# Geometric Sub-bundles

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**Abstract**

Let $\mathfrak{X}$ be a topological stack, and $LocSys(\mathfrak{X})$ a local system taking varieties $v \in \mathfrak{X}$ to their projective resolutions over an affine coordinate system. Let $\alpha$ and $\beta$ be smooth charts encompassing non-degenerate loci of the upper-half plane, and let $\varphi$ be the map $\beta \circ \alpha^{-1}$. Our goal is to describe a class of vector bundles, called *geometric sub-bundles*, which provide holonomic transport for n-cells (for small values of n) over a $G_\delta$-space which models the passage $\mathfrak{X} \rightrightarrows LocSys(\mathfrak{X})$. We will first establish the preliminary definitions before advancing our core idea, which succinctly states that for a pointed, stratified space $Strat^*_M$, there is a canonical selection of transition maps $[\varphi]$ which preserves the intersection of a countable number of fibers in some sub-bundle of the bundle $Bun_V$ over $LocSys(\mathfrak{X})$.

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# 1 Preliminaries

## 1.1 Preamble

One of the motivating considerations in writing this paper, was that of a *geometric multiverse*, in which there was an orthogonal time direction. **Stack** seemed to be a very poignant word here. For each wordline, $\mathfrak{W}_n$ for a particle $p_n$, there seemed to be some sort of intuitively visualizable "stack" of moments in $\mathbb{R}^3$, represented as flat screens. Orthogonal to this stack would then be a collection of things and events which coexisted with what we would call the "actual" ones.

A further interest of mine, which I had established very early on, is in "negative dimensions", more specifically in *negative-dimensional spaces*. Keenly enough, the idea of an *inertia group* of a *topological stack*, as presented by Noohi, seems to yearn to be woven into this strand of thinking.

It was in a recent, very stimulating conversation, with sir Oliver Hancock, that it became very apparent to me that the novelty of negative dimensioms extended to something much further, and much more serious. Indeed, if we were to take such an idea seriously, at face value, we would require an entire *algebra of dimensions*! Moving forward, I work with the mission to develop such a thing in the back of my mind, although I am ill-equipped to make real progress here at the moment. I believe that topological stacks are the right generalization to make if we are aiming for this sort of development.

## 1.2 Curvature

For any space $S$, we may select a point $Cent_S$ which is its barycenter. Thus, any point $p \neq Cent_S$, there is a non-zero distance $d(p, Cent_S)$ which parameterizes the infimum of the set of paths between the two. The collection of such distance functions over the set $P \ni p$ shall be written

$$D_P = \sum_{p=Cent_S+\varepsilon}^{p\cap\partial S} d(p, Cent_S) \tag{1}$$

Now, let us suppose we have an arbitrary comparison line, $\ell$, which has zero sectional curvature everywhere. We can express the curvature of a given geodesic $g \in S$ by $C_g - l$, where

$$C_g = \int_0^1 \frac{d\theta}{dL}$$

where $dL$ represents the difference in lengths along a path whose interval is [0,1], and where

$$g = [0, 1] \mapsto \ell'$$

Write $g_\phi$ for the filter obtained by normalizing the value of every $C_g$ to a singular element in an overring $R \supset (r \in rep(S))$. We can construct a map $R \longrightarrow F_p$ to a p-adic field by assigning to each $C_g$ a valuation $\mathfrak{v}$, such that each $p \in S$ receives a projective resolution $p \longrightarrow \mathfrak{v}(p) \subset [Cent_S, \partial S]$. We shall do so now.

**Definition 1.** *An intrinsic fiber of a space $X$ is a fiber whose completion lies in the interior of $X$.*

Fix a bundle $Bun_{int(S)}$ consisting of the intrinsic fibers of S. Denote a standard fiber of this bundle by $f : Cent_S \to I \to P$, where I is the standard interval. The composition of f with a metric $\mu$ produces a sub-bundle

$$\widetilde{Bun_{int(S)}} \subseteq Bun_{int(S)}$$

such that isometries of the standard bundle may be decomposed into sections of f together with some gluing condition which tells us how to refine our section under composition. Let $a, b \subset f$, and let $b \circ a = a \oplus_{k\varepsilon} b$. Suppose that $k > 1$. Then,

**Proposition 1.** *$\widetilde{Bun_{int(S)}}$ is piecewise linear.*

Now, let

$$\{\sum_{k=1}^{\infty} a \oplus_{k\varepsilon} b \mid k \in \mathbb{Z}^+\} = 0 \longrightarrow 1 \tag{2}$$

We can then define a map $dL \overset{\propto}{\mapsto} d(\mathfrak{v}(p))$, such that the valuation of the point increases proportionally with respect to the distance along a closed geodesic.

**Proposition 2.** *Equation (2) defines a frame.*

*Proof.* Since we have a bottom (k=1), and a top (k=$\infty$), we have an object in the category of toposes. Inverting this gives us an object in the category of frames. $\square$

The inclusion

$$(dL \overset{\propto}{\mapsto} d(\mathfrak{v}(p))) \hookrightarrow Frm$$

effectively allows us to work with a Heyting algebra, equipped with whatever additional structure the class of curvatures measured along [0,1] provides. In addition, this gives us a natural stratification $Frm \mapsto Strat_M$ to a (possibly anabelian) topological space. We shall now consider generalizations of this state of affairs.

