi, \quad \zeta(4) = \pi^4/90, \ where \ \zeta(k) = \sum_n 1/n^k.$$

Modular Group Representation of Reals

• A CF of a real number r defines a unique sequence $W = (k_1, k_2, k_3 \cdots)$ of $PSL_2(Z)$ elements, such that $r = \ldots T^{k_3} S T^{k_2} S T^{k_1}(0)$.; e.g.:

$$W = (3, 2, 2) : r = T^3 \circ S \circ T^2 \circ S \circ T^2(0).$$

Computing: $T^2(0) = 2$, $S(2) = -1/2$, $T^2(1/2) = 1/2 + 2$ etc.

• Conversely, each $ST - sequence$ W defines a *Dedekind cut*, hence a real number [3].

Theorem

The correspondence $W \to r(W)$ is a bijection and compatible with the usual topology of R.

… and Euclid's Algorithm

[Connections with elementary Math ...]

This is just encoding *Euclid's algorithm*, when comparing two integers m and n, to find $gcd(m,n)$ [3]:

$$m = q_1 \ n + r_1, \ n = q_2 \ r_1 + r_2 \quad etc. \quad Ex. \frac{7}{5} = 1 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2}.$$

Denote $E(m,n) = (n,r) \Leftrightarrow m = qn + r$. Here the fraction m/n is represented as it should, as a pair (m,n)[3]. Equivalently:

$$\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$W = TST^2ST^2S \leftrightarrow W = (1,2,2) \leftrightarrow \frac{7}{5} = [1;2,2] \ CF.$$

[3]The *monoid-to-group* construction: $Q = Z \times Z^\times / \sim$.

In other words ...

Rational numbers, as equivalence classes of pairs (irreducible representatives), have an MG-representation, with *two digits* S, T, or even resembling the decimal rep.:

$$W = TSTTSTT = 1.2.2, \quad \text{“}multiple\ \text{“}dot'' - rep.$$

A *dot* signifies the inversion S; compare with, e.g. 3.14, from integer to fractional.

This provides a *resolution depth*, similar to a *p-adic valuation*, except it is *universal* (geometric), base independent:

$$\nu(W) = \#\ of\ \ S's, i.e.\ \text{“}dots''.$$

• The well-known theory of CF ensures convergence: even products and odd products define a Dedekind cut, when the *MG Word* (MG-sequence) is infinite.

• The canonical family of MG-Sequences define a *sequencial topology* implying *Cauchy convergence* and usual topology, EXCEPT it has more

Work to be done ...

• Define in analogy with convergence of series:

$$s = \sum_{1}^{\infty} a_n : \quad s_n = \sum_{1}^{n} a_j, \; s = \lim_{n \to \infty} s_n,$$

but for *partial products* of the MG-word W, and *check* the Axioms of a Sequential Topology:

$$q_n = \prod_{j=1..n} W_j, \quad \lambda(W) := Lim\ q_n\ (Dedekind\ cut).$$

• The extension and intrinsic interpretation in terms of *complex integral Mobius transformations* $PSL_2(Z[i])$ on the Riemann sphere $S^2 = CP^1$ is left for later developments ...

$$Rethinking: \; Complex\ Numbers\ C = R[i]\ via\ MG.$$

... and p-adic Numbers (Sketch / Skip for now ...)

The p-adic numbers Q_p are the "other completions of Q, conform *Ostrovski Theorem*. They are *filtered fields* .

They also have CF, hence expressible in terms of the Modular Group. The completion of Q relative to the absolute value $|x - y|$ (usual Real Numbers) is also referred to as the norm corresponding to the *prime at infinity*.

One expects to have a unification by considering the Riemann sphere and inversion; similar to "point at infinity" for meromorphic functions ... Indeed, the projective space CP^1 (or RP^1, the circle) can be view as defining two charts (S^2 *North Pole* $= \infty$ and S^2 *South Pole* $= 0$), and two isomorphic fields (about 0 ind ∞), isomorphic under inversion $S(z) = 1/z$.

