Probability waves in a classical experiment.

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Observing quantum interference properties in macroscopic objects is difficult because the size of the diffraction grating must be of the same order of magnitude as the de Broglie wavelength. The article shows a simple method of experimental observation and theoretical description of the interference pattern in experiments with macroscopic objects. The results obtained coincide with the solutions of the Schrödinger equation for quantum particles and can complement these solutions for macroscopic objects.

1. Introduction

Interference effects play a decisive role in quantum mechanics; the principle of superposition is one of the basic quantum postulates. De Broglie wave functions do not have a physical meaning, but they well control the behavior of microparticles at the quantum level. The wave properties of microparticles have been confirmed by all experiments. One of the most famous experiments is the observation of interference of electrons through a screen with two slits [1][2].

The behavior of macroscopic particles in a double-slit experiment differs from the behavior of electrons. This discrepancy leads to the fact that for objects with large masses, the wave properties are, as a rule, neglected and the concept of the trajectory of the body is used. It is believed that a body that has a trajectory cannot show interference effects in an experiment and vice versa [3]. It can be shown that for macroscopic particles, the use of corpuscular and wave approaches is possible simultaneously, for the same physical objects.

Let's consider several thought experiments in which a football player kicks a ball inside a certain room. In the same room there is an observer who can catch the ball at a certain, precisely specified moment in time. In all experiments we will assume that the initial coordinate of the ball is known. We will also assume that the absolute value of the initial speed V_0 with which a football player can hit the ball is known. For simplicity, we will assume that the only way we have to find out something about the coordinates and speed of the ball is visual observation.

2. Thought experiments

Experiment 1. If there is lighting in the room, then the observer can measure any coordinate (x_1, y_1) at an arbitrary moment in time, then calculate the direction of the speed and trajectory of the ball.

$$\vec{r}_1 = x_1\vec{i} + y_1\vec{j}$$
$$\vec{V} = V_0 \frac{\vec{r}_1}{|r_1|} \tag{1}$$
$$\vec{r}(t) = \vec{V}t = V_0 \frac{\vec{r}_1}{|r_1|} t$$

In this experiment, we can place the goal as accurately as possible at the right time in the right place so that the probability of catching the ball is as high as possible.

$$P(\vec{r}, t) = \begin{cases} 1 & \vec{r} = V_0 \dfrac{\vec{r}_1}{|r_1|} t \\ 0 & \vec{r} \neq V_0 \dfrac{\vec{r}_1}{|r_1|} t \end{cases} \tag{2}$$

Experiment 2. If there is no lighting in the room, but the exact time $t_0 = 0$ of hitting the ball is known. The observer cannot in any way know the initial direction of the

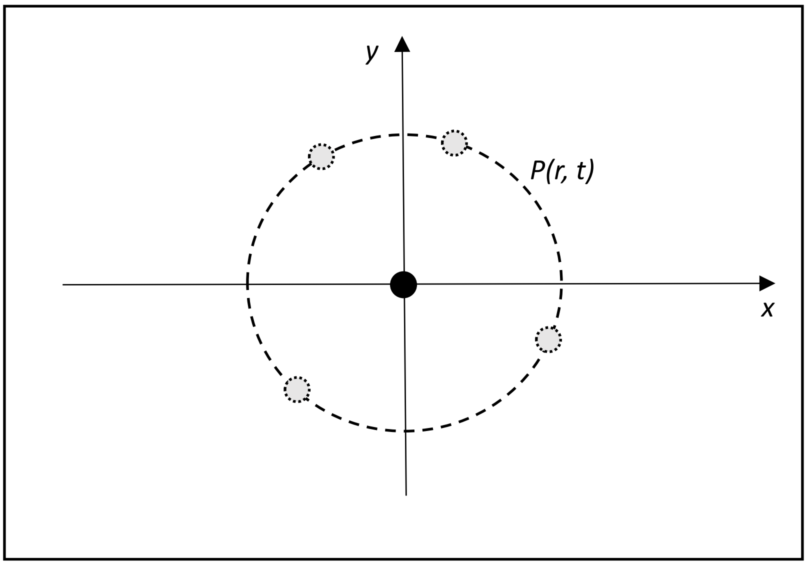

Figure 1

velocity; in this case it is impossible to calculate the exact trajectory. In this experiment, at any moment in time, it is possible to determine the geometric location of the points at which the ball can be with equal probability $P(r,t)$.

