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The Isomorphism of H_4 and E_8

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This paper gives an explicit isomorphic mapping from the 240 real $\mathbb{R}^8$ roots of the E_8 Gossett 4_{21} 8-polytope to two golden ratio scaled copies of the 120 root H_4 600-cell quaternion 4-polytope using a traceless 8×8 rotation matrix $\mathbb{U}$ with palindromic characteristic coefficients and a unitary form $e^{i\mathbb{U}}$. It also shows the inverse map from a single H_4 600-cell to E_8 using a 4D↪8D chiral left↔right mapping function, φ scaling, and $\mathbb{U}^{-1}$. This approach shows that there are actually four copies of each 600-cell living within E_8 in the form of chiral $H_{4L}\oplus\varphi H_{4L}\oplus H_{4R}\oplus\varphi H_{4R}$ roots. In addition, it demonstrates a quaternion Weyl orbit construction of H_4-based 4-polytopes that provides an explicit mapping between E_8 and four copies of the tri-rectified Coxeter-Dynkin diagram of H_4, namely the 120-cell of order 600. Taking advantage of this property promises to open the door to as yet unexplored E_8-based Grand Unified Theories or GUTs.

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I. INTRODUCTION

Fig. 1 is the Petrie projection of the Gosset 4_{21} 8-polytope derived from the Split Real Even (SRE) form of the E_8 Lie group with unimodular lattice in $\mathbb{R}^8$. It has 240 vertices and 6,720 edges of 8-dimensional (8D) length $\sqrt{2}$. E_8 is the largest of the exceptional simple Lie algebras, groups, and lattices. An important and related higher dimensional structure is the $\mathbb{R}^{24}$ ($\mathbb{C}^{12}$) Leech lattice ($\Lambda_{24} \supset E_8 \otimes E_8 \otimes E_8$), with its binary (ternary) Golay code construction.

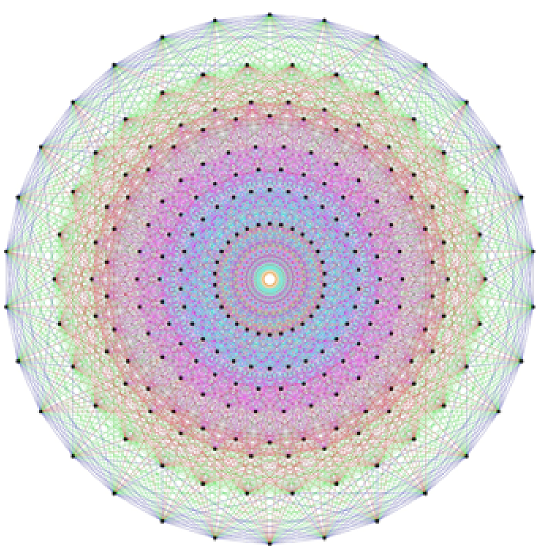

FIG. 1. E_8 4_{21} Petrie projection

It is widely known [1]-[14] that the E_8 can be projected, mapped, or "folded" (as shown in Fig. 2) to two golden ratio ($\varphi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618$) scaled copies of the 4 dimensional 120 vertex 720 edge H_4 600-cell. Folding an 8D object into a 4 dimensional one can be done by projecting each vertex using its dot product with a 4×8 matrix[11]. This produces $H_4 \oplus \varphi H_4$, where H_4 is the binary icosahedral group $2I$ of order 120, a subgroup of Spin(3). It covers H_3 as the full icosahedral group I_h of order 120, a subgroup of SO(3). The binary icosahedral group can be considered as the double cover of the alternating group alternating group A_5.

Despite others'[2][9] recent attempts, the inverse morphism or "unfolding" from H_4 to E_8 is less trivial given that the matrix is not square and lacks an inverse. Yet, a real ($\mathbb{R}$) symmetric volume preserving Det($\mathbb{U}$)=1 rotation matrix(1) was derived in 2012 and documented[11][12][13]. The quadrant structure of $\mathbb{U}$ rotates E_8 into four 4D copies of H_4 600-cells, with the original two (L)eft and (R)ight side unit scaled 4D copies related to the two L/R φ scaled copies which we now identify as H_4(L⊕R⊕1⊕φ). This traceless form of $\mathbb{U}$ has palindromic characteristic coefficients and provides for an explicit isomorphic mapping of $E_8 \leftrightarrow H_4$(L⊕R⊕1⊕φ). This involves using a bidirectional L↔R mapping function (`mapLR`) and $\mathbb{U}^{-1}$(2). The process is described and visualized in Section II.

$$\mathbb{U} = \begin{pmatrix} 1-\varphi & 0 & 0 & 0 & 0 & 0 & 0 & -\varphi^2 \\ 0 & -1 & \varphi & 0 & 0 & \varphi & 1 & 0 \\ 0 & \varphi & 0 & 1 & -1 & 0 & \varphi & 0 \\ 0 & 0 & -1 & \varphi & \varphi & 1 & 0 & 0 \\ 0 & 0 & 1 & \varphi & \varphi & -1 & 0 & 0 \\ 0 & \varphi & 0 & 1 & -1 & 0 & \varphi & 0 \\ 0 & 1 & \varphi & 0 & 0 & \varphi & -1 & 0 \\ -\varphi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\varphi \end{pmatrix} /(2\sqrt{\varphi}) \tag{1}$$

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$$\mathbb{U}^{-1}=\begin{pmatrix} \varphi-1 & 0 & 0 & 0 & 0 & 0 & 0 & -\varphi^2 \\ 0 & -\varphi & 1 & 0 & 0 & 1 & \varphi & 0 \\ 0 & 1 & 0 & \varphi & -\varphi & 0 & 1 & 0 \\ 0 & 0 & -\varphi & 1 & 1 & \varphi & 0 & 0 \\ 0 & 0 & \varphi & 1 & 1 & -\varphi & 0 & 0 \\ 0 & 1 & 0 & \varphi & -\varphi & 0 & 1 & 0 \\ 0 & \varphi & 1 & 0 & 0 & 1 & -\varphi & 0 \\ -\varphi^2 & 0 & 0 & 0 & 0 & 0 & 0 & \varphi-1 \end{pmatrix} /(2\sqrt{\varphi}) \tag{2}$$

A. Generating Polytopes

The quaternion ($\mathbb{H}$) Weyl group orbit O(Λ)=W(H_4)=I of order 120 is constructed from the parent orbit (1000) of the Coxeter-Dynkin diagram for H_4 shown in Fig. 2b. This results in the 600-cell 4-polytope of order 120 labeled here and in [3] as I. In addition, $\mathbb{U}$ provides for a direct mapping from E_8 to four L⊕R⊕1⊕φ copies of the tri-rectified parent of H_4 (i.e. the filled node 1 is shifted right 3 times giving 0001), which is the 120-cell of order 600 labeled here and in [3] as J. Both of these 4-polytopes are shown in Appendix A Figs. 14-16. The detail of the quaternion Weyl orbit construction is described in Section III.

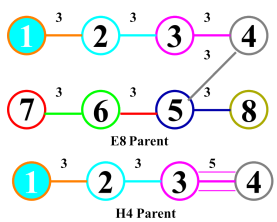


FIG. 2. a) E_8 Dynkin diagram in folding orientation
b) The associated Coxeter-Dynkin diagram of H_4

In addition to the 240 root 4_{21} E_8 8-polytope identified by its Coxeter-Dynkin diagram in Fig. 3a, there are 2^8 possible orbits using only 0's↔1's, empty↔filled, or ringed nodes of the E_8 Coxeter-Dynkin diagram, including the snub (00000000) orbit. Several other orbit permutations are commonly represented visually using the Petrie projection basis. They are the 2,160 root 2_{41} and 17,280 root 1_{42} 8-polytopes, which are constructed by generating the resulting roots by moving the filled (or ringed) node to each of the two other ends of the Dynkin diagram, as shown in Figs. 3b and 3c respectively.

B. 8D Platonic Rotation

Interestingly from [13], $\mathbb{U}$ can be generated using a combination of the unimodular matrices commonly used for Quantum Computing (QC) qubit logic, namely those of the 2 qubit CNOT (3) and SWAP (4) gates. Taking these patterns, combined with the recursive functions

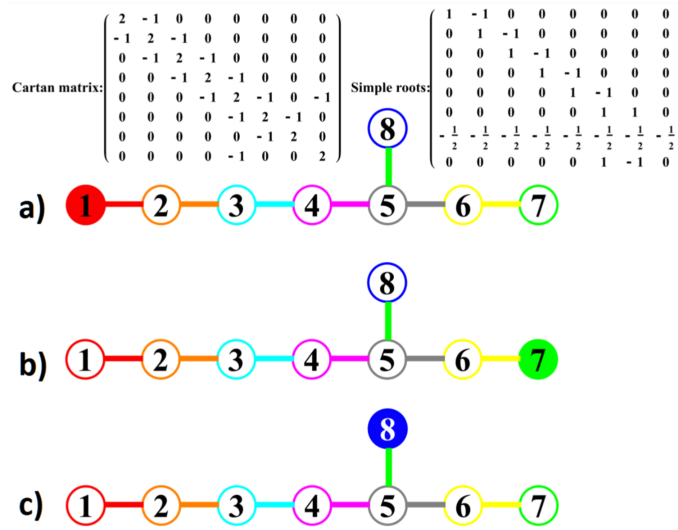


FIG. 3. E_8 Dynkin diagrams a) 4_{21}, b) 2_{41}, c) 1_{42}
Also shown are the Cartan and simple root matrices which correspond to the common Coxeter-Dynkin representation of the diagrams.

that build φ from the Fibonacci sequence, it is straightforward to derive $\mathbb{U}$ from scaled QC logic gates.[14]

$$\text{CNOT}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3}$$

$$\text{SWAP}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

C. 2D and 3D Projection

Projection of E_8 to 2D (or 3D) requires 2 (or 3) basis vectors $\{X, Y, Z\}$. For the Petrie projection shown in Fig. 1, we start with the basis vectors in (5), which are simply the two 2D Petrie projection basis vectors of the 600-cell (a.k.a. the Van Oss projection), with an optional 3rd (z) basis vector added for an interesting 3D projection[11].

$$\begin{array}{lccccc} x=\{ & 0, & \varphi 2\text{Sin}\frac{2\pi}{15}, & 2\text{Sin}\frac{2\pi}{15}, & 0, & 0,0,0,0\} \\ y=\{ & -\varphi 2\text{Sin}\frac{2\pi}{30}, & 0, & 0, & 1, & 0,0,0,0\} \\ z=\{ & 1, & 0, & 0, & \varphi 2\text{Sin}\frac{2\pi}{30}, & 0,0,0,0\} \end{array} \tag{5}$$

$\{X, Y, Z\} = \mathbb{U}.\{x, y, z\}$ as shown in (6).

$$\begin{array}{llllllll} X=\{0 & .252 & .427 & -.319 & .319 & .427 & .781 & 0\} \\ Y=\{.821 & 0 & -.393 & .636 & .636 & .393 & 0 & .348\} \\ Z=\{-.242 & 0 & -.132 & .215 & .215 & .132 & 0 & -1.03\} \end{array} \tag{6}$$

D. 3D Platonic Solid Projection

This basis is derived from the icosahedral symmetry of H_3-based Platonic solid. The twelve vertices of the icosahedron can be decomposed into three mutually-perpendicular golden rectangles (as shown in Fig. 4), whose boundaries are linked in the pattern of the Borromean rings. Rows (or columns) 2-4 (or 5-8) of $\mathbb{U}$ contain 6 of the 12 vertices of this icosahedron, including 2 at the origin with the other 6 of 12 icosahedron vertices being the antipodal reflection of these through the origin. These 2 (or 3) rows can then used as a kind of "Platonic solid projection prism" to form the 2 (or 3) 8D basis vectors used in the 2D (or 3D) projection of 4_{21}, 2_{41}, and 1_{42}.

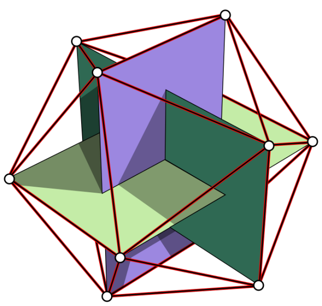
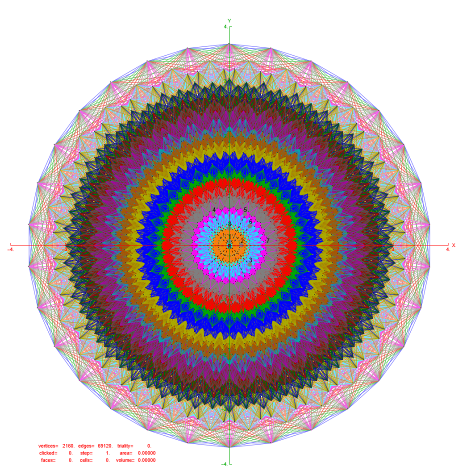
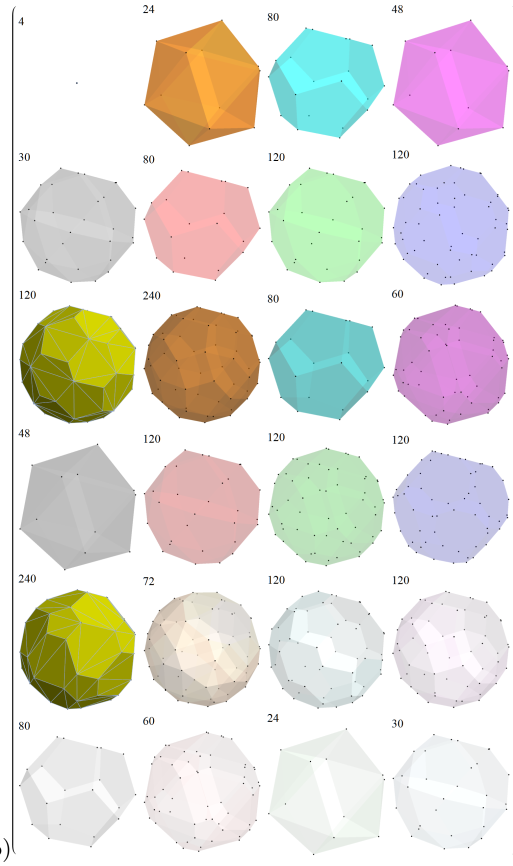


FIG. 4. The icosahedron formed from 3 mutually-perpendicular golden rectangles

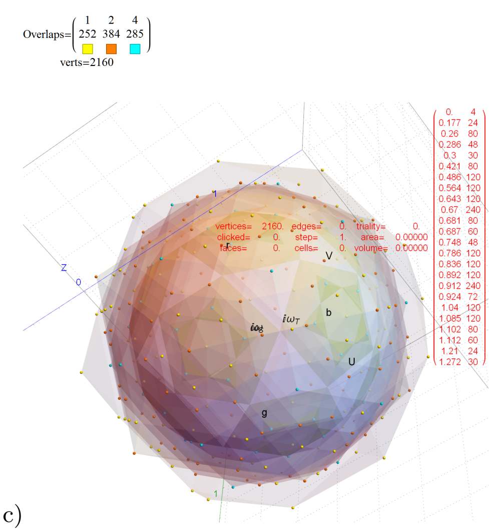
Orthogonal projection to 3D after $\mathbb{U}$ folding (i.e. selecting one of 56 unique subsets of 3 (of 8) dimensions, here we use $\{1, 2, 3\}$) manifests a large number of concentric hulls with Platonic and Archimedean solid related structures. The eight projected 3D hulls of 4_{21} include two φ scaled sets of four hulls from two 600-cells ($H_4 \oplus \varphi H_4$) as shown in Appendix A Fig. 14. 2_{41} and 1_{42} projections of E_8 are shown in Figs. 5-6.



a)



b)



c)

FIG. 5. 2_{41} projections of its 2,160 vertices
a) 2D to the E_8 Petrie projection using basis vectors X and Y from (6) with 8-polytope radius $2\sqrt{2}$ and 69,120 edges of length $\sqrt{2}$
b) 3D projections with vertices sorted and tallied by their 3D norm generating the increasingly transparent hulls for each set of tallied norms. Notice the last two outer hulls are a combination of two overlapped Icosahedrons (24) and a Icosidodecahedron (30).
c) Combined 3D hulls with the overlapping vertices color coded by overlap count. Also shown is a list (in red) the normed hull distance and the number of vertices in the group.

