A Truly Easy Proof: Pi is Irrational

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Abstract

Using the derivative of an integer polynomial composed with Euler's formula we prove that π is irrational.

Proof

Proofs of the irrationality of π are numerous [1], but none are as easy and direct as the following.

Theorem 1. *π is irrational.*

Proof. A simple case generalizes. Suppose $f_3(x) = x^3$ and consider the sum of its derivatives:

$$F_3(x) = x^3 + 3x^2 + 3!x + 3!.$$

It follows that $F_3(0) = 3!$. Now consider

$$\begin{aligned} F_3(0)e^x &= 3!\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \sum_{k=4}^{\infty} \frac{x^k}{k!}\right) \\ &= F_3(x) + 3! \sum_{k=4}^{\infty} \frac{x^k}{k!} \\ &= F_3(x) + 3!(e^x - s_3(x)), \end{aligned}$$

where $s_3(x)$ is a partial sum of e^x.

Adding $F(0)$ and imagining $x = \pi i$, we have

$$(e^x + 1)F_3(0) = F_3(0) + F_3(x) + 3!(e^x - s_3(x)) \tag{1}$$

$$0 = \frac{F_3(0) + F_3(x)}{3!} + (e^x - s_3(x)). \tag{2}$$

There is no reason to believe that for a general term of any polynomial this pattern would change. Nor is there any reason that all surviving non-zero coefficients of $F_n(r)$, r a root of $f_n(x)$ would not have factors of the multiplicity of the root, if the coefficients of $f_n(x)$ are integers. Thus assuming $\pi = p/q$, we can use $x^3(qx-pi)^3$, for example, and these conditions are met. So (2) gives, using Euler's formula in (1), 0 is an integer plus a fraction less than 1, a contradiction. □

References

[1] Eymard, P., Lafon, J.-P. (2004). *The Number* π. Providence, RI: American Mathematical Society.