This is a probability wave, but the wave with a single maximum. The observer can figure out in what neighborhood the gate should be installed but cannot know the specific point at any given time unless the lighting is turned on. There is no point in using the concept of trajectory in this experiment. However, the probability of catching the ball is still high, this probability can be calculated.

$$\begin{aligned} &|r(t)| = V_0 t \\ &P(r,t) \sim \frac{1}{r^2} \end{aligned} \tag{3}$$

With increasing time, the probability wave in a small neighborhood can be considered flat, then the probability of detecting the object in this neighborhood can be considered a positive real constant, less than one.

$$\lim_{t\to\infty} P(r,t) = \begin{cases} P_0 = const & r = V_0 t \\ 0 & r \neq V_0 t \end{cases} = P_0 \int_0^\infty \delta\left(r - V_0 t\right) dr \tag{4}$$

Further in the text we will always assume that the wave is plane and we will not write a limit.

Experiment 3. There is no lighting in the room, the football players are running towards the ball, one by one, at equal intervals $\Delta t = const$, and trying to hit it in the dark. Only one strikes, but who exactly is unknown. In this case, the observer does not have the exact time of impact. A probability wave with many maxima diverges from the point of impact. There will be as many maximums as there are football players participating in the experiment. By analogy, we can write the formula for probability:

$$\begin{aligned} &|r_n| = V_0 t + V_0 n \Delta t \quad n = 1,2,3... \\ &P(r_n,t) = \begin{cases} P_0 & r = V_0 t + V_0 n\Delta t \\ 0 & r \neq V_0 t + V_0 n \Delta t \end{cases} = P_0 \int_0^\infty \delta\left(r - V_0 t - V_0 n \Delta t\right) dr \end{aligned} \tag{5}$$

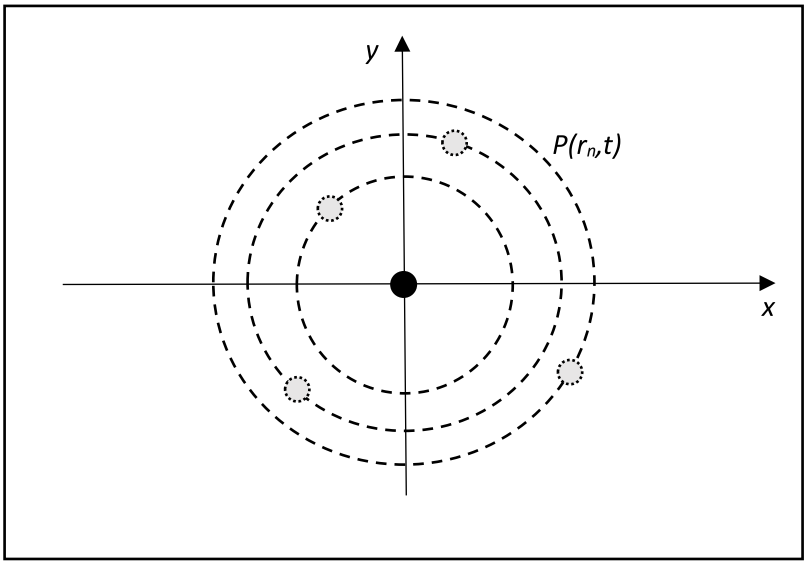

Figure 2

Let's move to a reference frame in which the plane wave is fixed $V_0 t = 0$, then the probability of finding particles at points $r_n = V_0 n \Delta t$ will be constant P_0, as shown in Fig. 3.