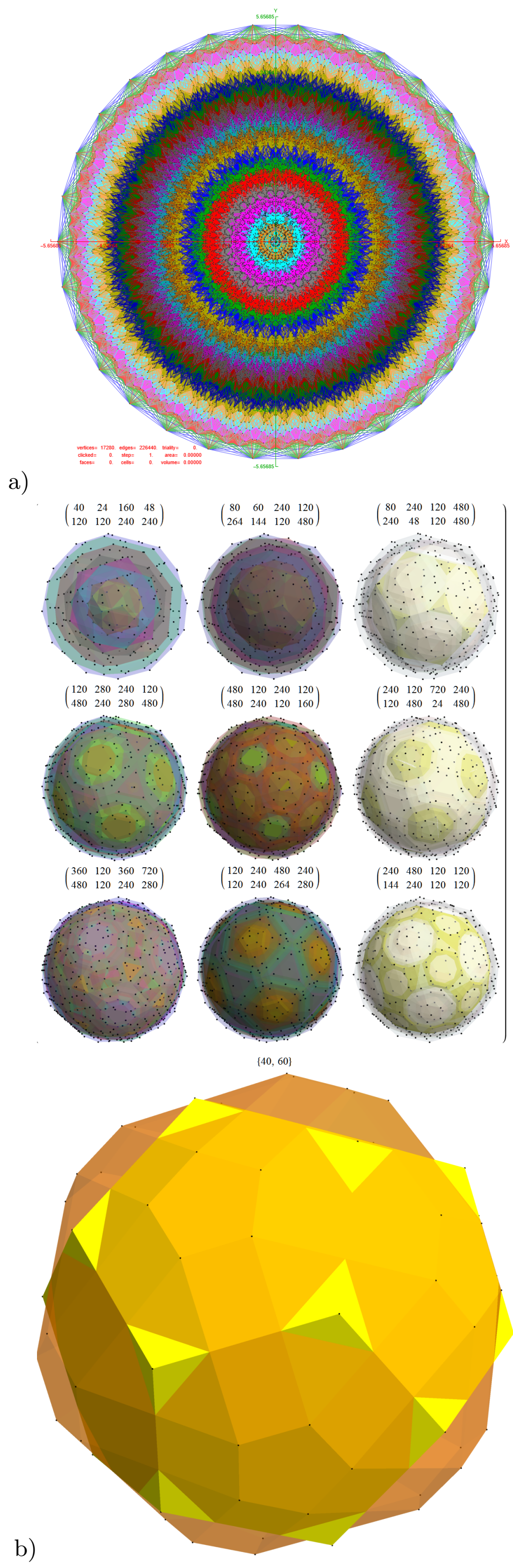


FIG. 6. 1_{42} projections of its 17,280 vertices
a) 2D to the E_8 Petrie projection using basis vectors X and Y from (6) with 8-polytope radius $4\sqrt{2}$ and 483,840 edges of length $\sqrt{2}$ (with 53% of inner edges culled for display clarity)
b) 3D projections with vertices sorted and tallied by their 3D norm generating the increasingly transparent hulls for each set of tallied norms. Notice the last two outer hulls are a combination of two overlapped Dodecahedrons (40) and a irregular Rhombicosidodecahedron (60).

II. THE PALINDROMIC UNITARY MATRIX

The particular maximal embedding of E_8 at height 248 that we are interested in for this work is shown in Appendix C Fig. 19 as the special orthogonal group of SO(16)=D_8 at height (120=112+4+4)+128', where 112 is interpreted as the subgroup embeddings of SO(8)⊗SO(8)=$D_4 \otimes D_4$ and 128' is interpreted as symplectic subgroup embeddings of C_8 where Sp(8)⊗Sp(8)=$C_4 \otimes C_4$ at height 136=128+4+4. These selected embeddings correspond to the 112 integer D_8 vertices and the 128 half-integer BC_8 vertices given by SRE E_8, in addition to the $8 \oplus \overline{8}$ generator roots for a total of 2^8. This is in 1::1 correspondence with the canonical root vertex ordering from the 9th row of the palindromic Pascal triangle $\{1, 8, 28, 56, 35\overline{35}, \overline{56}, \overline{28}, \overline{8}, \overline{1}\}$, where each entry in the list gives the number of vertices that alternate between half-integer BC_8 and integer D_8 vertex sets, with the right 5 overbar sets of 128 vertices being the negated vertices of the left 5 sets of 128 in reverse order.

It is these embeddings that have an isomorphic connection to $\mathbb{U}$ and provide the $E_8 \leftrightarrow H_4$(L⊕R⊕1⊕φ) mapping via `mapLR`. The *Mathematica*™ code for `mapLR` and the code to validate the $E_8 \leftrightarrow H_4$ isomorphism is shown in Appendix D Fig. 21. It demonstrates that E_8 rotates into four 4D copies of H_4 600-cells, with the original two (L)eft side φ scaled 4D copies related to the two (R)ight side unscaled 4D copies.

Due to the palindromic structure of $\mathbb{U}$, the H_{4L} and H_{4R} are also palindromic with each R vertex being the reverse order of the L vertex, along with `mapLR` exchanges in the (S)nub 24-cell vertices. For each L vertex that is not a member of the (T)etrahedral group's self-dual D_4 24-cell (or φT), the R vertex will be a member of the scaled φS (or S) respectively. This is due to the exchange of $\varphi^{3/2} \leftrightarrow \varphi^{-3/2}$ in `mapLR` which changes the norm (i.e. to/from a large norm=1 or a small norm=1/φ). The 24-cell T vertices are unaffected by `mapLR` exchange and have L and R vertex values of the palindromic opposite sign and the same norm.

It is clear that $\mathbb{U}$ is traceless, but it is not unitary. Since $\mathbb{U}$ is Hermitian, it is easily made unitary as $e^{i\mathbb{U}}$. While that is unitary it is not traceless, so it is not an A_7 group SU(8) symmetry. For the identification of their palindromic characteristic polynomial coefficients, see Figs. 7-8.

See Appendix D Figs. 22-23 showing the detail of the $E_8 \leftrightarrow H_4$(L⊕R⊕1⊕φ) isomorphism for each vertex.

III. QUATERNIONIC WEYL ORBIT CONSTRUCTION

The content within this paper was generated using a computational environment the author has written in *Mathematica™ by Wolfram Research, Inc.*. In order to deal effectively with quaternions, it supplants the native

```
(* Show the Determinant of 𝕌=1 *)
Det@𝕌
N[% /. φRep]
```

Out[∘]= $\dfrac{64\,\varphi^9 - 64\,\varphi^3}{256\,\varphi^6}$

Out[∘]= 1.

```
(* Show the Trace of 𝕌=0 *)
octSimplify /@ Tr@𝕌
Chop@N[% /. φRep]
```

Out[∘]= $-\dfrac{1}{\varphi^{3/2}} - \sqrt{\dfrac{1}{\varphi}} + \sqrt{\varphi}$

Out[∘]= 0

```
In[∘]:= (* Show the Eigensystem of 𝕌 *)
octSimplify /@ FullSimplify[Eigenvalues@𝕌,
  Assumptions → φAssumptions]
Total@N[% /. φRep]
FullSimplify[Eigenvectors@𝕌, Assumptions → φAssumptions]
```

Out[∘]= $\left\{-\sqrt{\frac{1}{\varphi}}, -\sqrt{\frac{1}{\varphi}}, \sqrt{\frac{1}{\varphi}}, -\sqrt{\varphi}, \sqrt{\varphi}, \sqrt{\varphi}, -\sqrt{\varphi}, \sqrt{\frac{1}{\varphi}}\right\}$

Out[∘]= 0.

Out[∘]=
$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
In[∘]:= (* Get the Characteristic coefficients *)
FullSimplify[CharacteristicPolynomial[𝕌, x],
 Assumptions → φAssumptions]
```

Out[∘]=
$$\frac{1}{\varphi^{9/2}}\Big(\varphi^3 x^7 (\varphi(1.-1.\varphi)+1.) + \varphi^{3/2}(0.25\varphi^6 - 0.25) + 1.\varphi^{9/2}x^8 + \varphi^{3/2}(\varphi(\varphi(\varphi(-0.25\varphi^3 - 1.\varphi - 1.) - 2.) + 1.) + 0.25)x^6 + \varphi(\varphi(0.25\varphi^6 - 0.25\varphi^5 + 1.\varphi^4 - 1.\varphi^3 - 2.\varphi - 1.25) + 0.25)x^5 + \sqrt{\varphi}(\varphi(0.25\varphi^7 + 0.25\varphi^6 + 0.25\varphi^5 + 1.\varphi^4 + 2.\varphi^3 + 0.75\varphi - 1.25) - 0.25)x^4 + (\varphi(\varphi(\varphi(\varphi(-0.25\varphi(1.\varphi - 1.)(1.\varphi^2 + 1.) - 1.) + 2.) + 1.25) + 0.75) + 0.25) - 0.25)x^3 + \sqrt{\varphi}(\varphi(\varphi(-0.25\varphi^5 - 0.25\varphi^4 - 0.25\varphi^3 - 1.\varphi - 1.) + 1.25) + 0.25) + 0.25)x^2 + \varphi(\varphi(0.25\varphi^6 - 0.25\varphi^5 - 1.\varphi - 0.25) + 0.25)x\Big)$$

```
In[∘]:= (* Collect and compare them *)
((1 + σ x²) (1 - τ x²))² /. slRep;
FullSimplify[% == %%];
Expand@%%;
Collect[%, x]
```

Out[∘]= $x^8 + \left(-2\varphi - \dfrac{2}{\varphi}\right)x^6 + \left(\varphi^2 + \dfrac{1}{\varphi^2} + 4\right)x^4 + \left(-2\varphi - \dfrac{2}{\varphi}\right)x^2 + 1$

```
In[∘]:= (* The palindrome of coefficients in the characteristic
 matrix of 𝕌 *)
  {1, 0, 2 (σ - τ), 0, 7, 0, 2 (σ - τ), 0, 1};
c𝕌 = 1 + 0 x + 2 (σ - τ) x² + 0 x³ + 7 x⁴ + 0 x⁵ + 2 (σ - τ) x⁶ + 0 x⁷ + x⁸ /. slRep;
FullSimplify[% == %%%, Assumptions → φAssumptions]
```

Out[∘]= $\varphi^2 x = (\varphi + 1)x$

```
In[∘]:= (* Verify the simplification is True *)
Chop@N[% /. φRep]
```

Out[∘]= True

FIG. 7. The trace, determinant, Eigenvalues, Eigenvector matrix, and characteristic polynomial coefficients of 𝕌

```
In[∘]:= octSimplify /@ FullSimplify[Eigenvalues@ei𝕌, Assumptions → φAssumptions]
Total@N[% /. φRep]
FullSimplify[Eigenvectors@ei𝕌, Assumptions → φAssumptions]
```

Out[∘]= $\left\{e^{-i\sqrt{\varphi}}, e^{\frac{i}{\sqrt{\varphi}}}, e^{-\frac{i}{\sqrt{\varphi}}}, e^{-\frac{i}{\sqrt{\varphi}}}, e^{\frac{i}{\sqrt{\varphi}}}, e^{-i\sqrt{\varphi}}, e^{i\sqrt{\varphi}}, e^{i\sqrt{\varphi}}\right\}$

Out[∘]= 4.0037 + 0. *i*

Out[∘]=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

```
In[∘]:= Cf1 = Cos[1/√ϕ] + Cos[√ϕ];
Cf2 = Cos[1/√ϕ] Cos[√ϕ];
Cf3 = Cos[1/√ϕ]² Cos[√ϕ] + Cos[1/√ϕ] Cos[√ϕ]²;
Cf4 = Cos[1/√ϕ]² + Cos[√ϕ]²;
```

```
In[∘]:= (* The palindrome of coefficients in the characteristic matrix of ei𝕌 *)
  {1, -4 Cf1, 4 (1 + 4 Cf2 + Cf4), -4 (3 Cf1 + 4 Cf3), 2 (3 + 4 (Cf4 + 2 Cf2 (Cf2 + 2))), -4 (3 Cf1 + 4 Cf3), 4 (1 + 4 Cf2 + Cf4), -4 Cf1, 1};
cei𝕌 = 1 - 4 Cf1 x + 4 (1 + 4 Cf2 + Cf4) x² - 4 (3 Cf1 + 4 Cf3) x³ + 2 (3 + 4 (Cf4 + 2 Cf2 (Cf2 + 2))) x⁴ - 4 (3 Cf1 + 4 Cf3) x⁵ + 4 (1 + 4 Cf2 + Cf4) x⁶ - 4 Cf1 x⁷ + x⁸ /. slRep;
N[% /. φRep]
```

Out[∘]= $x^8 - 4.0037\,x^7 + 9.67125\,x^6 - 15.3419\,x^5 + 18.0346\,x^4 - 15.3419\,x^3 + 9.67125\,x^2 - 4.0037\,x + 1.$

```
In[∘]:= (* Re@ei𝕌 *)
FullSimplify[Re@ei𝕌, Assumptions → φAssumptions] // MatrixForm
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} \cos\left(\frac{1}{2\varphi^{3/2}}\right)\cos\left(\frac{\varphi^{3/2}}{2}\right) & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\left(\frac{1}{2\varphi^{3/2}}\right)\sin\left(\frac{\varphi^{3/2}}{2}\right) \\ 0 & \frac{1}{2}\left(\cos\left(\frac{1}{\sqrt{\varphi}}\right)+\cos(\sqrt{\varphi})\right) & 0 & 0 & 0 & 0 & \frac{1}{2}\left(\cos(\sqrt{\varphi})-\cos\left(\frac{1}{\sqrt{\varphi}}\right)\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(\cos\left(\frac{1}{\sqrt{\varphi}}\right)+\cos(\sqrt{\varphi})\right) & 0 & 0 & \frac{1}{2}\left(\cos(\sqrt{\varphi})-\cos\left(\frac{1}{\sqrt{\varphi}}\right)\right) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(\cos\left(\frac{1}{\sqrt{\varphi}}\right)+\cos(\sqrt{\varphi})\right) & \frac{1}{2}\left(\cos(\sqrt{\varphi})-\cos\left(\frac{1}{\sqrt{\varphi}}\right)\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(\cos(\sqrt{\varphi})-\cos\left(\frac{1}{\sqrt{\varphi}}\right)\right) & \frac{1}{2}\left(\cos\left(\frac{1}{\sqrt{\varphi}}\right)+\cos(\sqrt{\varphi})\right) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(\cos(\sqrt{\varphi})-\cos\left(\frac{1}{\sqrt{\varphi}}\right)\right) & 0 & 0 & \frac{1}{2}\left(\cos\left(\frac{1}{\sqrt{\varphi}}\right)+\cos(\sqrt{\varphi})\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(\cos(\sqrt{\varphi})-\cos\left(\frac{1}{\sqrt{\varphi}}\right)\right) & 0 & 0 & 0 & 0 & \frac{1}{2}\left(\cos\left(\frac{1}{\sqrt{\varphi}}\right)+\cos(\sqrt{\varphi})\right) & 0 \\ -\sin\left(\frac{1}{2\varphi^{3/2}}\right)\sin\left(\frac{\varphi^{3/2}}{2}\right) & 0 & 0 & 0 & 0 & 0 & 0 & \cos\left(\frac{1}{2\varphi^{3/2}}\right)\cos\left(\frac{\varphi^{3/2}}{2}\right) \end{pmatrix}$$

```
In[∘]:= Tr@%
N[% /. φRep]
```

Out[∘]= $2\cos\left(\dfrac{1}{2\varphi^{3/2}}\right)\cos\left(\dfrac{\varphi^{3/2}}{2}\right) + 3\left(\cos\left(\dfrac{1}{\sqrt{\varphi}}\right) + \cos(\sqrt{\varphi})\right)$