This allows to add additional structure to the *adeles* ...

Why bother rethinking R? (in brief)

• The *measurement process*, e.g. in Quantum Physics, shows the inadequacy of choosing an arbitrary unit of measurement[4]
• Physics quantities have a *Natural Unit*, e.g. Planck's constant h, electric charge e etc., as if a *greatest common divisor* ...
• *Physics Laws* have evolved into *Algebraic-Geometric Period Laws* [5]:

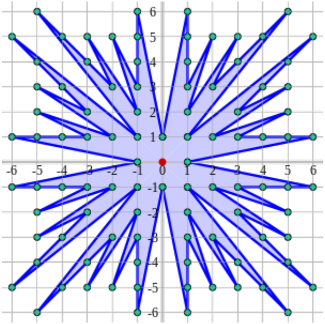
$$Foundations: \ Cohomological\ Physics\ “+'' \ Number\ Theory.$$

What physicists measure in experiments are *AG-Periods*, disguised as "general real numbers", via the *quantity / unit traditional approach*! (e.g. Feynman scattering amplitudes, charges, actions, angular momentum etc.). The role of periods in Physics is supported by the presence of *fundamental dimensional constants* (see *Buckingham's Pi-Theorem* [4]), e.g. $\alpha, R\#$ etc. These are naturally expressible via the Modular Group approach to "numbers", as *encoding a comparison process*.

[4]Incorporating Pythagorean's philosophy: "Number Rules the Univers", atomism, Zeno etc.

Farey Filtration and Projective Line

Farey sequence is a filtration of rationals in the interval $[0, 1]$. When a farey fraction $r = (p, q) \in Q$ (irreducible) is plotted as a pair $(p, q) \in Z \times Z$, it encodes the *rays of the projective integral line* $Q \to P^1 Z$

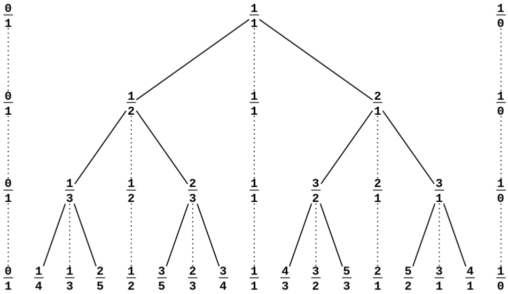


• There are interesting connections with Pythagorean triples and rational circle, Pell's equation.

• It relates with density of visible points and Riemann zeta function ...

Better: 2/3 trees (other filtrations)

- Stern-Brocot Tree is a filtration for *all rationals*:

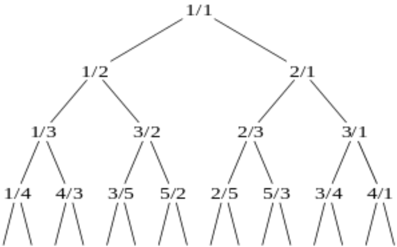


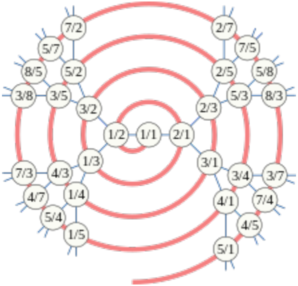
$$\textit{Property}: \quad \sum_k \frac{1}{p_k q_k} = 1, \quad \textit{Level } N \textit{ fractions}.$$

[Relation with serial/parallel addition?]

Calkin-Wilf Tree

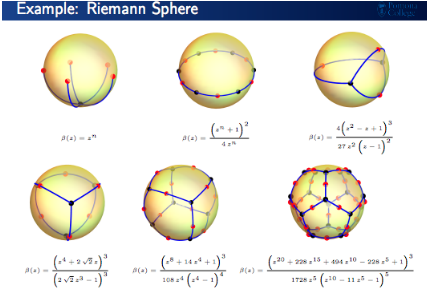
- Calkin-Wilf tree (see Wiki):





Modular Group, Platonic solids, Dessins d'Enfant ...

