Guessing that the Riemann Hypothesis is unprovable

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 164 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I guess the independence (unprovability) of the Riemann Hypothesis. In this, I deal with propositions equivalent to the Riemann Hypothesis regarding the Möbius function. First, I define something called "the distorted Möbius function". Finally, since "the distorted Möbius function" can be used to make two models at infinity in two consistent ways, I guess that the Riemann Hypothesis is unprovable.

1

I define Möbius function $\mu(n)$ as

$$\mu(n) := \begin{cases} 1 & \text{product of even primes} \\ -1 & \text{product of odd primes} \\ 0 & \text{when divisible by the square of a prime} \end{cases}$$

The Riemann Hypothesis is that the real part of the non-trivial zero of the ζ function is 1/2. (Ivić[2]p44)

Theorem 1. *(Ivić[2]p48,Titchmarsh[5] p370, Theorem 14.25)*

$$\text{the Riemann Hypothesis} \Leftrightarrow \sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

The following is known as a short proof.
$M(x)$ is defined as follows.

$$M(x) := \sum_{n=1}^{[x]} \mu(n)$$

Then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \int_{x=0.1}^{\infty} x^{-s} d(M(x))$$

$d(M(x))$ is the Stieltjes' integral of $M(x)$.

$$= [M(x)x^{-s}]_{0.1}^{\infty} + s\int_{x=0.1}^{\infty} M(x)x^{-s-1}dx = s\int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

Here, I analytically continue $\frac{1}{\zeta(s)}$ until this value is finite. When $\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$ holds, this integral is finite when $Re(s) > 1/2$. I obtain that there are no zeros of the zeta function when $Re(s) > 1/2$. Furthermore, by combining this with a discussion of functional equations, I obtain the Riemann Hypothesis.

Conversely, if I assume the Riemann Hypothesis,

$$s\int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

This is not infinite at $Re(s) > 1/2 + \epsilon > 1/2$

$$|M(x)| < Km^{\frac{1}{2}+\epsilon}$$

I obtain that ϵ can be arbitrarily small.

Proposition 1.

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

I guess this proposition's unprovability. This proposition is equivalent to the Riemann Hypothesis (by theorem 1).

From now on, we will consider two axiom systems to create two models: one where the Riemann hypothesis is true and one where it is false. First, the axiom system that real and complex numbers satisfy. This satisfies the completion axioms. On the other hand, the axiom system that Non-Standard Analysis satisfies. The concepts of infinity and infinitesimals ([1] p.36), whose existence is allowed by nonstandard analysis, do not satisfy the completion axioms. Here, infinitesimals are positive real number smaller than any positive number, and infinity is the reciprocal of infinitesimals, a number larger than any finite number. Clearly, Cauchy sequences never converge in the infinitesimals.

From now on I argue in the Non-Standard Analysis. I define "the sum of the distorted Möbius function" as $\sum_{n\le P-1}\mu(n)+f(P)$ that is in the summation of Möbius function (up to some large prime P) add (or subtract) $f(P)$ instead of $\mu(P)$. (The sign of $f(P)$ can be chosen freely. This is due to the multiplicative nature of the Möbius function.) Using non-standard analysis, let $P\to\infty$(P is a prime number). I define "the distorted Möbius function" at $P*M$ as $f(P)*\mu(M)$. (When M is divisible by P, I set it to 0.) This value can be determined from P to infinity with no contradiction. Since only the value at P is needed from now on, I only consider that the sum of the Möbius function up to P can be determined freely. The sum of the Möbius function at $P\to\infty$(P is a prime number) can be determined arbitrarily. (By the way, the sum of the Möbius function can be taken arbitrarily for any m that satisfies $P<m<2P$.) (There are two forms that will be used later. First, $|\sum_{n\le P-1}\mu(n)+f(P)|<KP^{\frac{1}{2}+\epsilon_0}$ and $P\to\infty$, $\epsilon_0>0$ is the infiniesimal. (P is a prime number). Next, $|\sum_{n\le P-1}\mu(n)+f(P)|=KP^{\frac{2}{3}}$ and $P\to\infty$(P is a prime number).

The validity of "the sum of distorted Möbius functions" is shown by the short proof of Theorem 1 above: when $|f(P)|=P^\sigma$, for $Re(s)>\sigma+\epsilon'$

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\lim_{P\to\infty}(\frac{\mu(1)}{1^s}+\frac{\mu(2)}{2^s}+\cdots+\frac{f(P)}{P^s})$$

is true. In other words, if I take $f(P)$ at P, then $\zeta(s)$ has a zero point at $Re(s)>1/2$. It is assumed that the zero point is $s=\sigma+\infty i$, but this argument is not accurate. Here, I will only discuss the fact that it is possible to take "the distorted Möbius function" instead of the Möbius function.

conjecture 1. *The Riemann Hypothesis is unprovable. In other words, the Riemann Hypothesis is an "independent proposition".*

Consider the Riemann Hypothesis for the Möbius function as stated in Proposition 1.

First, consider the case where the negative proposition of the Riemann Hypothesis can be proven in normal axiom system. This is one possibility. In the following, I assume that there is no counterexample to Proposition 1 in a finite range.

Next, I show that the positive proposition of the Riemann Hypothesis is unprovable.

As Model A, consider the case where

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

is true for all m.

As Model B, consider in non-standard analysis.