Out[∘]= 4.0037

```
In[∘]:= (* Im@ei𝕌 *)
FullSimplify[Im@ei𝕌, Assumptions → φAssumptions] // MatrixForm
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} \sin\left(\frac{1}{2\varphi^{3/2}}\right)\left(-\cos\left(\frac{\varphi^{3/2}}{2}\right)\right) & 0 & 0 & 0 & 0 & 0 & 0 & \sin\left(\frac{\varphi^{3/2}}{2}\right)\left(-\cos\left(\frac{1}{2\varphi^{3/2}}\right)\right) \\ 0 & -\frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & \frac{\sin(\sqrt{\varphi})}{2} & 0 & 0 & \frac{\sin(\sqrt{\varphi})}{2} & \frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & 0 \\ 0 & \frac{\sin(\sqrt{\varphi})}{2} & 0 & -\frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & \frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & 0 & \frac{\sin(\sqrt{\varphi})}{2} & 0 \\ 0 & 0 & -\frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & \frac{\sin(\sqrt{\varphi})}{2} & \frac{\sin(\sqrt{\varphi})}{2} & \frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & \frac{\sin(\sqrt{\varphi})}{2} & \frac{\sin(\sqrt{\varphi})}{2} & -\frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & 0 & 0 \\ 0 & \frac{\sin(\sqrt{\varphi})}{2} & 0 & \frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & -\frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & 0 & \frac{\sin(\sqrt{\varphi})}{2} & 0 \\ 0 & \frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & \frac{\sin(\sqrt{\varphi})}{2} & 0 & 0 & \frac{\sin(\sqrt{\varphi})}{2} & -\frac{1}{2}\sin\left(\frac{1}{\sqrt{\varphi}}\right) & 0 \\ \sin\left(\frac{\varphi^{3/2}}{2}\right)\left(-\cos\left(\frac{1}{2\varphi^{3/2}}\right)\right) & 0 & 0 & 0 & 0 & 0 & 0 & \sin\left(\frac{1}{2\varphi^{3/2}}\right)\left(-\cos\left(\frac{\varphi^{3/2}}{2}\right)\right) \end{pmatrix}$$

```
In[∘]:= Tr@%
Chop@N[% /. φRep]
```

Out[∘]= $-2\sin\left(\dfrac{1}{2\varphi^{3/2}}\right)\cos\left(\dfrac{\varphi^{3/2}}{2}\right) - \sin\left(\dfrac{1}{\sqrt{\varphi}}\right) + \sin(\sqrt{\varphi})$

Out[∘]= 0

```
In[∘]:= (* Re@ei𝕌 *)
N@FullSimplify[Re@ei𝕌 /. φRep, Assumptions → φAssumptions] // MatrixForm
```

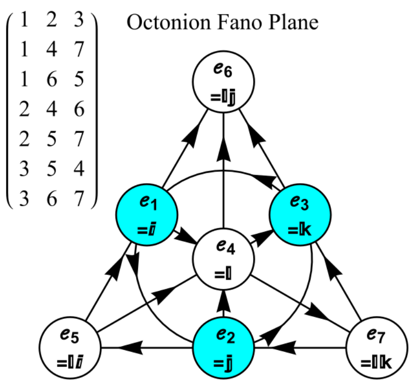
Out[∘]//MatrixForm=
$$\begin{pmatrix} 0.500463 & 0. & 0. & 0. & 0. & 0. & 0. & -0.206111 \\ 0. & 0.500463 & 0. & 0. & 0. & 0. & -0.206111 & 0. \\ 0. & 0. & 0.500463 & 0. & 0. & -0.206111 & 0. & 0. \\ 0. & 0. & 0. & 0.500463 & -0.206111 & 0. & 0. & 0. \\ 0. & 0. & 0. & -0.206111 & 0.500463 & 0. & 0. & 0. \\ 0. & 0. & -0.206111 & 0. & 0. & 0.500463 & 0. & 0. \\ 0. & -0.206111 & 0. & 0. & 0. & 0. & 0.500463 & 0. \\ -0.206111 & 0. & 0. & 0. & 0. & 0. & 0. & 0.500463 \end{pmatrix}$$

```
In[∘]:= (* Im@ei𝕌 *)
N@FullSimplify[Im@ei𝕌 /. φRep, Assumptions → φAssumptions] // MatrixForm
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} -0.124029 & 0. & 0. & 0. & 0. & 0. & 0. & -0.831668 \\ 0. & -0.35382 & 0.477849 & 0. & 0. & 0.477849 & 0.35382 & 0. \\ 0. & 0.477849 & 0. & -0.35382 & 0.35382 & 0. & 0.477849 & 0. \\ 0. & 0. & -0.35382 & 0.477849 & 0.477849 & 0.35382 & 0. & 0. \\ 0. & 0. & 0.35382 & 0.477849 & 0.477849 & -0.35382 & 0. & 0. \\ 0. & 0.477849 & 0. & 0.35382 & -0.35382 & 0. & 0.477849 & 0. \\ 0. & 0.35382 & 0.477849 & 0. & 0. & 0.477849 & -0.35382 & 0. \\ -0.831668 & 0. & 0. & 0. & 0. & 0. & 0. & -0.124029 \end{pmatrix}$$

FIG. 8. The Eigenvalues, Eigenvector matrix, and characteristic polynomial coefficients of the unitary form of 𝕌 as $e^{i\mathbb{U}}$ showing a Tr@Re@$e^{i\mathbb{U}} \approx 4$ and a traceless imaginary part

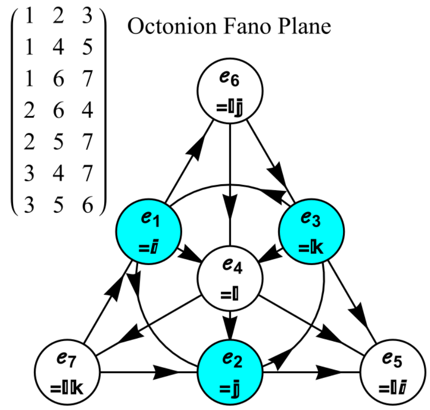
Quaternion package with a more flexible symbolic octonion ($\mathbb{O}$) capability. This allows for the selection of a multiplication table from any of the 480 possible octonion tables, including their split and bi-octonion forms. It also handles the sedenion forms as well and has been used to verify the octonion forms of E_8 from Koca[1], Dixon[15], Pushpa and Bisht[16], R. A. Wilson, Dray, and Monague[17], including the complexified octonions of Günaydin-Gürsey[18] and Furey[19]. To ensure that our quaternion (and bi-quaternion) math is consistent with the standard multiplication convention related to quaternions, we need to select one of the 48 octonions with a first triad of 123 and a Cayley-Dickson construction where e_4-e_7 quadrant multiplication remains within the quadrant. See Fig. 9 showing the selected triads, Fano plane, and multiplication table of the octonion used in this and several of the referenced papers[1].



0↦1	1	2	3	4	5	6	7
1	−1	3	−2	7	−6	5	−4
2	−3	−1	1	6	7	−4	−5
3	2	−1	−1	−5	4	7	−6
4	−7	−6	5	−1	−3	2	1
5	6	−7	−4	3	−1	−1	2
6	−5	4	−7	−2	1	−1	3
7	4	5	6	−1	−2	−3	−1

FIG. 9. The selected octonion Fano plane mnemonic and multiplication table based on its 7 structure constant triads . The first triad (123) defines standard convention for quaternions.

It has been shown that the 3D symmetry groups of A_3, B_3, and H_3[3] and 4D symmetry groups of A_4, D_4, F_4, and H_4 are related to the higher dimensional groups of D_6 and E_8[5][9]. A quaternionic Weyl group orbit O(Λ)=W(H_4)=I of order 120 can be constructed from H_3 which generates some of the Platonic, Archimedean and dual Catalan solids shown in Appendix B Fig. 18, including their irregular and chiral forms[4]. The polytopes for a particular orbit of O(Λ)=W(`group`) are generated using a function `Λ[group_, orbit_, perm_ : "Rotate"]`, where `perm` can be one of 18 combinations of sign and position permutation functions (e.g. "oSign" gives all odd sign permutations and cyclic rotations of position and the default "Rotate" gives all sign permutations of cyclicly rotated positions). The first column in these figures show



0↦1	1	2	3	4	5	6	7
1	−1	3	−2	5	−4	7	−6
2	−3	−1	1	−6	7	4	−5
3	2	−1	−1	7	6	−5	−4
4	−5	6	−7	−1	1	−2	3
5	4	−7	−6	−1	−1	3	2
6	−7	−4	5	2	−3	−1	1
7	6	5	4	−3	−2	−1	−1

FIG. 10. An alternative set of structure constant triads, octonion Fano plane mnemonic, and multiplication table, with decorations showing the palindromic multiplication.

the set of calls to the Λ function. This same method is used to generate the H_4-based 4-polytopes of the 120-cell and 600-cell shown in Appendix A Figs. 14-16.

The A_3 in A_4 group embedding of SU(5)⊃SU(4)⊗U_1[5] are shown in Appendix C Fig. 20 in combination with these 3 and 4-polytope visualizations. [2]

We identify the parent orbit (1000) of W(D_4) as the self-dual 24-cell T, which is the combination of the 4D octahedron (aka. 16-cell) and the 4D cube (aka. 8-cell with a 3D hull of the cuboctahedron derived from the tri-rectified (0001) W(BC_4)). T' is identified as the set of 3 orbits {(0100), (0010), (0001)} of W(D_4) with 8 vertices each made of 2-component (vector) quaternions and has a 3D hull of the rhombic dodecahedron. See Fig. 11 for their specific symbolic and numeric values.

From T (and T') we can take any one vertex to define a c (and c'=cp) respectively. For this paper, we use as an example c=t_1 from eq. (18) from Koca[3] which is our 13'th T (and T') shown in Fig.11 such that c=$\frac{1}{2}(1+e_1-e_2-e_3)$ (and c'=$\frac{e_2-e_3}{\sqrt{2}}$). Here c' is used with A' to generate the parent W(A_4), or simply A as the 5-cell[3]. Specifically, A=$(c' \circ A')^*$ with

[1] It is interesting to note that this particular octonion is close to (but not) palindromic. Using an algorithmic identification and construction of all of the possible 480 unique permutations of octonions[20], we find that a small change in triads to {123,145,167,264,257,347,356} with 5↔7 ordering swaps creates a palindromic E_8. This octonion is shown in Fig. 10

[2] In the methods and coding descriptions, since Mamone[6] identifies the 5-cell as S, but Koca uses S to identify the (S)nub 24-cell (a convention which we use here), Mamone's A_4-based 5-cell is now identified as A which is the 4D version of the tetrahedron.

```
(* Show T vertices *)
checkVertices[T, False, True, True, False, False, False]
```

Out[∘]= List length= 24 and it is symbolic octonion

	Math	Numeric
1	$\frac{1}{2}(-1-e_1-e_2-e_3)$	$-0.5-0.5e_1-0.5e_2-0.5e_3$
2	$\frac{1}{2}(-1-e_1-e_2+e_3)$	$-0.5-0.5e_1-0.5e_2+0.5e_3$
3	$\frac{1}{2}(-1-e_1+e_2-e_3)$	$-0.5-0.5e_1+0.5e_2-0.5e_3$
4	$\frac{1}{2}(-1-e_1+e_2+e_3)$	$-0.5-0.5e_1+0.5e_2+0.5e_3$
5	$\frac{1}{2}(-1+e_1-e_2-e_3)$	$-0.5+0.5e_1-0.5e_2-0.5e_3$
6	$\frac{1}{2}(-1+e_1-e_2+e_3)$	$-0.5+0.5e_1-0.5e_2+0.5e_3$
7	$\frac{1}{2}(-1+e_1+e_2-e_3)$	$-0.5+0.5e_1+0.5e_2-0.5e_3$
8	$\frac{1}{2}(-1+e_1+e_2+e_3)$	$-0.5+0.5e_1+0.5e_2+0.5e_3$
9	$\frac{1}{2}(1-e_1-e_2-e_3)$	$0.5-0.5e_1-0.5e_2-0.5e_3$
10	$\frac{1}{2}(1-e_1-e_2+e_3)$	$0.5-0.5e_1-0.5e_2+0.5e_3$
11	$\frac{1}{2}(1-e_1+e_2-e_3)$	$0.5-0.5e_1+0.5e_2-0.5e_3$
12	$\frac{1}{2}(1-e_1+e_2+e_3)$	$0.5-0.5e_1+0.5e_2+0.5e_3$
13	$\frac{1}{2}(1+e_1-e_2-e_3)$	$0.5+0.5e_1-0.5e_2-0.5e_3$
14	$\frac{1}{2}(1+e_1-e_2+e_3)$	$0.5+0.5e_1-0.5e_2+0.5e_3$
15	$\frac{1}{2}(1+e_1+e_2-e_3)$	$0.5+0.5e_1+0.5e_2-0.5e_3$
16	$\frac{1}{2}(1+e_1+e_2+e_3)$	$0.5+0.5e_1+0.5e_2+0.5e_3$
17	$-e_3$	$0.-1.e_3$
18	$-e_2$	$0.-1.e_2$
19	$-e_1$	$0.-1.e_1$
20	-1	$-1.$
21	e_3	$0.+1.e_3$
22	e_2	$0.+1.e_2$
23	e_1	$0.+1.e_1$
24	1	$1.$

```
(* Show T' vertices *)
checkVertices[Tp, False, True, True, False, False, False]
```

Out[∘]= List length= 24 and it is symbolic octonion

	Math	Numeric
1	$-\frac{1+e_1}{\sqrt{2}}$	$-0.70711-0.70711e_1$
2	$-\frac{1+e_2}{\sqrt{2}}$	$-0.70711-0.70711e_2$
3	$-\frac{1+e_3}{\sqrt{2}}$	$-0.70711-0.70711e_3$
4	$\frac{-1+e_3}{\sqrt{2}}$	$-0.70711+0.70711e_3$
5	$\frac{-1+e_2}{\sqrt{2}}$	$-0.70711+0.70711e_2$
6	$\frac{-1+e_1}{\sqrt{2}}$	$-0.70711+0.70711e_1$
7	$-\frac{e_1+e_2}{\sqrt{2}}$	$0.-0.70711e_1-0.70711e_2$
8	$-\frac{e_1+e_3}{\sqrt{2}}$	$0.-0.70711e_1-0.70711e_3$
9	$\frac{-e_1+e_3}{\sqrt{2}}$	$0.-0.70711e_1+0.70711e_3$
10	$\frac{-e_1+e_2}{\sqrt{2}}$	$0.-0.70711e_1+0.70711e_2$
11	$-\frac{e_2+e_3}{\sqrt{2}}$	$0.-0.70711e_2-0.70711e_3$
12	$\frac{-e_2+e_3}{\sqrt{2}}$	$0.-0.70711e_2+0.70711e_3$
13	$\frac{e_2-e_3}{\sqrt{2}}$	$0.+0.70711e_2-0.70711e_3$
14	$\frac{e_2+e_3}{\sqrt{2}}$	$0.+0.70711e_2+0.70711e_3$
15	$\frac{e_1-e_2}{\sqrt{2}}$	$0.+0.70711e_1-0.70711e_2$
16	$\frac{e_1-e_3}{\sqrt{2}}$	$0.+0.70711e_1-0.70711e_3$
17	$\frac{e_1+e_3}{\sqrt{2}}$	$0.+0.70711e_1+0.70711e_3$
18	$\frac{e_1+e_2}{\sqrt{2}}$	$0.+0.70711e_1+0.70711e_2$
19	$-\frac{-1+e_1}{\sqrt{2}}$	$0.70711-0.70711e_1$
20	$-\frac{-1+e_2}{\sqrt{2}}$	$0.70711-0.70711e_2$
21	$-\frac{-1+e_3}{\sqrt{2}}$	$0.70711-0.70711e_3$
22	$\frac{1+e_3}{\sqrt{2}}$	$0.70711+0.70711e_3$
23	$\frac{1+e_2}{\sqrt{2}}$	$0.70711+0.70711e_2$
24	$\frac{1+e_1}{\sqrt{2}}$	$0.70711+0.70711e_1$

FIG. 11. The values of the D_4 24-cell T and its alternate T'

```
In[∘]:= (* Generate the A' parent Weyl orbit *)
ΛA4[{0, 1, 4, 2, 3}, {1, 0, 0, 0}]
```