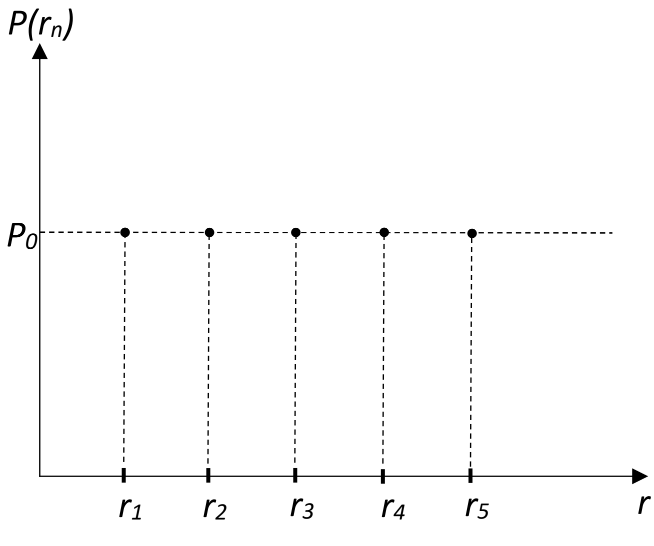

Figure 3

Representing the probability function as a set of individual points is mathematically inconvenient and has no physical meaning, since any real experiment contains non-zero errors. It can be considered that in a non-ideal experiment, the resulting maxima should be blurred. Indeed, if we assume that the time to swing a leg, the time to hit the ball, the time to accelerate and lift the ball off the foot, etc. is finite, then the probability wave is more correctly described by a normal distribution:

$$P(r_n) = P_0 e^{-\frac{1}{2}\left(\frac{r - V_0 n\Delta t}{\sigma}\right)^2} \tag{6}$$

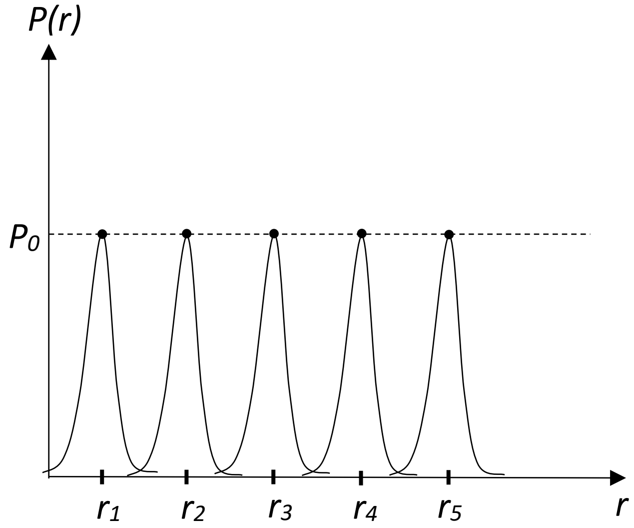

Figure 4

It is possible to select experimental conditions so that distribution (6) is approximately described by the squared cosine function. To do this, the standard deviation must satisfy the condition:

$$\sigma = \frac{V_0 \Delta t}{\sqrt{2\pi}} \tag{7}$$

Substituting (7) into (6), we get:

$$\begin{gathered}\left(\frac{r - V_0 n\Delta t}{\sigma}\right)^2 = \left(\frac{\sqrt{2}\pi r}{V_0 \Delta t} - \sqrt{2}\pi n\right)^2 \\ P(r,n) = P_0 e^{-\frac{1}{2}\left(\frac{\sqrt{2}\pi r}{V_0 \Delta t} - \sqrt{2}\pi n\right)^2} \approx P_0\left(1 - \left(\frac{\pi r}{V_0 \Delta t} - \pi n\right)^2\right) \approx P_0 \cos^2\left(\frac{\pi r}{V_0 \Delta t} - \pi n\right)\end{gathered} \tag{8}$$