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon_0})$$

holdss for some infinitesimal $\epsilon_0 > 0$ and $m = \infty$ case. amd the case where

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

is not true for $m = \infty$.

Model A is the model the Riemann Hypothesis is true. Model B is the model the Riemann Hypothesis is false.

There is no counterexample to Proposition 1 in a finite range. Model A holds in a finite range. Model A is "consistent" in ZFC(K.Kunen[3] Introduction§1). (Model A consists of equations that are not contradictory to each other. In such a case, it is said to be "consistent". (K.Kunen[3] Introduction§1))

Let take "the sum of distorted Möbius functions" for $m = P$ well. Then, Model B is true when $m = \infty$. Model B is "consistent" in Non-Standard Analysis.

In such a case, the Riemann Hypothesis is independent of theory and cannot be proven. (K.Kunen[3] Introduction§1)

It is possible that the negative proposition of the Riemann Hypothesis is provable, or the positive proposition of the Riemann Hypothesis is unprovable.

2 Equivalent Propositions

I will state that the three equivalent propositions of the Riemann Hypothesis are unprovable.

Proposition 2.

$$\pi(x) = li(x) + O(x^{\frac{1}{2}+\epsilon})$$

In the Non-standard Analysiis, I will consider the situation at $P \to \infty$, assuming that $\pi(x)$ is summed $g(P) > 1$ for a certain prime number P. Or

$$\pi(x) = O(m^{\frac{1}{2}+\epsilon_0})$$

holds some infinitesimal $\epsilon_0 > 0$ and $m = \infty$.

As with the Möbius function, it is also clear that two opposing models exist and that they are consistent. Therefore, it is impossible to prove. Of course, it is equally possible to disprove it.

Proposition 3.

$$\sigma(n) < e^{\gamma} n \log\log n (for\ n > 5040)$$

In the Non-Standard Analysiis, For a sufficiently large prime factor P of $\sigma(n)$, replace all divisors AP^s related to P with $AP^s \times h(P)^s(h(P) > 1)$. Note that this proposition is an "inequality", so a model in which only one number does not satisfy the inequality is sufficient. This is because, unlike the previous two, $\sigma(n)$ should not be treated as "quantitative". Similarly, two consistent models can be created. Of course, this can also be disproved by finding a case in which the inequality does not hold within a finite range.

Proposition 4. *The only non-trivial zero point of the Riemann Zeta function is on $Re(s) = 1/2$.*

In this case, it is a "conjecture" half. I think in Non-Standard Analysis.

conjecture 2. *For $Re(s) > 1/2$, there exists a zero point of the Riemann Zeta function that can be written in the form $s = Re(s) + \infty i$.*

I can only see in some calculations.

$$\zeta(2/3 + 1020.9 \cdots i) = 0.21 \cdots + 0i$$

$$\zeta(2/3 + 100,049.4239 \cdots i) = 0.19 \cdots + 0i$$

$$\zeta(2/3 + 10,000,111.93 \cdots i) = 0.21 \cdots + 0i$$

$$\zeta(2/3 + 1,000,000,395.2 \cdots i) = 0.189 \cdots + 0i$$

$$\zeta(2 + 1009.9753 \cdots i) = 0.73 \cdots + 0i$$

$$\zeta(2 + 1,000,000,402.2 \cdots i) = 0.71 \cdots + 0i$$

Of course, I have not examined all pure real values, and the result only seems to gradually approach zero. If I replace the real value part $(2/3, 2)$ with a number smaller than $1/2$, the result becomes even worse and it is hard to imagine that it approaches zero.

If $\zeta(s) = 0, Re(s) = 1/2$, then for infinitesimal ϵ_0, we can set $\zeta(s + \epsilon_0) = 0, Re(s + \epsilon_0) \neq 1/2$. We used the continuity of the Riemann Zeta function. This condition alone is sufficient to show that the Riemann hypothesis is unprovable. This impossibility of proving includes a scenario that I do not want to think about. It is impossible to prove, but it is possible to disprove it. This means that I can find a counterexample to the zero point from a small number, but the Riemann Hypothesis cannot be proven in this direction.

Let's look at one more thing

Proposition 5.

$$\int_0^\infty \int_{\frac{1}{2}}^\infty \frac{1 - 12t^2}{(1 + 4t^2)^3} log|\zeta(\sigma + it)| d\sigma dt = \frac{\pi(3 - \gamma)}{32}$$

In this case,

$$\sum_\rho \frac{Re(\rho)}{|\rho|^2} = \sum_\rho \frac{\frac{1}{2}}{|\rho|^2} \Leftrightarrow \textit{the Riemann Hypothesis}$$

is the basis, so

$$\lim_{Im(\rho) \to \infty} \frac{Re(\rho)}{|\rho|^2} = 0$$

Therefore, (in the sense of this paper) in cases where the Riemann Hypothesis does not hold and where the Riemann Hypothesis does hold,

$$\sum_\rho \frac{Re(\rho)}{|\rho|^2} = \sum_\rho \frac{\frac{1}{2}}{|\rho|^2}$$

holds. (Note that the only two zero points where the imaginary part of the model B's secondary case, the infinity are $s = Re(s) + \infty i, Re(s) - \infty i$.) In other words, this is insufficient as a judgment condition (at least within this paper). That is, it is not necessary to discuss it.

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References

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