Out[∘]=

$$\left(\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{\varphi}{\sqrt{10}} & \frac{1}{\sqrt{10}\,\varphi} \\ \frac{1}{\sqrt{2}} & 0 & \sqrt{\frac{2}{5}}\,\varphi-\frac{\varphi}{\sqrt{10}} & \frac{\sqrt{\frac{2}{5}}}{\varphi}-\frac{1}{\sqrt{10}\,\varphi} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{\varphi^2}{\sqrt{10}}-\sqrt{\frac{2}{5}}\,\varphi & -\frac{1}{\sqrt{10}\,\varphi^2}-\frac{\sqrt{\frac{2}{5}}}{\varphi} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{\frac{2}{5}}-\frac{\varphi^2}{\sqrt{10}} & \frac{1}{\sqrt{10}\,\varphi^2}-\sqrt{\frac{2}{5}} \\ 0 & 0 & -\sqrt{\frac{2}{5}} & \sqrt{\frac{2}{5}} \end{pmatrix} \ \text{NoneNoSign} \right)$$

```
In[∘]:= (* Scale by -√5/2 to match conventional A scaling *)
ApList = oct2List@biQuaternion[-(√5/2) #] & /@ %[[1, 1]]
```

Out[∘]=

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{2} & 0 & -\frac{\varphi}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}\,\varphi} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{\frac{5}{2}}}{2} & 0 & -\frac{\varphi}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}\,\varphi} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{(\varphi-2)\varphi}{2\sqrt{2}} & \frac{2\varphi+1}{2\sqrt{2}\,\varphi^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{\frac{5}{2}}}{2} & \frac{\varphi^2-2}{2\sqrt{2}} & -\frac{\frac{1}{\varphi^2}-2}{2\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[∘]:= (* Put it into symbolic octonion form *)
Ap = octSimplify@octonion /@ %;
% // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{\varphi e_2}{2\sqrt{2}}-\frac{e_3}{2\sqrt{2}\,\varphi}+\frac{\sqrt{\frac{5}{2}}}{2} \\ -\frac{\varphi e_2}{2\sqrt{2}}-\frac{e_3}{2\sqrt{2}\,\varphi}-\frac{\sqrt{\frac{5}{2}}}{2} \\ \frac{(2\varphi+1)e_3}{2\sqrt{2}\,\varphi^2}-\frac{(\varphi-2)\varphi e_2}{2\sqrt{2}}+\frac{1}{2}\sqrt{\frac{5}{2}}\,e_1 \\ \frac{(\varphi^2-2)e_2}{2\sqrt{2}}-\frac{\left(\frac{1}{\varphi^2}-2\right)e_3}{2\sqrt{2}}-\frac{1}{2}\sqrt{\frac{5}{2}}\,e_1 \\ \frac{e_2}{\sqrt{2}}-\frac{e_3}{\sqrt{2}} \end{pmatrix}$$

```
In[∘]:= (* Display vertex valuse *)
checkVertices[%, False, True, True, False, False, False]
```

Out[∘]= List length= 5 and it is symbolic octonion

Math =

1	$-\frac{\frac{1}{4}(1+\sqrt{5})^2 e_2+e_3-\frac{1}{2}\sqrt{5}(1+\sqrt{5})}{\sqrt{2}(1+\sqrt{5})}$
2	$-\frac{\frac{1}{4}(1+\sqrt{5})^2 e_2+e_3+\frac{1}{2}\sqrt{5}(1+\sqrt{5})}{\sqrt{2}(1+\sqrt{5})}$
3	$\frac{\sqrt{2}\left(\frac{1}{4}\sqrt{5}(1+\sqrt{5})^2 e_1-\frac{1}{8}(1+\sqrt{5})^3\left(-2+\frac{1}{2}(1+\sqrt{5})\right)e_2+(2+\sqrt{5})e_3\right)}{(1+\sqrt{5})^2}$
4	$\frac{\sqrt{2}\left(\frac{1}{4}(1+\sqrt{5})^2\left(-\sqrt{5}\,e_1+\left(-2+\frac{1}{4}(1+\sqrt{5})^2\right)e_2+2e_3\right)-e_3\right)}{(1+\sqrt{5})^2}$
5	$\frac{e_2-e_3}{\sqrt{2}}$

Numeric =

1	$-0.57206e_2-0.21851e_3+0.79057$
2	$-0.57206e_2-0.21851e_3-0.79057$
3	$0.79057e_1+0.21851e_2+0.57206e_3+0.$
4	$-0.79057e_1+0.21851e_2+0.57206e_3+0.$
5	$0.70711e_2-0.70711e_3+0.$

```
In[∘]:= (* Simplify quaternion multiplication using p∘r∘q which also handles lists,
    We scale up/down by 4 for symbolic clarity.
   Please note the double struck A to avoid stepping on LieArt *)
𝔸 = (1/4 octonion[biQuaternion[# /. φRep]]* /. slRep) & /@
  (* φRep replaces the symbolic forms ϕ→(√5+1)/2, also note the conjugation
  (4 oct2Quat@# & /@ Flatten@prq[cp, 1, Ap]);
In[∘]:= checkVertices[𝔸 /. e0 → 1, False, True, True, False, False, False]
```

Out[∘]= List length= 5 and it is symbolic octonion

	Math	Numeric
1	$\frac{1}{4}(\sqrt{5}\,e_1-\sqrt{5}\,e_2+\sqrt{5}\,e_3+1)$	$0.55902e_1-0.55902e_2+0.55902e_3+0.25$
2	$\frac{1}{4}(\sqrt{5}\,e_1+\sqrt{5}\,e_2-\sqrt{5}\,e_3+1)$	$0.55902e_1+0.55902e_2-0.55902e_3+0.25$
3	$\frac{1}{4}(-\sqrt{5}\,e_1+\sqrt{5}\,e_2+\sqrt{5}\,e_3+1)$	$-0.55902e_1+0.55902e_2+0.55902e_3+0.25$
4	$\frac{1}{4}(-\sqrt{5}\,e_1-\sqrt{5}\,e_2-\sqrt{5}\,e_3+1)$	$-0.55902e_1-0.55902e_2-0.55902e_3+0.25$
5	-1	$-1.$

FIG. 12. Explicit *Mathematica*^{TM} computation of A from the `ΛA4[Λ_, orbit_]` generated A'

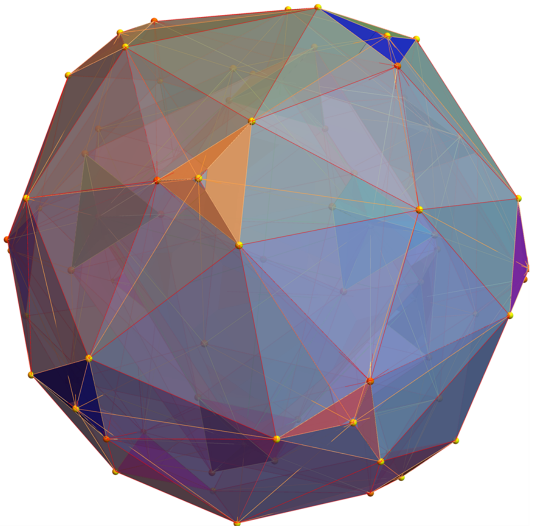


FIG. 13. Visualization of the 144 root vertices of S'+T+T' now identified as the dual snub 24-cell

A'=ΛA4[{0, 1, 4, 2, 3}, {1, 0, 0, 0}]. [3] See Fig. 12 for the explicit *Mathematica*^{TM} computation related to A and A'.

The snub orbit (0000) of W(D_4) will generate the vertices of the snub 24-cell or S=I-T, as with the alternate snub 24-cell S'=I'-T' as shown in (7) and (8). We can generate S (or S') by taking the odd (or even) sign and cyclic position permutations of a seed quaternion p∈I (or I') to be assigned to α (or β) for generating S (or S') respectively. There are only 48 that satisfy the necessary constraint where $p^5 = \pm 1$. Those quaternions that satisfy the constraint are identified with an * in Appendix D. For this paper, we took the 8'th permutation of the generated S for $\alpha = \frac{1}{2}\left(\frac{1}{\varphi} + \varphi e_2 + e_1\right)$ (and S' for $\beta = \frac{-\varphi - \frac{e_2}{\varphi} + \sqrt{5}e_1}{\sqrt{8}}$). This process of generating the snub 24-cell can be visualized as generating four quaternion 4D rotations of T (and T'). The 3D hulls of I' is shown in Fig. 15.

$$\begin{aligned} &S = I - T = \textstyle\sum_{i=1}^{4} \alpha^i \circ T \\ &\texttt{or} \\ &I = \texttt{prq}[\alpha^{0-4}, \texttt{1}, \texttt{T}] \end{aligned} \tag{7}$$

$$\begin{aligned} &S' = I' - T' = \textstyle\sum_{i=1}^{4} \beta^i \circ T' \\ &\texttt{or} \\ &I' = \texttt{prq}[\beta^{0-4}, \texttt{1}, \texttt{T}'] \end{aligned} \tag{8}$$

The 3D hulls for one copy of I (or φI) are represented in Fig. 14 hulls {2,3,5} (or {6,7,8}) respectively plus 1/2 of the vertices in hull 4. The vertex values of I are listed in either of the center columns of either Appendix D Fig. 22 or Fig. 23.

Koca[3] has also identified the dual to the snub 24-cell as being made up of the 144 root vertices of S'+T+T'. This 4-polytope is visualized in Fig. 13.

The equations for the generation of J (and J') are shown in (9) and (10). As it was for I (and I') vertices each mapping to 5 quaternion rotations of T (and T'), J (and J') vertices each map to 5 quaternion rotations of I (and I') or 25 quaternion rotations of T (and T'). Given the isomorphism between each E_8 root vertex and 4 copies of I (i.e. L and R each at unit and $1/\varphi$ scales) as demonstrated in Section II, this means quaternionic Weyl orbit construction, when used with $\mathbb{U}$ and `mapLR`, provides for an explicit map between each of the 240 E_8 root vertices and 20 J (or J') vertices (i.e. 20=4 L⊕R⊕1⊕φ copies of each I (or I') vertex⊗5 quaternion rotations).

$$\begin{aligned} &J = \textstyle\sum_{i=0}^{4} c' \circ \bar{\alpha}^{\dagger i} \circ \alpha^i \circ T \\ &\texttt{or} \\ &J = \texttt{prq}[\texttt{A}', \alpha^{0-4}, \texttt{T}] \end{aligned} \tag{9}$$

$$\begin{aligned} &J' = \textstyle\sum_{i=0}^{4} c \circ \bar{\beta}^{\dagger i} \circ \beta^i \circ T' \\ &\texttt{or} \\ &J' = \texttt{prq}[\texttt{A}', \beta^{0-4}, \texttt{T}'] \end{aligned} \tag{10}$$

See Figs. 16-17 for the 120-cell (J) and its alternate (J') as generated by J=prq[A', 1, I] and J'=prq[A', 1, I'] respectively.

IV. CONCLUSION

This paper has given an explicit isomorphic mapping from the 240 $\mathbb{R}^8$ root E_8 Gossett 4_{21} 8-polytope to two φ scaled copies of the 120 root H_4 600-cell quaternion 4-polytope using $\mathbb{U}$. It has also shown the inverse map from a single H_4 600-cell to E_8 using a 4D↪8D chiral L↔R mapping function, φ scaling, and $\mathbb{U}^{-1}$. This approach has shown that there are actually four copies of each 600-cell living within E_8 in the form of chiral $H_{4L} \oplus \varphi H_{4L} \oplus H_{4R} \oplus \varphi H_{4R}$ roots. In addition, it has

---

[3] The 4-polytopes for a particular orbit of O(Λ)=W(`group`) are generated using a function Λ[`group_`, `orbit_`, `perm_`] which is called by ΛA4[Λ_, `orbit_`] for the subgroup embeddings in A_4 as described in [5]. In addition, `SmallCircle` (∘) is the symbolic operator for quaternion (octonion) multiplication that operates across lists, along with the expected symbolic exponentials (* and †) for Conjugate and ConjugateTranspose respectively. The function `prq[p_, r_, q_, left:False] := If[left, q∘(p∘r), p∘(r∘q)]` implements the operation of [p,q]:r from eq. (6) in [3], which is defined for any combinations of inputs as elements or lists in order to add flexibility to quaternion and octonion operators, including left or right (default) non-commutative multiplication ordering. Other operators are also available for scalar product+(⊕), scalar product-(⊖), commutator(⊙), anti-commutator(∧), derivation(□), Kronecker product(⊗), and `octExp` for exponential powers of octonions.

demonstrated a quaternion Weyl orbit construction of H_4-based 4-polytopes that provides an explicit map from E_8 to four copies of the tri-rectified Coxeter-Dynkin diagram of H_4, namely the 120-cell of order 600. Taking advantage of this property promises to open the door to as yet unexplored chiral E_8-based Grand Unified Theories or GUTs. It is anticipated that these visualizations and connections will be useful in discovering new insights into unifying the mathematical symmetries as they relate to unification in theoretical physics.

ACKNOWLEDGMENTS

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Appendix A: *Concentric hulls from Platonic 3D projection with numeric and symbolic norm distances*

Figs. 14-17

Appendix B: *Archimedean and dual Catalan solids*

Fig. 18

Appendix C: *Maximal SO(16)=D_8 related embeddings of E_8 at height 248*

Figs. 19-20

Appendix D: *Mathematica^{TM} code and output showing $E_8 \leftrightarrow H_4$ isomorphism*

Figs. 21-23

ListName= C600atE8

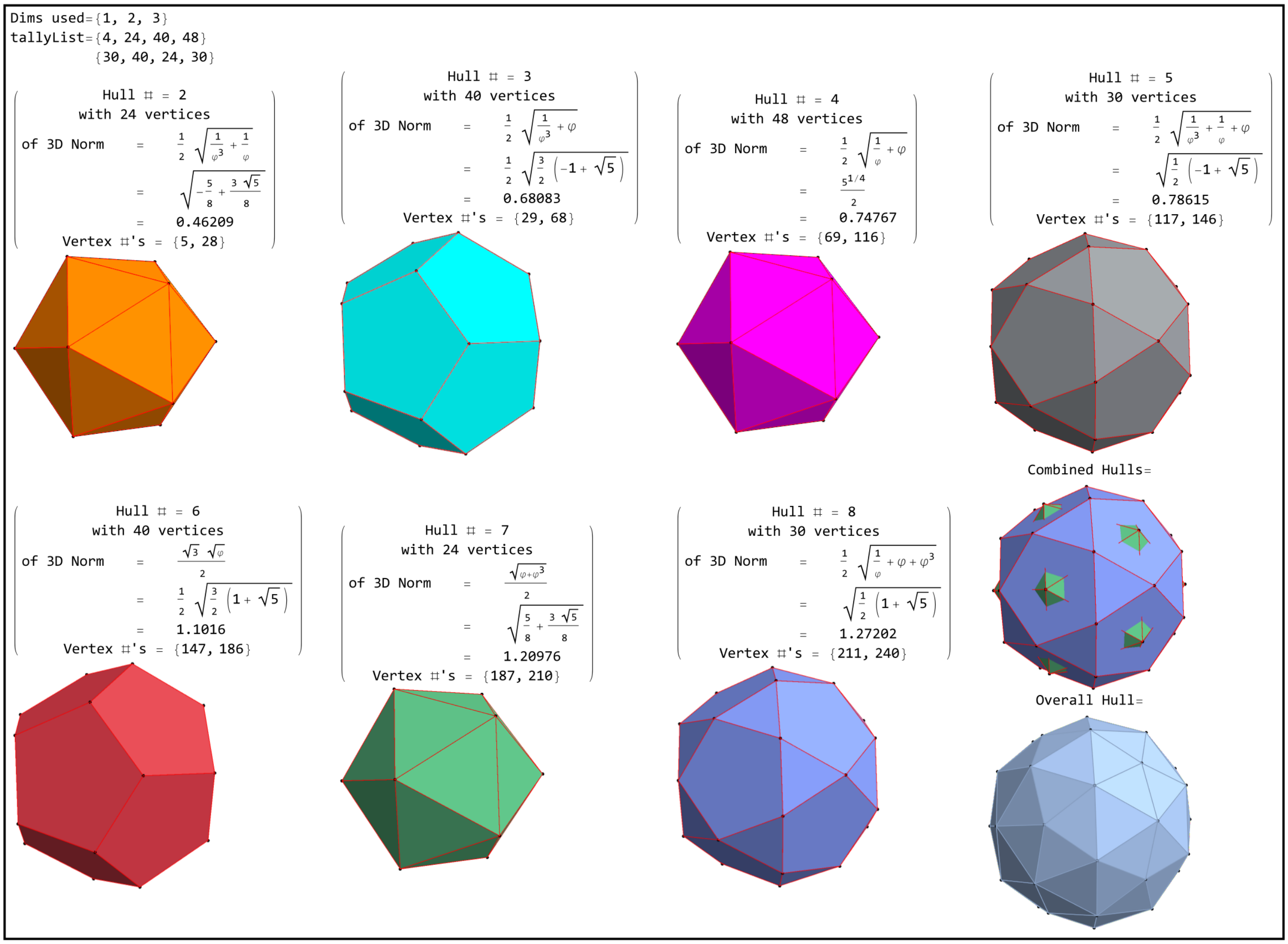
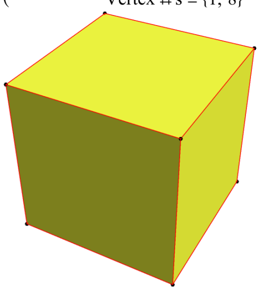