Thus, a simple approximate formula can be used for probability waves:

$$P(r) = P_0 \cos^2\left(\frac{\pi r}{V_0 \Delta t}\right) \tag{9}$$

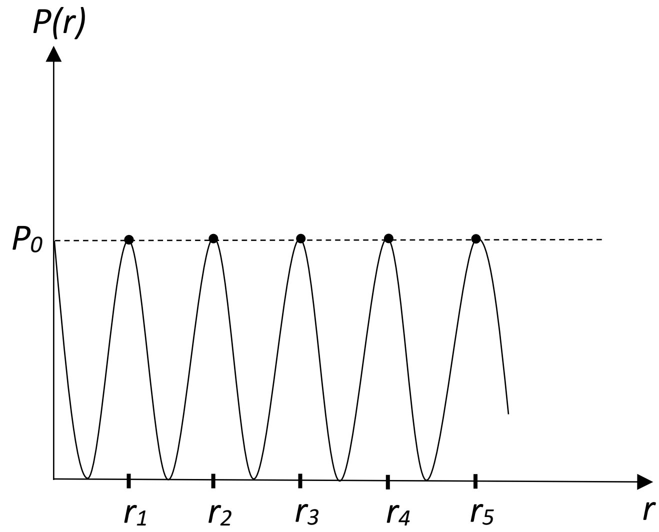

Figure 5

In the third experiment we still can try to determine the probability of where the ball would be at a particular time. The maximums of the probability wave will give the geometric location of the points where the gates need to be placed to have at least some chance of success. Now it is not difficult to find a wave equation whose solution is functions (9):

$$\begin{aligned} P(r) &= P_0 \cos^2\left(\frac{\pi r}{V_0 \Delta t}\right) = \frac{P_0}{2} + \frac{P_0}{2}\cos\left(\frac{2\pi r}{V_0 \Delta t}\right) \\ P(r) - \frac{P_0}{2} &= \frac{P_0}{2}\cos\left(\frac{2\pi r}{V_0 \Delta t}\right) \equiv \phi \end{aligned} \tag{10}$$

From here we get:

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= -\frac{2\pi}{V_0 \Delta t}\frac{P_0}{2}\sin\left(\frac{2\pi r}{V_0 \Delta t}\right) \\ \frac{\partial^2 \phi}{\partial r^2} &= -\left(\frac{2\pi}{V_0 \Delta t}\right)^2 \frac{P_0}{2}\cos\left(\frac{2\pi r}{V_0 \Delta t}\right) = -\left(\frac{2\pi}{V_0 \Delta t}\right)^2 \phi \\ \frac{\partial^2 \phi}{\partial r^2} &+ \left(\frac{2\pi}{V_0 \Delta t}\right)^2 \phi = 0 \end{aligned} \tag{11}$$

Let's move in (11) from speed to kinetic energy:

$$E_k \frac{\partial^2 \phi}{\partial r^2} + 2m\left(\frac{\pi}{\Delta t}\right)^2 \phi = 0 \tag{12}$$

Let's move from kinetic energy to total and potential energies:

$$\frac{1}{2m}\frac{\partial^2\phi}{\partial r^2}+\frac{\pi^2}{(E-U)\Delta t^2}\phi=0 \tag{13}$$

It can be noted that (13) completely coincides with the stationary Schrödinger equation if the following equality is achieved in the experiment:

$$V_0\Delta t=\frac{2\pi\hbar}{mV_0}=\lambda_D \tag{14}$$

Where λ_D is the de Broglie wavelength.

Indeed, let us combine (13) and (14):

$$\begin{cases}\dfrac{1}{2m}\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{\pi^2}{(E-U)\Delta t^2}\phi=0\\ \Delta t=\dfrac{2\pi\hbar}{mV_0^2}=\dfrac{\pi\hbar}{E_k}=\dfrac{\pi\hbar}{(E-U)}\end{cases} \tag{15}$$

From here we get:

$$\begin{gathered}\frac{1}{2m}\frac{\partial^2\phi}{\partial r^2}+\frac{\pi^2(E-U)^2}{(E-U)\pi^2\hbar^2}\phi=0\\ \boxed{\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial r^2}+(E-U)\phi=0}\end{gathered} \tag{16}$$

Equation (16) coincides with the Schrödinger equation, however, the solution here is not the complex amplitude, but the real function ϕ, which is related to the probability by relation (10):

$$P(r)=\phi+\frac{P_0}{2} \tag{17}$$

The maximum probability P_0 must be determined so that for any values of ϕ the inequality $0\le P(r)\le P_0$.