Using $SL_2(Z)$ and its subgroups as additional structure (modular curves), provides models in Elementary Particle Physics [6].



Problem. When considering a *congruence subgroup* $\Gamma \to SL_2(Z)$ and its geometry in C, defines "special real numbers" *CF mod* Γ. Are these *periods* of the associated Belyi map? Is there a connection with *modular forms*?

Applications to Algebraic-Geometry and the SM

The use of Reals obtained via the modular group has ties to important Algebraic-Geometric objects with applications to *Number Theory* and *Elementary Particle Physics*.

• For example, the *Farey map* (the above triangulation $\mathcal{M}_3$ of C_+) has reductions *mod n* (as pairs and group action) which include *Platonic Solids* for $n = 3, 4, 5$ (see [12]):

Appendix A. Pictures of $\mathcal{M}_3(n)$

For $n = 3, \ldots, 8$.

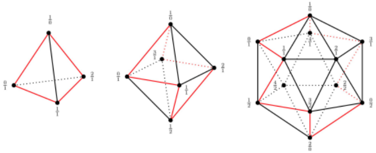


Figure 2. Tetrahedron, Octahedron and Icosahedron respectively.

For the relation with quark flavors: u, d, s, c, t, b see [6, 7].

From Numbers to Mathematical Structures and Objects

• Numbers are "shadows" of Sets (Cantor's cardinal numbers), Abelian Groups and other *algebraic-geometric structures*, which are used as Physics models ...

Various classes of numbers are in fact grouped as belonging to Algebraic or Geometric Theories!
1) Rational numbers;
2) Algebraic numbers;
3) Periods etc.

“Arithmetic’s 5 Operations”

So, what the German mathematician Martin Eichler supposedly said [10]:
“There are five fundamental operations in mathematics: Addition, subtraction, multiplication, division and modular forms.”
(jokingly) is quite for “real” ...

Bridging the “gap”: Some fun topics in elementary Arithmetic and Number Theory:

• Congruence arithmetic and divisibility tests; e.g. 700s AD Arabic *Casting out Nines Error Test*;
• Euclid’s gcd algorithm and Continued Fractions representations of numbers;
• Rational numbers, Farey fractions and graph (mix of arithmetic, graphs, linear transformations).

Conclusions

A different approach to Real Numbers allows to extend Fourier Analysis and Wavelet Theory, with implications in Math and Physics.
This allows to "bring char 0 to Number Theory": a new theory of Adeles.
Helps understand the practical Number Systems:

$$N \to Z \to Q \to Algebraic \to Geometric\ (Periods)\ \ldots$$

[Note: e and π are *really special* ... (To be continued)]
The general use of the new structure, MG-sequences representing real numbers is:

$$To\ understand\ a\ Real\ Number \quad \to \quad Study\ it's\ CF\ in\ SL_2(Z)!$$

i.e. don't forget R, just "translate into modern language".

Further developments

Some suggestions are included:

• Study classes of "real numbers" with periodic ST-representations modulo a congruence subgroup $\Gamma \to SL_2(Z)$. e.g. at level p: $\Gamma = SL_2(F_p)$; relations with the *modular curves*? its Hodge-de Rham periods?

• The MG representation of the Reals provides a *fractional representation*, extending the one for rational numbers; consequences? How is the extended real line related to adeles, via this MG/ ST-representation?

• *Modular forms* of level n are *translation invariant* $f(T(z)) = f(z)$ (Fourier periodic) and intertwines the antipodal inversion with a shift in the grading of its series (integration vs. $1/n!\ d^n/dz^n$):

$$f(S(z)) = z^n f(z), \quad f(W(z)) = z^{n\ \nu(W(z))} f(z),$$

where $\nu(W(z)) = \#\ of\ S's$ (inversions), is a generalization of a valuation. Consequences?

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