FIG. 14. Concentric hulls of 4_{21} in Platonic 3D projection with numeric and symbolic norm distances

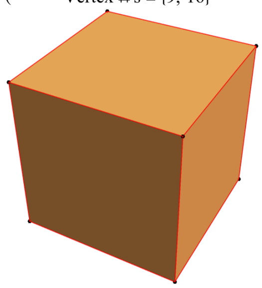
In[ ]:=
```
Ip = Flatten@prq[octExpα, 1, Tp];
IpRnd = rndOct /@ %;
IpList = oct2List@# & /@ %%;
hulls3DPerms["IpList", False, , 1]
```
ListName= IpList

Dims used={1, 2, 3}
tallyList={8, 8, 12, 24}
{24, 24, 8, 12}

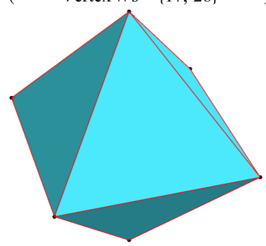
Hull ⌗ = 1
with 8 vertices
of 3D Norm $= \sqrt{\frac{1}{8}(\varphi-1)^2+\frac{1}{4\varphi^2}}$
$= \frac{1}{4}\sqrt{9-3\sqrt{5}}$
$= 0.37847$
Vertex ⌗'s = {1, 8}



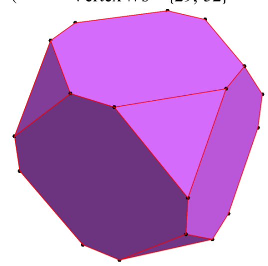
Hull ⌗ = 2
with 8 vertices
of 3D Norm $= \frac{\sqrt{\varphi^2+\frac{1}{\varphi^2}}}{2\sqrt{2}}$
$= \frac{\sqrt{\frac{3}{2}}}{2}$
$= 0.61237$
Vertex ⌗'s = {9, 16}



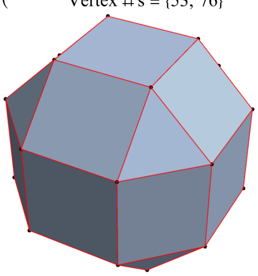
Hull ⌗ = 3
with 12 vertices
of 3D Norm $= \frac{1}{\sqrt{2}}$
$= \frac{1}{\sqrt{2}}$
$= 0.70711$
Vertex ⌗'s = {17, 28}



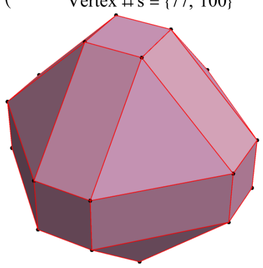
Hull ⌗ = 4
with 24 vertices
of 3D Norm $= \frac{\sqrt{\varphi^2+2+\frac{2}{\varphi^2}}}{2\sqrt{2}}$
$= \frac{\sqrt{13-\sqrt{5}}}{4}$
$= 0.82021$
Vertex ⌗'s = {29, 52}



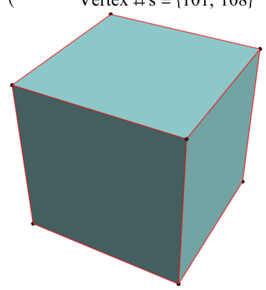
Hull ⌗ = 5
with 24 vertices
of 3D Norm $= \frac{\sqrt{2\varphi^2+1+\frac{2}{\varphi^2}}}{2\sqrt{2}}$
$= \frac{\sqrt{\frac{7}{2}}}{2}$
$= 0.93541$
Vertex ⌗'s = {53, 76}



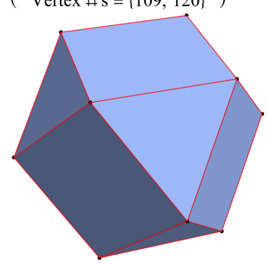
Hull ⌗ = 6
with 24 vertices
of 3D Norm $= \frac{\sqrt{4(\varphi^4+\varphi^2)+2}}{4\varphi}$
$= \frac{\sqrt{13+\sqrt{5}}}{4}$
$= 0.97584$
Vertex ⌗'s = {77, 100}



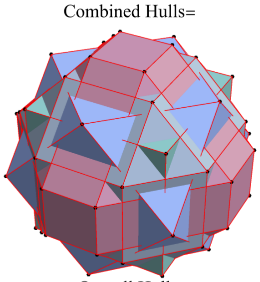
Hull ⌗ = 7
with 8 vertices
of 3D Norm $= \frac{\sqrt{\varphi(2\varphi^3+\varphi+2)+1}}{2\sqrt{2}\varphi}$
$= \frac{1}{8}\left(\sqrt{6}+\sqrt{30}\right)$
$= 0.99084$
Vertex ⌗'s = {101, 108}



Hull ⌗ = 8
with 12 vertices
of 3D Norm $= 1$
$= 1$
$= 1.$
Vertex ⌗'s = {109, 120}



Combined Hulls=



Overall Hull=

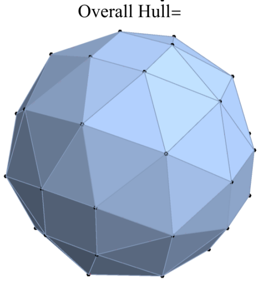


FIG. 15. Concentric hulls of I' as the parent H_4 600-cell of order 120 in Platonic 3D projection with numeric and symbolic norm distances. This is generated by $\mathtt{I}' = \mathtt{prq}[\alpha^{0-4}, \mathtt{1}, \mathtt{T}']$.

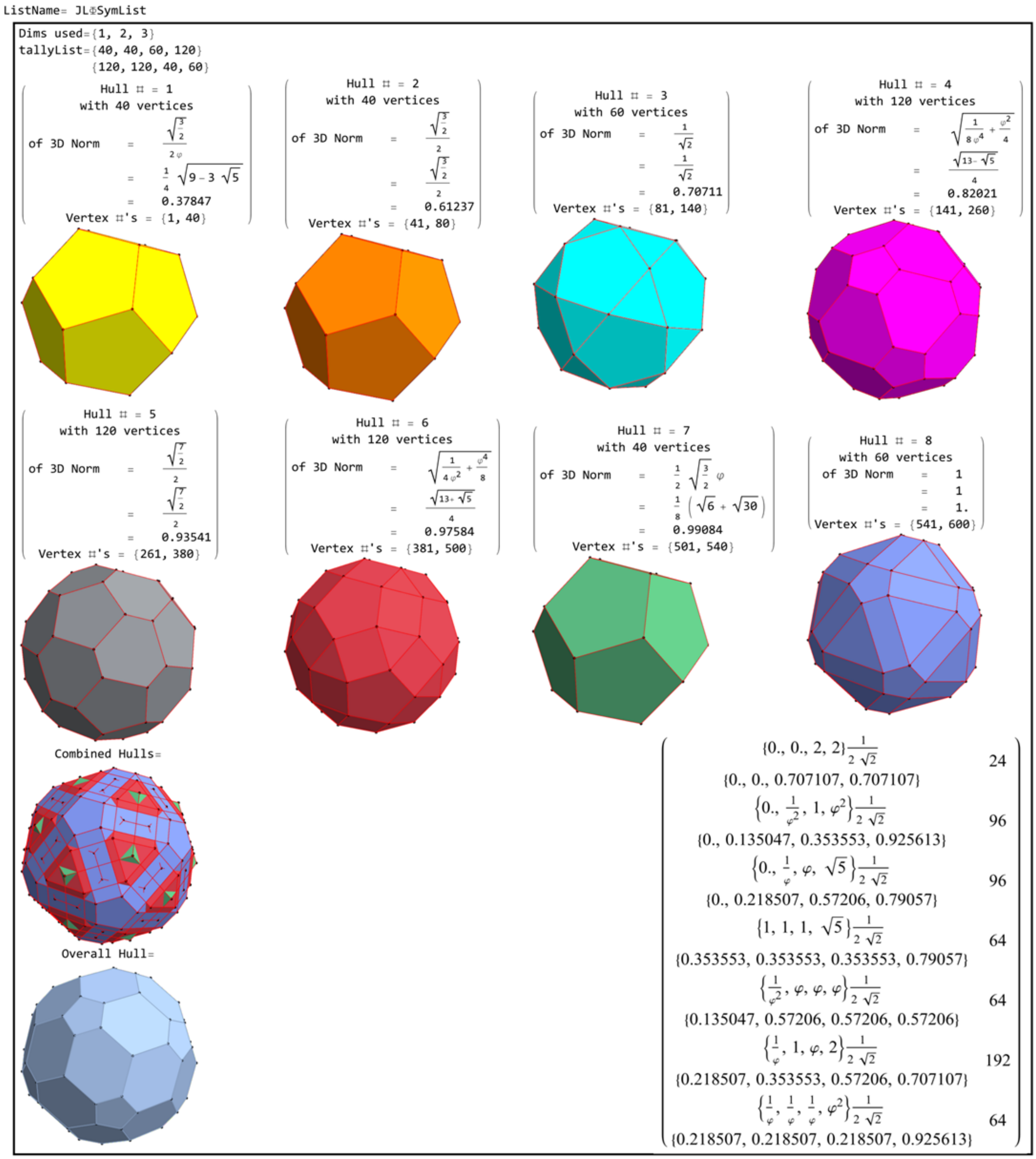


FIG. 16. Concentric hulls of J as the tri-rectified H_4 120-cell of order 600 in Platonic 3D projection with numeric and symbolic norm distances. This is generated by $\mathtt{J} = \mathtt{prq}[\mathtt{A}', 1, \mathtt{I}] = \mathtt{prq}[\mathtt{A}', \alpha^{0-4}, \mathtt{T}]$.
Note: The numeric and symbolic tally list of unpermuted vertex values in the lower-right corner

In[∘]:=
```
(* J'=prq[A',1,I'] *)
Jp = Flatten@prq[Ap, 1, Ip];
```

In[∘]:=
```
hulls3DPerms["JpList", False, , 1]
```

ListName= JpList

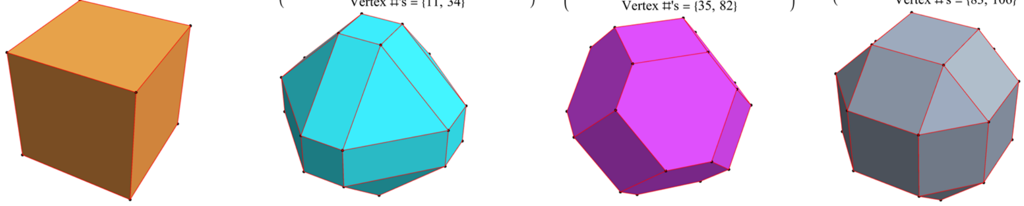
Dims used={1, 2, 3}
tallyList={2, 8, 24, 48}
{24, 8, 48, 48}
{64, 48, 24, 48}
{56, 48, 48, 54}

Hull ⌗ = 2 with 8 vertices of 3D Norm $= \frac{\sqrt{3}}{4\varphi} = \frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right) = 0.26762$; Vertex ⌗'s = {3, 10}

Hull ⌗ = 3 with 24 vertices of 3D Norm $= \frac{\sqrt{\varphi^6+2}}{4\varphi^2} = \frac{1}{4}\sqrt{\frac{1}{2}\left(17-5\sqrt{5}\right)} = 0.42646$; Vertex ⌗'s = {11, 34}

Hull ⌗ = 4 with 48 vertices of 3D Norm $= \frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}} = \sqrt{\frac{1}{4}+\frac{1}{(1+\sqrt{5})^2}} = 0.58779$; Vertex ⌗'s = {35, 82}

Hull ⌗ = 5 with 24 vertices of 3D Norm $= \frac{1}{8}\sqrt{\frac{(\varphi^2+\varphi+3)^2}{\varphi^2}+8} = \frac{\sqrt{7}}{4} = 0.66144$; Vertex ⌗'s = {83, 106}

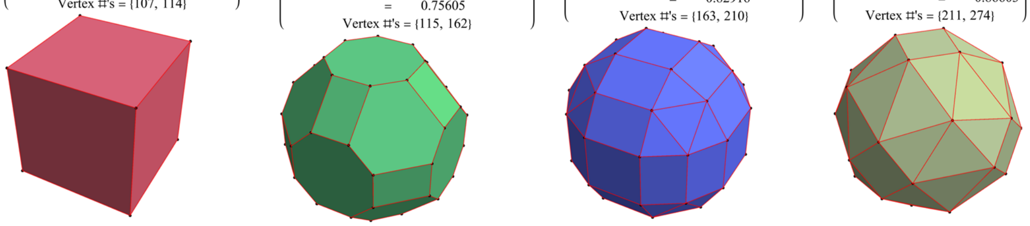


Hull ⌗ = 6 with 8 vertices of 3D Norm $= \frac{\sqrt{3}\,\varphi}{4} = \frac{1}{8}\sqrt{3}\left(1+\sqrt{5}\right) = 0.70063$; Vertex ⌗'s = {107, 114}

Hull ⌗ = 7 with 48 vertices of 3D Norm $= \frac{\sqrt{\varphi^6+\varphi^2+3+\frac{1}{\varphi^2}}}{4\varphi} = \frac{1}{4}\sqrt{\frac{1}{2}\left(25-3\sqrt{5}\right)} = 0.75605$; Vertex ⌗'s = {115, 162}

Hull ⌗ = 8 with 48 vertices of 3D Norm $= \sqrt{\frac{(\varphi^2+\varphi+3)^2}{32\varphi^2}+\frac{1}{16}} = \frac{\sqrt{11}}{4} = 0.82916$; Vertex ⌗'s = {163, 210}

Hull ⌗ = 9 with 64 vertices of 3D Norm $= \frac{\sqrt{\varphi^4+1}}{2\varphi} = \frac{\sqrt{3}}{2} = 0.86603$; Vertex ⌗'s = {211, 274}

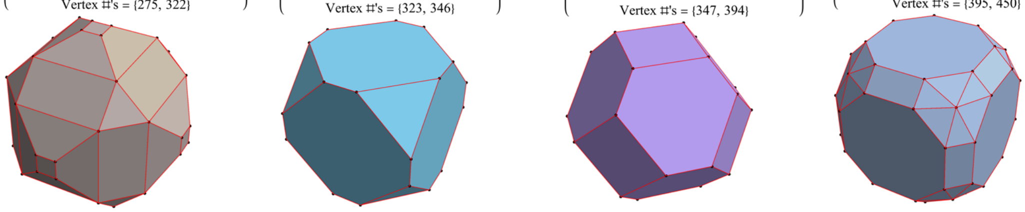


Hull ⌗ = 10 with 48 vertices of 3D Norm $= \frac{1}{4}\sqrt{2\varphi^2+\frac{(\varphi+3)^2}{\varphi^2}} = \frac{1}{8}\sqrt{58-2\sqrt{5}} = 0.91453$; Vertex ⌗'s = {275, 322}

Hull ⌗ = 11 with 24 vertices of 3D Norm $= \frac{\sqrt{2\varphi^6+1}}{4\varphi} = \sqrt{\frac{17}{32}+\frac{5\sqrt{5}}{32}} = 0.93842$; Vertex ⌗'s = {323, 346}

Hull ⌗ = 12 with 48 vertices of 3D Norm $= \frac{\sqrt{\varphi^2+1}}{2} = \frac{1}{2}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)} = 0.95106$; Vertex ⌗'s = {347, 394}