Experiment 4. Let's modify the third experiment. Let us place an impenetrable obstacle with two slits in the path of the ball. The ball can fly through slits, but not through obstacles. If the ball flies through the slit, its trajectory changes randomly. An observer behind the slits can predict the points where the probability of finding the ball will be maximum and where this probability will be minimum. The probability wave will superimpose on itself, giving a pattern like the interference pattern of a light wave.

3. Case of a rectangular potential well

Let's consider the solution to equation (13) for an experiment with a macroscopic object in a potential well with infinitely high walls.

$$\frac{1}{2m}\frac{\partial^2\phi}{\partial r^2}+\frac{\pi^2}{E\Delta t^2}\phi=0 \tag{18}$$

We will look for a solution in the form of an exponential function:

$$\phi=C_1e^{i\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}r}+C_2e^{-i\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}r}=a\cos\left(\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}r+\varphi\right) \tag{19}$$

We impose boundary conditions so that the probability of finding the object at the edges of the well will be zero:

$$\begin{aligned}&P(0)=\phi(0)+\frac{P_0}{2}=0\Rightarrow\phi(0)=-\frac{P_0}{2}\\&P(l)=\phi(l)+\frac{P_0}{2}=0\Rightarrow\phi(l)=-\frac{P_0}{2}\end{aligned} \tag{20}$$

Substitute (20) into (19):

$$\begin{cases}\phi(0)=a\cos(\varphi)=-\frac{P_0}{2}\\ \phi(l)=a\cos\left(\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}l+\varphi\right)=-\frac{P_0}{2}\end{cases} \tag{21}$$

Assuming the initial phase to be zero, now we can find the energy levels in the potential well:

$$\begin{aligned}&\begin{cases}a=-\frac{P_0}{2}\\ -\frac{P_0}{2}\cos\left(\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}l\right)=-\frac{P_0}{2}\end{cases}\\&\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}l=2\pi n\Rightarrow\boxed{E_n=\frac{ml^2}{2n^2\Delta t^2}}\end{aligned} \tag{22}$$

If we substitute the time from (14) into (22), we get:

$$\begin{cases} E_n = \dfrac{ml^2}{2n^2\Delta t^2} \\ \Delta t = \dfrac{2\pi\hbar}{mV_0^2} = \dfrac{\pi\hbar}{E_n} \end{cases} \tag{23}$$

$$E_n = \frac{ml^2E_n^2}{2n^2\pi^2\hbar^2} \Rightarrow E_n = \frac{2n^2\pi^2\hbar^2}{ml^2}$$

As can be seen, energies (23) differ from the energy levels of a quantum particle in the potential well. However, if instead of (14), choose a wavelength half as large:

$$V_0\Delta t = \frac{\pi\hbar}{mV_0} = \frac{\lambda_D}{2} \tag{24}$$

then we get complete agreement with the classical result, which is obtained by solving the Schrödinger equation:

$$\begin{cases} E_n = \dfrac{ml^2}{2n^2\Delta t^2} \\ \Delta t = \dfrac{\pi\hbar}{mV_0^2} = \dfrac{\pi\hbar}{2E_n} \end{cases} \tag{25}$$

$$E_n = \frac{ml^2 4E_n^2}{2n^2\pi^2\hbar^2} \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}$$

Let's find the probability distribution:

$$\begin{aligned} &P(r) = \phi + \frac{P_0}{2} = a\cos\left(\frac{\pi}{\Delta t}\sqrt{\frac{2m}{E}}r + \varphi\right) + \frac{P_0}{2} \\ &P(r) = -\frac{P_0}{2}\cos\left(\frac{2\pi n}{l}r\right) + \frac{P_0}{2} = \frac{P_0}{2}\left(1-\cos\left(\frac{2\pi n}{l}r\right)\right) = P_0\sin^2\left(\frac{\pi n}{l}r\right) \end{aligned} \tag{26}$$

The obtained result completely coincides with the solution of the stationary Schrödinger equation for a particle in an infinitely deep potential well.

4. Conclusion

Calculations show that if the experiment is set up correctly, wave properties can also be observed in macroscopic particles. The probability distribution can be obtained from solving the stationary Schrödinger equation directly, without resorting to complex amplitudes.

Reference

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