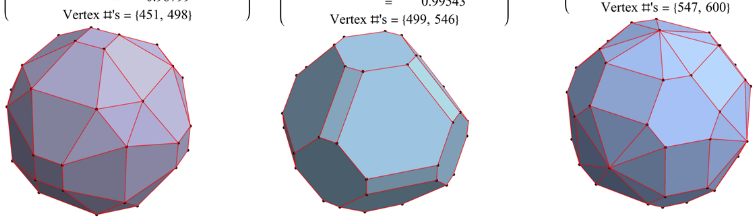
Hull ⌗ = 13 with 56 vertices of 3D Norm $= \frac{1}{8}\sqrt{3}\left(\varphi+1+\frac{3}{\varphi}\right) = \frac{\sqrt{15}}{4} = 0.96825$; Vertex ⌗'s = {395, 450}



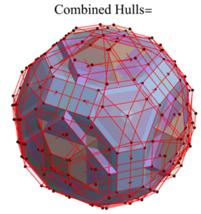
Hull ⌗ = 14 with 48 vertices of 3D Norm $= \frac{\sqrt{(2\varphi+3)^2+2}}{4\varphi} = \frac{1}{4}\sqrt{\frac{1}{2}\left(29+\sqrt{5}\right)} = 0.98799$; Vertex ⌗'s = {451, 498}

Hull ⌗ = 15 with 48 vertices of 3D Norm $= \frac{\sqrt{\varphi^6+(\varphi+2)^2\varphi^4+1}}{4\varphi^2} = \sqrt{\frac{25}{32}+\frac{3\sqrt{5}}{32}} = 0.99543$; Vertex ⌗'s = {499, 546}

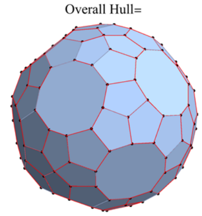
Hull ⌗ = 16 with 54 vertices of 3D Norm $= \frac{\sqrt{\varphi^4+\varphi^2+1}}{2\varphi} = 1 = 1.$; Vertex ⌗'s = {547, 600}



Combined Hulls=



Overall Hull=



In[∘]:=
```
tallyGroupPerms[Jp, 2]
```

Out[∘]//MatrixForm=

$\{0., 0., 0., 2\}\frac{1}{2}$ {0., 0., 0., 1.}	8
$\left\{0., \frac{1}{\varphi}, 1, \varphi\right\}\frac{1}{2}$ {0., 0.309015, 0.5, 0.809015}	192
$\left\{\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{5}}{2}, \frac{3}{2}\right\}\frac{1}{2}$ {0.25, 0.25, 0.559015, 0.75}	96
$\left\{\frac{1}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right\}\frac{1}{2}$ {0.25, 0.559015, 0.559015, 0.559015}	32
$\{1, 1, 1, 1\}\frac{1}{2}$ {0.5, 0.5, 0.5, 0.5}	16
$\left\{\frac{1}{2\varphi^2}, \frac{1}{2\varphi^2}, \frac{\varphi}{2}, \frac{\varphi+2}{2}\right\}\frac{1}{2}$ {0.09549, 0.09549, 0.40451, 0.90451}	96
$\left\{\frac{1}{2\varphi}, \frac{1}{2}\left(\frac{1}{\varphi^2}+1\right), \frac{\varphi^2}{2}, \frac{\varphi^2}{2}\right\}\frac{1}{2}$ {0.15451, 0.34549, 0.65451, 0.65451}	96
$\left\{\frac{1}{2\varphi}, \frac{1}{2\varphi}, \frac{1}{2\varphi}, \frac{1}{2}\left(\frac{3}{\varphi}+2\right)\right\}\frac{1}{2}$ {0.15451, 0.15451, 0.15451, 0.963525}	32
$\left\{\frac{\varphi}{2}, \frac{\varphi}{2}, \frac{\varphi}{2}, \frac{1}{2}\left(\frac{3}{\varphi}+1\right)\right\}\frac{1}{2}$ {0.40451, 0.40451, 0.40451, 0.713525}	32

FIG. 17. Concentric hulls of J' as the tri-rectified H_4 120-cell of order 600 in Platonic 3D projection with numeric and symbolic norm distances. This is generated by $\mathtt{J'} = \mathtt{prq[A', 1, I']} = \mathtt{prq[A',}\beta^{0-4}\mathtt{, T']}$.
Note: The numeric and symbolic tally list of unpermuted vertex values in the lower-right corner

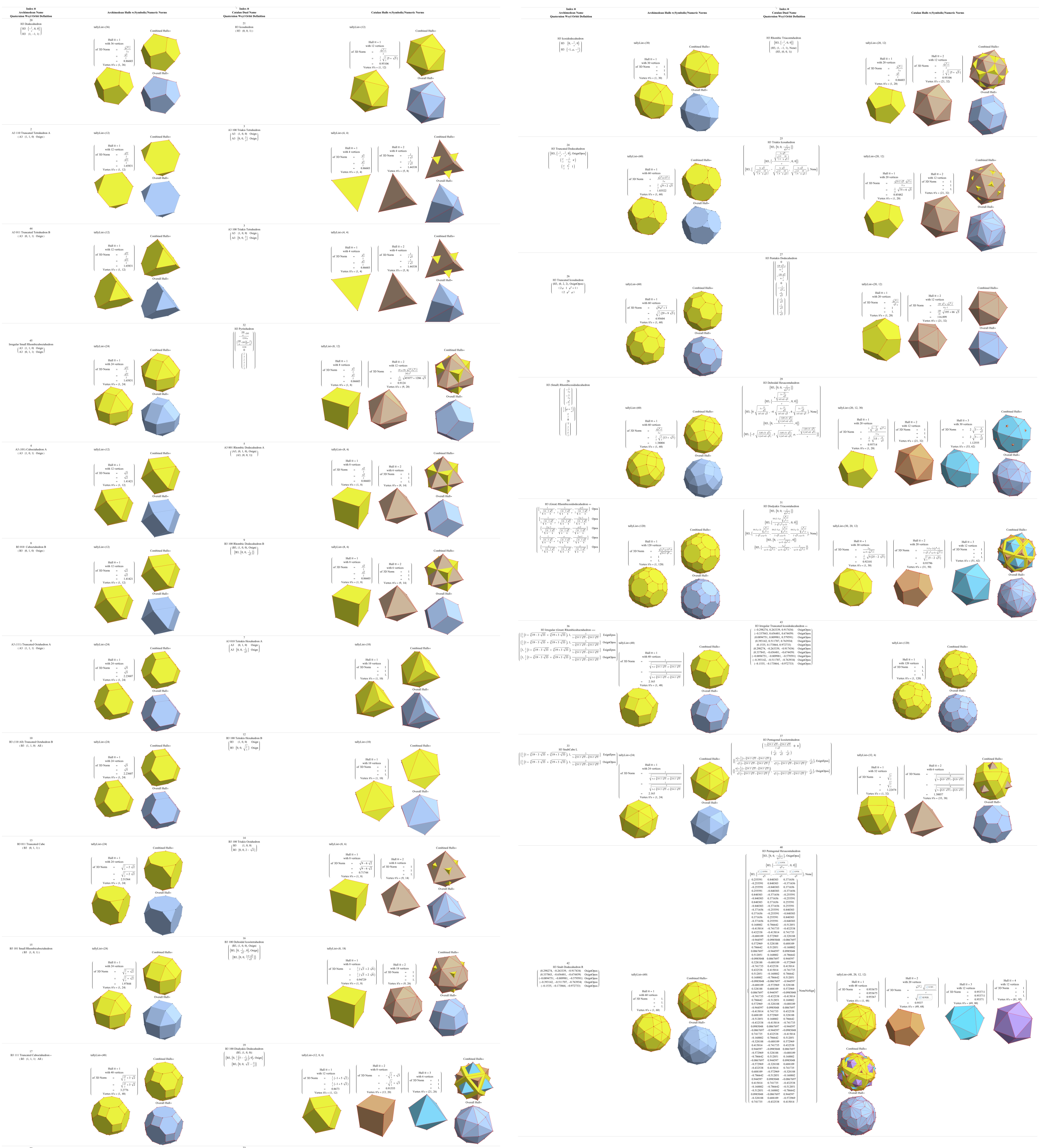
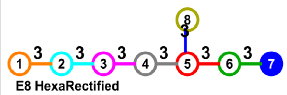
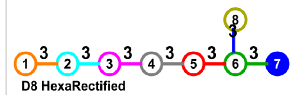


FIG. 18. *Archimedean and dual Catalan solids, including their irregular and chiral forms. These were created using quaternion Weyl orbits directly from the A_3, B_3, and H_3 group symmetries[4] listed in the first column.*

a)

SO16 Projection Matrix

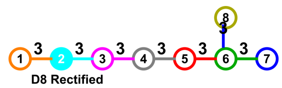
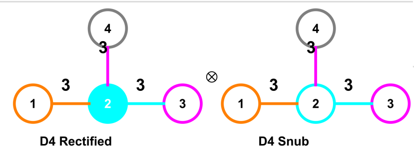
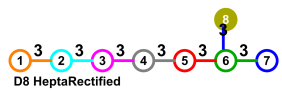
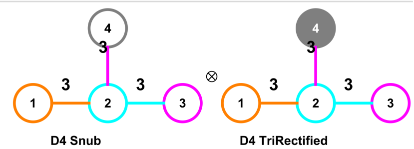
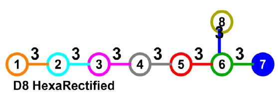
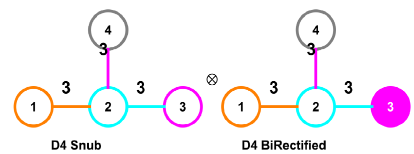
$$\begin{pmatrix} -2 & -4 & -6 & -5 & -4 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Group Coxeter–Dynkin Diagram	Coxeter–Dynkin Weyl Orbit	E8 Rep	SO16 Content	Subgroup Coxeter–Dynkin Diagrams
	{0, 0, 0, 0, 0, 0, 1, 0}	**248**	**{120, 128'}**	

b)

SO8⊗SO8 Projection Matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -2 & -2 & -2 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

	{0, 1, 0, 0, 0, 0, 0, 0}	**120**	$\{28\otimes 1,\ 1\otimes 28,\ 8_v^{\otimes 2}\}$	
	{0, 0, 0, 0, 0, 0, 0, 1}	**128**	$\{8_c^{\otimes 2},\ 8_s^{\otimes 2}\}$	
	{0, 0, 0, 0, 0, 0, 1, 0}	**128'**	$\{8_c\otimes 8_s,\ 8_s\otimes 8_c\}$	

c)

SP8⊗SP8 Projection Matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

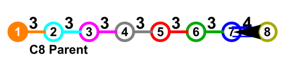
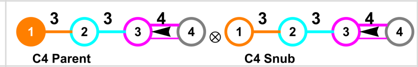
Group Coxeter–Dynkin Diagram	Coxeter–Dynkin Weyl Orbit	SP16 Rep	SP8⊗SP8 Content	Subgroup Coxeter–Dynkin Diagrams
	{2, 0, 0, 0, 0, 0, 0, 0}	**136**	$\{36\otimes 1,\ 1\otimes 36,\ 8^{\otimes 2}\}$	

FIG. 19. Breakdown of E_8 maximal embeddings at height 248 of content SO(16)=D_8 (120,128')
a) Height 248 SO(16) content 120=(112+4+4)+128'
b) Height 120 and 128' SO(8)⊗SO(8) content w/$8^{\otimes 2}_{v,c,s}$ triality
c) Height 136 Sp(8)⊗Sp(8) content (32+4)⊗1, 1⊗(32+4), $8^{\otimes 2}$
Note: This output was created in *Mathematica*™ with support from the GroupMath[21] and SuperLie[22] packages.

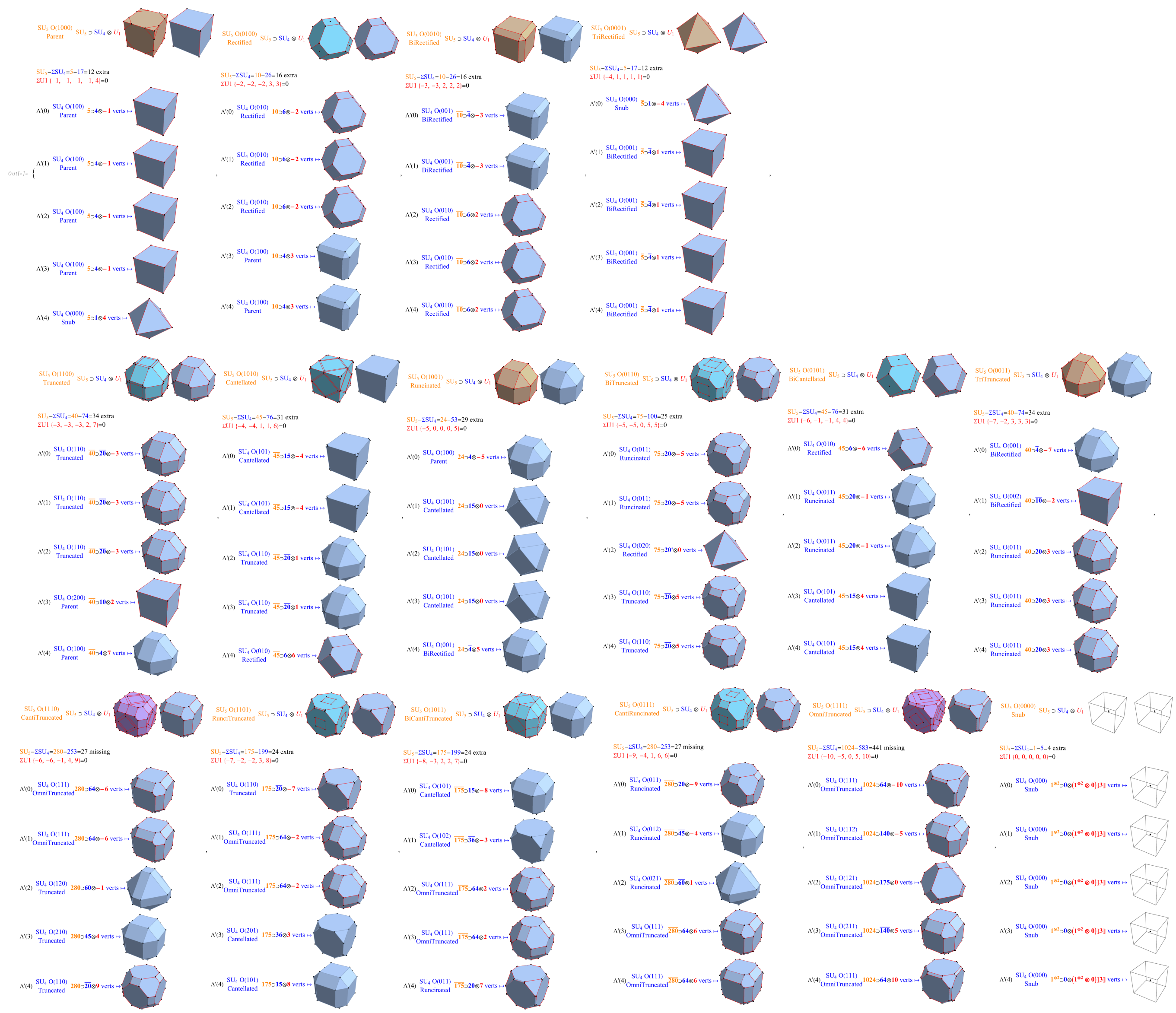


FIG. 20. A_3 in A_4 embeddings of SU(5)⊃SU(4)⊗U_1
These include the specified 3D quaternion Weyl orbit hulls for each subgroup identified.

```
(* This switches the H4 (L)eft side scale to the (R)ight side scale (and vice-versa).
   We don't use scaleBy if it is a snub 24-cell vertices. *)
switchScale[in_, scaleBy_ : 1] := (* We don't use scaleBy if it is a snub 24-cell vertices. *)
  If[Length@Union@Flatten@Abs@oct2List@N[in /. φRep] == 2,
   in, scaleBy in /. slRep];

In[∘]:= (* Replacement order is critical *)
mapLRrep = # /. slRep & /@ {
    (* φ^±3 Scale changing: Exchange the ±φ^2↔±1/φ  and ±φ^±3/2↔±φ^∓3/2 *)
    1/ϕsw^2 → φ, ϕsw^2 → 1/φ, ϕsw^-3/2 → φ^3/2, ϕsw^3/2 → φ^-3/2,
    (* Sign changing: Exchange the ±√φ↔∓√φ & ±1/√φ↔∓1/√φ, and ±1/φ↔∓1/φ*)
    √ϕsw → -√φ, √(1/ϕsw) → -√(1/φ), 1/ϕsw → -1/φ,
    (* Final φ^±3 Scale changing: ±φ↔±1/φ^2 *)
    ϕsw → 1/φ^2 (**)};

In[∘]:= (* This processes only individual vertices with a symbolic list input. *)
mapLR[in_, scaleBy_ : 1, 𝕌Det1fCorrection_ : True] := Module[{(**)input, output(**)},
   (* Correct for use of √ϕ in 𝕌,
   which produces i values (which may be desired?) *)
   input = If[curr𝕌 == 11 || ! 𝕌Det1fCorrection, in, FullSimplify[in 𝕌Det1f /. slRep, Assumptions → φAssumptions]];
   output = FullSimplify[switchScale[octSym@input /. φ → ϕsw /. mapLRrep, scaleBy] ×
      (* Correct back *)
      If[curr𝕌 == 11 || ! 𝕌Det1fCorrection, 1, 1 / 𝕌Det1f] /. slRep, Assumptions → φAssumptions];
   (* curr𝕌<9 don't reverse the L↔R ordering *)
   If[curr𝕌 < 9, output, Join[Reverse[output⟦ ;; 4⟧], Array[0 &, Length[output] - 4]]]];

(* List and verify the operation of mapLR - one for h4⊕ and one for h4 *)
genE8fromH4@in_String := Module[{indx, inH4 = If[in == "H4⊕", h4⊕, h4], i, j, left, right, h4LR},
   (* Style the Heading in Bold, 24-cell rows in Red, and p48 constraint members marked with an * *)
   Style[#,
       {If[MemberQ[If[in == "H4⊕", h4⊕cell24, h4cell24], indx], Red, Black],
        If[Head@indx === String, Bold, Plain]}] & /@ (indx = #⟦2⟧; #) & /@
    Join[
     (* The Heading row *)
     {{"#", in <> " #", If[labels, "pLbl", Nothing],
       Column[{"E8 vertex", "E8.𝕌=" <> in <> "_L" <> "⊕" <> in <> "_R"}, Center],
       Column[{If[curr𝕌 == 11, "", "2 "] <> in <> "_L", "mapLR(" <> in <> "_L)=" <> in <> "_R"}, Center],
       Column[{If[curr𝕌 == 11, "", "2 "] <> in <> "_R", "mapLR(" <> in <> "_R)=" <> in <> "_L"}, Center],
       Column[{"", "(" <> in <> "_L" <> "⊕" <> in <> "_R" <> ").𝕌^-1=E8 vertex"}, Center],
       Column[{"E8→" <> in <> "_L" <> "⊕" <> in <> "_R" <> "≡", in <> "_L" <> "⊕" <> in <> "_R" <> "→E8"}, Center]}},
     (* Generate data row content *)
     {ToString@# <> If[MemberQ[p48L, #], "*", " "], (* h4⊕⟦#⟧ is an E8 index number to an E8 element in h4⊕ *)
        inH4⟦#⟧, If[labels, pLbl@inH4⟦#⟧, Nothing],
        (* Show the E8 vertex *)
        i = pE8@inH4⟦#⟧,
        (* pC600 is converts from E8→H4 using 𝕌, here we take the H4 4D left side *)
        If[curr𝕌 == 11, 1, 2] (left = octSym[pC600[inH4⟦#⟧]⟦ ;; 4⟧] /. φRuleList),
        (* mapLR converts the H4 4D left side vertex to its corresponding H4 4D right-side vertex,
        which when Joined gives the 8D H4 that can be converted back to E8 by using 𝕌Inv *)
        If[curr𝕌 == 11, 1, 2] (right = mapLR@left /. φRuleList),
        (* Conditionally print some cross-checks *)
        print["#=", #, " h4⟦#⟧=", inH4⟦#⟧, " E8.𝕌=", octSym[pC600[inH4⟦#⟧]] /. φRuleList, " left=", left, " right=", Reverse@right];
        print["   E8.𝕌==Join[left,mapLR@left]=", N@Join[left, right] == N@octSym[pC600[inH4⟦#⟧]] /. φRuleList];
        print["   mapLR@right", If[curr𝕌 == 11, 1, 2] (mapLR@right /. φRuleList)];
        print["   left==mapLR@right=", N@left == N@mapLR@right /. φRuleList];
        (* Show the H4_L⊕H4_R.𝕌Inv vertex *)
        h4LR = Join[left, right];
        j = Rationalize@FullSimplify[Chop[h4LR.𝕌Inv /. φRep, chop], Assumptions → φAssumptions],
        (* Check that E8→H4→H4_L⊕H4_R→E8 *)
        j == N@i} /. slRep & /@ Range@120] // MatrixForm];
```

FIG. 21. *Mathematica*™ code to generate the output showing $E_8 \leftrightarrow H_4$ isomorphism

```
In[]:= currU = 9; setU;
(octSym@# → mapLR@# /. slRep) & /@ ({
    {1 / τ, 0, -1, 1 / τ^2},
    {-1 / τ, 0, 1, -1 / τ^2},
    {τ, 0, -1, 1 / τ},
    {-τ, 0, 1, -1 / τ}} / If[currU == 11, 1, UDet1f] /. slRep) // MatrixForm

(* 24-cell rows in Red and p48 constraint members marked with an * *)
genE8fromH4@"H4"
```

Out[]//MatrixForm=

#	H4 #	E8 vertex E8.U=H4$_L$⊕H4$_R$	2 H4$_L$ mapLR(H4$_L$)=H4$_R$	2 H4$_R$ mapLR(H4$_R$)=H4$_L$	(H4$_L$⊕H4$_R$).U^{-1}=E8 vertex	E8→H4$_L$⊕H4$_R$≡ H4$_L$⊕H4$_R$→E8
1	13	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0.\}$	$\{0,-\varphi^{3/2},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
2*	14	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},-\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
3*	15	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
4*	20	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},0,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
5	24	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
6	30	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
7	32	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
8	33	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
9	34	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0.\}$	$\{0,-\varphi^{3/2},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
10*	35	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},0,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
11	36	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},-\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
12	37	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,-\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
13	38	$\{-1,-1,0,0,0,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{-1,-1,0,0,0,0,0,0\}$	True
14*	39	$\{-1,0,-1,0,0,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},\varphi^{3/2}\}$	$\{-1,0,-1,0,0,0,0,0\}$	True
15	40	$\{-1,0,0,-1,0,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},0.,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,\varphi^{3/2}\}$	$\{-1,0,0,-1,0,0,0,0\}$	True
16	41	$\{-1,0,0,0,-1,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},0.,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,\varphi^{3/2}\}$	$\{-1,0,0,0,-1,0,0,0\}$	True
17*	42	$\{-1,0,0,0,0,-1,0,0\}$	$\{\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},\varphi^{3/2}\}$	$\{-1,0,0,0,0,-1,0,0\}$	True
18	43	$\{-1,0,0,0,0,0,-1,0\}$	$\{\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{-1,0,0,0,0,0,-1,0\}$	True
19	45	$\{-1,0,0,0,0,0,0,1\}$	$\{-2\sqrt{\frac{1}{\varphi}},0.,0.,0.\}$	$\{0,0,0,\frac{2}{\sqrt{\varphi}}\}$	$\{-1,0,0,0,0,0,0,1\}$	True
20	46	$\{-1,0,0,0,0,0,1,0\}$	$\{\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{-1,0,0,0,0,0,1,0\}$	True
21*	47	$\{-1,0,0,0,0,1,0,0\}$	$\{\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},\varphi^{3/2}\}$	$\{-1,0,0,0,0,1,0,0\}$	True
22*	48	$\{-1,0,0,0,1,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},0.,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,\varphi^{3/2}\}$	$\{-1,0,0,0,1,0,0,0\}$	True
23	49	$\{-1,0,0,1,0,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},0.,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,\varphi^{3/2}\}$	$\{-1,0,0,1,0,0,0,0\}$	True
24	50	$\{-1,0,1,0,0,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},\varphi^{3/2}\}$	$\{-1,0,1,0,0,0,0,0\}$	True
25	51	$\{-1,1,0,0,0,0,0,0\}$	$\{\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{-1,1,0,0,0,0,0,0\}$	True
26	52	$\{0,-1,-1,0,0,0,0,0\}$	$\{0.,-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\varphi^{3/2},0\}$	$\{0,-1,-1,0,0,0,0,0\}$	True
27	53	$\{0,-1,0,-1,0,0,0,0\}$	$\{0.,\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0\}$	$\{0,-1,0,-1,0,0,0,0\}$	True
28*	55	$\{0,-1,0,0,0,-1,0,0\}$	$\{0.,-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\varphi^{3/2},0\}$	$\{0,-1,0,0,0,-1,0,0\}$	True
29*	59	$\{0,-1,0,0,0,0,1,0\}$	$\{0.,2\sqrt{\frac{1}{\varphi}},0.,0.\}$	$\{0,0,-\frac{2}{\sqrt{\varphi}},0\}$	$\{0,-1,0,0,0,0,1,0\}$	True
30	61	$\{0,-1,0,0,1,0,0,0\}$	$\{0.,\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0\}$	$\{0,-1,0,0,1,0,0,0\}$	True
31	64	$\{0,0,-1,-1,0,0,0,0\}$	$\{0.,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	$\{0,0,-1,-1,0,0,0,0\}$	True
32	65	$\{0,0,-1,0,-1,0,0,0\}$	$\{0.,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	$\{0,0,-1,0,-1,0,0,0\}$	True
33	70	$\{0,0,-1,0,0,0,1,0\}$	$\{0.,-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\varphi^{3/2},0\}$	$\{0,0,-1,0,0,0,1,0\}$	True
34*	71	$\{0,0,-1,0,0,1,0,0\}$	$\{0.,0.,0.,2\sqrt{\frac{1}{\varphi}}\}$	$\{-\frac{2}{\sqrt{\varphi}},0,0,0\}$	$\{0,0,-1,0,0,1,0,0\}$	True
35*	76	$\{0,0,0,-1,0,0,-1,0\}$	$\{0.,-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0\}$	$\{0,0,0,-1,0,0,-1,0\}$	True
36	80	$\{0,0,0,-1,0,1,0,0\}$	$\{0.,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	$\{0,0,0,-1,0,1,0,0\}$	True
37	81	$\{0,0,0,-1,1,0,0,0\}$	$\{0.,0.,2\sqrt{\frac{1}{\varphi}},0.\}$	$\{0,-\frac{2}{\sqrt{\varphi}},0,0\}$	$\{0,0,0,-1,1,0,0,0\}$	True
38	86	$\{0,0,0,0,-1,0,1,0\}$	$\{0.,\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0\}$	$\{0,0,0,0,-1,0,1,0\}$	True
39*	87	$\{0,0,0,0,-1,1,0,0\}$	$\{0.,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	$\{0,0,0,0,-1,1,0,0\}$	True
40*	91	$\{0,0,0,0,0,-1,1,0\}$	$\{0.,-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\varphi^{3/2},0\}$	$\{0,0,0,0,0,-1,1,0\}$	True
41	94	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
42*	95	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
43*	96	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
44*	100	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
45*	101	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0.\}$	$\{0,\varphi^{3/2},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
46*	103	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
47	106	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
48*	107	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},0,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
49	109	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},-\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
50*	111	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
51	113	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0.\}$	$\{0,-\varphi^{3/2},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
52	114	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
53*	115	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
54	120	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
55*	122	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
56	124	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
57	125	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
58*	126	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
59	127	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
60	128	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},0,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
61	129	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},0,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
62	130	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},\varphi^{3/2},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
63*	131	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
64	132	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
65	133	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
66*	135	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
67	137	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
68*	142	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
69	143	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
70	144	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0.\}$	$\{0,\varphi^{3/2},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
71*	146	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
72	148	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	$\{-\varphi^{3/2},\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
73*	150	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},0,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
74	151	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},-\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
75*	154	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
76*	156	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0.\}$	$\{0,-\varphi^{3/2},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
77*	157	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
78*	161	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,-\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
79*	162	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
80	163	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\sqrt{\varphi}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
81*	166	$\{0,0,0,0,0,1,-1,0\}$	$\{0.,\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\varphi^{3/2},0\}$	$\{0,0,0,0,0,1,-1,0\}$	True
82*	170	$\{0,0,0,0,1,-1,0,0\}$	$\{0.,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	$\{0,0,0,0,1,-1,0,0\}$	True
83	171	$\{0,0,0,0,1,0,-1,0\}$	$\{0.,-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0\}$	$\{0,0,0,0,1,0,-1,0\}$	True
84	176	$\{0,0,0,1,-1,0,0,0\}$	$\{0.,0.,-2\sqrt{\frac{1}{\varphi}},0.\}$	$\{0,\frac{2}{\sqrt{\varphi}},0,0\}$	$\{0,0,0,1,-1,0,0,0\}$	True
85	177	$\{0,0,0,1,0,-1,0,0\}$	$\{0.,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	$\{0,0,0,1,0,-1,0,0\}$	True
86*	181	$\{0,0,0,1,0,0,1,0\}$	$\{0.,\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0\}$	$\{0,0,0,1,0,0,1,0\}$	True
87*	186	$\{0,0,1,0,0,-1,0,0\}$	$\{0.,0.,0.,-2\sqrt{\frac{1}{\varphi}}\}$	$\{\frac{2}{\sqrt{\varphi}},0,0,0\}$	$\{0,0,1,0,0,-1,0,0\}$	True
88	187	$\{0,0,1,0,0,0,-1,0\}$	$\{0.,\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\varphi^{3/2},0\}$	$\{0,0,1,0,0,0,-1,0\}$	True
89	192	$\{0,0,1,0,1,0,0,0\}$	$\{0.,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	$\{0,0,1,0,1,0,0,0\}$	True
90	193	$\{0,0,1,1,0,0,0,0\}$	$\{0.,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	$\{0,0,1,1,0,0,0,0\}$	True
91	196	$\{0,1,0,0,-1,0,0,0\}$	$\{0.,-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0\}$	$\{0,1,0,0,-1,0,0,0\}$	True
92*	198	$\{0,1,0,0,0,0,-1,0\}$	$\{0.,-2\sqrt{\frac{1}{\varphi}},0.,0.\}$	$\{0,0,\frac{2}{\sqrt{\varphi}},0\}$	$\{0,1,0,0,0,0,-1,0\}$	True
93*	202	$\{0,1,0,0,0,1,0,0\}$	$\{0.,\frac{1}{\varphi^{3/2}},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\varphi^{3/2},0\}$	$\{0,1,0,0,0,1,0,0\}$	True
94	204	$\{0,1,0,1,0,0,0,0\}$	$\{0.,-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0\}$	$\{0,1,0,1,0,0,0,0\}$	True
95	205	$\{0,1,1,0,0,0,0,0\}$	$\{0.,\frac{1}{\varphi^{3/2}},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\varphi^{3/2},0\}$	$\{0,1,1,0,0,0,0,0\}$	True
96	206	$\{1,-1,0,0,0,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{1,-1,0,0,0,0,0,0\}$	True
97	207	$\{1,0,-1,0,0,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},-\varphi^{3/2}\}$	$\{1,0,-1,0,0,0,0,0\}$	True
98	208	$\{1,0,0,-1,0,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},0.,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,-\varphi^{3/2}\}$	$\{1,0,0,-1,0,0,0,0\}$	True
99*	209	$\{1,0,0,0,-1,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},0.,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,-\varphi^{3/2}\}$	$\{1,0,0,0,-1,0,0,0\}$	True
100*	210	$\{1,0,0,0,0,-1,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},-\varphi^{3/2}\}$	$\{1,0,0,0,0,-1,0,0\}$	True
101	211	$\{1,0,0,0,0,0,-1,0\}$	$\{-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{1,0,0,0,0,0,-1,0\}$	True
102	212	$\{1,0,0,0,0,0,0,-1\}$	$\{2\sqrt{\frac{1}{\varphi}},0.,0.,0.\}$	$\{0,0,0,-\frac{2}{\sqrt{\varphi}}\}$	$\{1,0,0,0,0,0,0,-1\}$	True
103	214	$\{1,0,0,0,0,0,1,0\}$	$\{-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{1,0,0,0,0,0,1,0\}$	True
104*	215	$\{1,0,0,0,0,1,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},-\varphi^{3/2}\}$	$\{1,0,0,0,0,1,0,0\}$	True
105	216	$\{1,0,0,0,1,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},0.,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,-\varphi^{3/2}\}$	$\{1,0,0,0,1,0,0,0\}$	True
106	217	$\{1,0,0,1,0,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},0.,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,-\varphi^{3/2}\}$	$\{1,0,0,1,0,0,0,0\}$	True
107*	218	$\{1,0,1,0,0,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},-\varphi^{3/2}\}$	$\{1,0,1,0,0,0,0,0\}$	True
108	219	$\{1,1,0,0,0,0,0,0\}$	$\{-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{1,1,0,0,0,0,0,0\}$	True
109	220	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
110	221	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
111*	222	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},0,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
112	223	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0.\}$	$\{0,\varphi^{3/2},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
113	224	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},\varphi^{3/2},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
114	225	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
115	227	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
116	233	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
117*	237	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},0,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
118*	242	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
119*	243	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	$\{\varphi^{3/2},\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
120	244	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0.\}$	$\{0,\varphi^{3/2},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True

FIG. 22. Output showing detail of $E_8 \leftrightarrow H_4$(L⊕R⊕1⊕φ) isomorphism for each vertex
Note: Red rows indicate D_4 24-cell membership and the * identifies those satisfying the constraint of p∈I where $p^5 = \pm1$.

In[∘]:= (* 24-cell rows in Red and p48 constraint members marked with an * *)
genE8fromH4@"H4Φ"

Out[∘]//MatrixForm=

#	H4Φ #	E8 vertex E8.𝕌=H4Φ $_L$⊕H4Φ $_R$	2 H4Φ $_L$ mapLR(H4Φ $_L$)=H4Φ $_R$	2 H4Φ $_R$ mapLR(H4Φ $_R$)=H4Φ $_L$	(H4Φ $_L$⊕H4Φ $_R$).𝕌$^{-1}$=E8 vertex	E8→H4Φ $_L$⊕H4Φ $_R$≡ H4Φ $_L$⊕H4Φ $_R$→E8
1	1	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
2*	10	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
3*	11	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
4*	12	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\varphi^{3/2},0.\}$	$\{0,-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
5	16	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
6	17	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},0,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
7	18	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
8	19	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
9	21	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
10*	22	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
11	23	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
12	25	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
13	26	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},0,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
14*	27	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
15	28	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
16	29	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,-\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
17*	31	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\varphi^{3/2},0.\}$	$\{0,-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
18	44	{−1, 0, 0, 0, 0, 0, 0, −1}	$\{2\sqrt{\varphi},0.,0.,0.\}$	$\{0,0,0,2\sqrt{\varphi}\}$	{−1, 0, 0, 0, 0, 0, 0, −1}	True
19	54	{0, −1, 0, 0, −1, 0, 0, 0}	$\{0.,\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0\}$	{0, −1, 0, 0, −1, 0, 0, 0}	True
20	56	{0, −1, 0, 0, 0, 0, −1, 0}	$\{0.,0.,-2\sqrt{\varphi},0.\}$	$\{0,-2\sqrt{\varphi},0,0\}$	{0, −1, 0, 0, 0, 0, −1, 0}	True
21*	57	{0, −1, 0, 0, 0, 0, 0, −1}	$\{\varphi^{3/2},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	{0, −1, 0, 0, 0, 0, 0, −1}	True
22*	58	{0, −1, 0, 0, 0, 0, 0, 1}	$\{-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	{0, −1, 0, 0, 0, 0, 0, 1}	True
23	60	{0, −1, 0, 0, 0, 1, 0, 0}	$\{0.,\varphi^{3/2},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0\}$	{0, −1, 0, 0, 0, 1, 0, 0}	True
24	62	{0, −1, 0, 1, 0, 0, 0, 0}	$\{0.,\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0\}$	{0, −1, 0, 1, 0, 0, 0, 0}	True
25	63	{0, −1, 1, 0, 0, 0, 0, 0}	$\{0.,\varphi^{3/2},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0\}$	{0, −1, 1, 0, 0, 0, 0, 0}	True
26	66	{0, 0, −1, 0, 0, −1, 0, 0}	$\{0.,-2\sqrt{\varphi},0.,0.\}$	$\{0,0,-2\sqrt{\varphi},0\}$	{0, 0, −1, 0, 0, −1, 0, 0}	True
27	67	{0, 0, −1, 0, 0, 0, −1, 0}	$\{0.,-\varphi^{3/2},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0\}$	{0, 0, −1, 0, 0, 0, −1, 0}	True
28*	68	{0, 0, −1, 0, 0, 0, 0, −1}	$\{\varphi^{3/2},-\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	{0, 0, −1, 0, 0, 0, 0, −1}	True
29*	69	{0, 0, −1, 0, 0, 0, 0, 1}	$\{-\varphi^{3/2},-\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	{0, 0, −1, 0, 0, 0, 0, 1}	True
30	72	{0, 0, −1, 0, 1, 0, 0, 0}	$\{0.,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	{0, 0, −1, 0, 1, 0, 0, 0}	True
31	73	{0, 0, −1, 1, 0, 0, 0, 0}	$\{0.,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	{0, 0, −1, 1, 0, 0, 0, 0}	True
32	74	{0, 0, 0, −1, −1, 0, 0, 0}	$\{0.,0.,0.,-2\sqrt{\varphi}\}$	$\{-2\sqrt{\varphi},0,0,0\}$	{0, 0, 0, −1, −1, 0, 0, 0}	True
33	75	{0, 0, 0, −1, 0, −1, 0, 0}	$\{0.,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	{0, 0, 0, −1, 0, −1, 0, 0}	True
34*	77	{0, 0, 0, −1, 0, 0, 0, −1}	$\{\varphi^{3/2},0.,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, −1, 0, 0, 0, −1}	True
35*	78	{0, 0, 0, −1, 0, 0, 0, 1}	$\{-\varphi^{3/2},0.,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, −1, 0, 0, 0, 1}	True
36	79	{0, 0, 0, −1, 0, 0, 1, 0}	$\{0.,\frac{1}{\sqrt{\varphi}},\varphi^{3/2},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0\}$	{0, 0, 0, −1, 0, 0, 1, 0}	True
37	82	{0, 0, 0, 0, −1, −1, 0, 0}	$\{0.,-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0\}$	{0, 0, 0, 0, −1, −1, 0, 0}	True
38	83	{0, 0, 0, 0, −1, 0, −1, 0}	$\{0.,-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0\}$	{0, 0, 0, 0, −1, 0, −1, 0}	True
39*	84	{0, 0, 0, 0, −1, 0, 0, −1}	$\{\varphi^{3/2},0.,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, −1, 0, 0, −1}	True
40*	85	{0, 0, 0, 0, −1, 0, 0, 1}	$\{-\varphi^{3/2},0.,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, −1, 0, 0, 1}	True
41	88	{0, 0, 0, 0, 0, −1, −1, 0}	$\{0.,-\varphi^{3/2},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0\}$	{0, 0, 0, 0, 0, −1, −1, 0}	True
42*	89	{0, 0, 0, 0, 0, −1, 0, −1}	$\{\varphi^{3/2},-\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, −1, 0, −1}	True
43*	90	{0, 0, 0, 0, 0, −1, 0, 1}	$\{-\varphi^{3/2},-\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,-\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, −1, 0, 1}	True
44*	92	{0, 0, 0, 0, 0, 0, −1, −1}	$\{\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, 0, −1, −1}	True
45*	93	{0, 0, 0, 0, 0, 0, −1, 1}	$\{-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},0.\}$	$\{0,-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, 0, −1, 1}	True
46*	97	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
47	98	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},0,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
48*	99	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
49	102	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
50*	104	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\varphi^{3/2},0.\}$	$\{0,\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
51	105	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
52	108	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
53*	110	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\varphi^{3/2},0.\}$	$\{0,-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
54	112	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
55*	116	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
56	117	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
57	118	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},0,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
58*	119	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
59	121	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
60	123	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,-\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
61	134	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
62	136	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
63*	138	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
64	139	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},0,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
65	140	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
66*	141	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
67	145	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},0.,\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0,\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
68*	147	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi},\varphi^{3/2},0.\}$	$\{0,\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
69	149	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	True
70	152	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
71*	153	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\varphi^{3/2},0.\}$	$\{0,-\frac{1}{\varphi^{3/2}},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
72	155	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
73*	158	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0.,-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},0,\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
74	159	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},0,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
75*	160	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,-\sqrt{\varphi},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},-\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
76*	164	{0, 0, 0, 0, 0, 0, 1, −1}	$\{\varphi^{3/2},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, 0, 1, −1}	True
77*	165	{0, 0, 0, 0, 0, 0, 1, 1}	$\{-\varphi^{3/2},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, 0, 1, 1}	True
78*	167	{0, 0, 0, 0, 0, 1, 0, −1}	$\{\varphi^{3/2},\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, 1, 0, −1}	True
79*	168	{0, 0, 0, 0, 0, 1, 0, 1}	$\{-\varphi^{3/2},\sqrt{\varphi},0.,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 0, 1, 0, 1}	True
80	169	{0, 0, 0, 0, 0, 1, 1, 0}	$\{0.,\varphi^{3/2},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0\}$	{0, 0, 0, 0, 0, 1, 1, 0}	True
81*	172	{0, 0, 0, 0, 1, 0, 0, −1}	$\{\varphi^{3/2},0.,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 1, 0, 0, −1}	True
82*	173	{0, 0, 0, 0, 1, 0, 0, 1}	$\{-\varphi^{3/2},0.,\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0,-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 0, 1, 0, 0, 1}	True
83	174	{0, 0, 0, 0, 1, 0, 1, 0}	$\{0.,\frac{1}{\sqrt{\varphi}},\varphi^{3/2},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},0\}$	{0, 0, 0, 0, 1, 0, 1, 0}	True
84	175	{0, 0, 0, 0, 1, 1, 0, 0}	$\{0.,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	{0, 0, 0, 0, 1, 1, 0, 0}	True
85	178	{0, 0, 0, 1, 0, 0, −1, 0}	$\{0.,-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0\}$	{0, 0, 0, 1, 0, 0, −1, 0}	True
86*	179	{0, 0, 0, 1, 0, 0, 0, −1}	$\{\varphi^{3/2},0.,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 1, 0, 0, 0, −1}	True
87*	180	{0, 0, 0, 1, 0, 0, 0, 1}	$\{-\varphi^{3/2},0.,-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0,-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 0, 1, 0, 0, 0, 1}	True
88	182	{0, 0, 0, 1, 0, 1, 0, 0}	$\{0.,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	{0, 0, 0, 1, 0, 1, 0, 0}	True
89	183	{0, 0, 0, 1, 1, 0, 0, 0}	$\{0.,0.,0.,2\sqrt{\varphi}\}$	$\{2\sqrt{\varphi},0,0,0\}$	{0, 0, 0, 1, 1, 0, 0, 0}	True
90	184	{0, 0, 1, −1, 0, 0, 0, 0}	$\{0.,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	{0, 0, 1, −1, 0, 0, 0, 0}	True
91	185	{0, 0, 1, 0, −1, 0, 0, 0}	$\{0.,\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},-\varphi^{3/2}\}$	$\{-\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0\}$	{0, 0, 1, 0, −1, 0, 0, 0}	True
92*	188	{0, 0, 1, 0, 0, 0, 0, −1}	$\{\varphi^{3/2},\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},\frac{1}{\varphi^{3/2}}\}$	{0, 0, 1, 0, 0, 0, 0, −1}	True
93*	189	{0, 0, 1, 0, 0, 0, 0, 1}	$\{-\varphi^{3/2},\sqrt{\varphi},0.,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},0,\sqrt{\varphi},-\frac{1}{\varphi^{3/2}}\}$	{0, 0, 1, 0, 0, 0, 0, 1}	True
94	190	{0, 0, 1, 0, 0, 0, 1, 0}	$\{0.,\varphi^{3/2},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\frac{1}{\varphi^{3/2}},0\}$	{0, 0, 1, 0, 0, 0, 1, 0}	True
95	191	{0, 0, 1, 0, 0, 1, 0, 0}	$\{0.,2\sqrt{\varphi},0.,0.\}$	$\{0,0,2\sqrt{\varphi},0\}$	{0, 0, 1, 0, 0, 1, 0, 0}	True
96	194	{0, 1, −1, 0, 0, 0, 0, 0}	$\{0.,-\varphi^{3/2},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0\}$	{0, 1, −1, 0, 0, 0, 0, 0}	True
97	195	{0, 1, 0, −1, 0, 0, 0, 0}	$\{0.,-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0\}$	{0, 1, 0, −1, 0, 0, 0, 0}	True
98	197	{0, 1, 0, 0, 0, −1, 0, 0}	$\{0.,-\varphi^{3/2},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},-\frac{1}{\varphi^{3/2}},0\}$	{0, 1, 0, 0, 0, −1, 0, 0}	True
99*	199	{0, 1, 0, 0, 0, 0, 0, −1}	$\{\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}}\}$	{0, 1, 0, 0, 0, 0, 0, −1}	True
100*	200	{0, 1, 0, 0, 0, 0, 0, 1}	$\{-\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},0.\}$	$\{0,\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},-\frac{1}{\varphi^{3/2}}\}$	{0, 1, 0, 0, 0, 0, 0, 1}	True
101	201	{0, 1, 0, 0, 0, 0, 1, 0}	$\{0.,0.,2\sqrt{\varphi},0.\}$	$\{0,2\sqrt{\varphi},0,0\}$	{0, 1, 0, 0, 0, 0, 1, 0}	True
102	203	{0, 1, 0, 0, 1, 0, 0, 0}	$\{0.,-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}},0\}$	{0, 1, 0, 0, 1, 0, 0, 0}	True
103	213	{1, 0, 0, 0, 0, 0, 0, 1}	$\{-2\sqrt{\varphi},0.,0.,0.\}$	$\{0,0,0,-2\sqrt{\varphi}\}$	{1, 0, 0, 0, 0, 0, 0, 1}	True
104*	226	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\varphi^{3/2},0.\}$	$\{0,\frac{1}{\varphi^{3/2}},\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
105	228	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,\varphi^{3/2},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
106	229	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
107*	230	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0,-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
108	231	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\frac{1}{\sqrt{\varphi}},0.,\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},0,\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
109	232	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
110	234	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},0.,\varphi^{3/2},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},0,-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
111*	235	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,\frac{1}{\varphi^{3/2}},-\frac{1}{\sqrt{\varphi}}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
112	236	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},-\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\}$	True
113	238	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\varphi^{3/2},-\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
114	239	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\varphi^{3/2},\frac{1}{\sqrt{\varphi}},0.\}$	$\{0,-\frac{1}{\sqrt{\varphi}},\frac{1}{\varphi^{3/2}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
115	240	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\frac{1}{\sqrt{\varphi}},0.,\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},0,-\frac{1}{\sqrt{\varphi}},-\sqrt{\varphi}\}$	$\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
116	241	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\}$	True
117*	245	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\sqrt{\varphi},\varphi^{3/2},0.\}$	$\{0,\frac{1}{\varphi^{3/2}},\sqrt{\varphi},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
118*	246	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},0.,\sqrt{\varphi},\varphi^{3/2}\}$	$\{\frac{1}{\varphi^{3/2}},\sqrt{\varphi},0,\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
119*	247	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\frac{1}{\sqrt{\varphi}},\varphi^{3/2},0.,\sqrt{\varphi}\}$	$\{\sqrt{\varphi},0,\frac{1}{\varphi^{3/2}},\frac{1}{\sqrt{\varphi}}\}$	$\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True
120	256	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{-\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi}\}$	$\{\sqrt{\varphi},\sqrt{\varphi},\sqrt{\varphi},-\sqrt{\varphi}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	True

FIG. 23. Output showing detail of $E_8 \leftrightarrow H_4(\mathrm{L}\oplus\mathrm{R}\oplus 1\oplus\varphi)$ isomorphism for each vertex.
Note: Red rows indicate D_4 24-cell membership and the * identifies those satisfying the constraint of p∈I where $p^5 = \pm 1$