### 1.3 Topological stacks

A topological stack is a generalization of a topological space, invented by B. Noohi. [1] An example of a topological stack is an orbifold. Essentially, a topological stack is an object *fibered in groupoids*, which replaces the usual map $Sets \to Top$ by the map $Grpd \to Stk$. Noohi showed [1, prop. 3.5] that any isomorphism of objects in the category of groupoids also admits an identical stackification of said objects, written

$$(\mathcal{X} \sim \mathcal{Y}) \in Grpd \to (\mathcal{X}^a = \mathcal{Y}^a) \tag{3}$$

where a superscripted $a$ denotes stackification. Substacks of a stack $\mathcal{X}$ are called *saturated*, meaning that, whenever they contain an object $o$, they contain the entire isomorphism class $o/\sim$.

The usual notion of a point becomes replaced by that of an *inertia group* of order n, written $I_x$. This allows us to effectively construct objects which are essentially $\frac{1}{n}$ of a single point. The following diagram:

$$\begin{tikzcd} \frac{1}{n} \arrow[rr, dashed, "\times_n"] & & \{*\} \\ \mathfrak{I}_1 \arrow[u, "Rep"] \arrow[r, "+\frac{1}{2}k"] & ... \arrow[r, "+\frac{1}{2}k"] & \mathfrak{I}_n \arrow[u, "Rep_*"'] \end{tikzcd}$$

is commutative. Here, *Rep* is the map from a W-group into a field of real-valued numbers, and $Rep_*$ is the association of the closure of said field with a solid point. By "*solid*," we mean effectively a topological point, although there may be some more subtleties with this terminology.

**Definition 2.** *A map in the category $St_{Top}$ of topological stacks, shall be called* **representable** *if a.) it is a morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks, and b.) the fiber product $Y := X \times_{\mathcal{X}} \mathcal{Y}$ is equivalent to a topological space for all maps $X \to \mathcal{X}$ out of a topological space.*

Essentially, the notion of a point is replaced by the notion of an "isomorphism class" of *quasi-points*, which are objects of a given $W$-group. The representation of the group $\mathcal{W}$ as a point is an epimorphism

$$p \in \mathcal{X}(\mathcal{W}) \times_n I_x \longrightarrow p \in Man \tag{4}$$

between a (possibly anabelian) group, $\mathcal{W}$, and a totally Abelian group $I_x$. Thus, the representation $p$ is classical, whereas $\mathcal{W}$ may be non-commutative, and thus models a quantum point. We thus have

$$(p \in Man) \simeq \mathcal{W}_{Ab}$$

where $\mathcal{W}_{Ab}$ is a $\mathcal{W}$-valued point in some arbitrary base category $C$.

In all of the practical applications, topological stacks are 2-categories. Maps out of every $\mathfrak{X} \in St_{Top}$ are 2-morphisms onto the set $Pr(\mathfrak{X})$, consisting of a stack $\mathcal{X}$ and a *path groupoid* $\mathcal{X}^{<n>}$, where n denotes the rank of the fiber $\mathcal{X} \xrightarrow{n} \mathcal{X}'$. The universal property of $St_{Top}$ is that the following diagram:

$$\begin{tikzcd} \mathcal{W} \arrow[r, "pr_n"'] \arrow[d, "pr_n"] & \mathcal{X} \arrow[d, "\Delta"'] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_{\mathfrak{X}} \mathcal{X} \end{tikzcd}$$

commutes.

# 2 Main Results

Let $\mathscr{G}$ be a $G_\delta$-space, and $\mathfrak{X}$ a topological space. Let there be an open subspace $\mathscr{G}|_o$ for every object o in the category of topological spaces. Let there be a group $\mathscr{W}$ such that $\mathscr{G} \cap \mathscr{W} \supset o$. Let $\mathscr{H}^2$ denote the upper half-plane, and let $Aut_\sim(\mathscr{W})$ denote the isogenies of the group $\mathscr{W}$ under the equivalence relation $\sim$.

**Lemma 1.** *If $\mathcal{X}$ is a topological stack, with a map $\mathcal{X} \rightrightarrows \mathscr{G}, \mathfrak{X}$, then a countable number of objects of $\mathcal{X}$ are representable.*

*Proof.* Since a $G_\delta$-set consists in the intersection of a countable number of open sets, we must show that open sets in some way correspond to objects in $\mathcal{X}$. We can do so using set methods by letting the finest topology on $\mathcal{X}$ be the one in which each object is a singleton set.

Then, for every point in $\mathfrak{X}$, there is a bijective morphism $(o \in \mathcal{X}) \leftrightarrows (o \in \mathscr{G})$. $\square$

**Definition 3** (Degeneracy)**.** *If $\Delta^\sharp$ is a simplicial complex, and if the map $(\Delta^\sharp)^{-1} \mapsto X^{-1}$. is surjective, but not invertible, on objects, then we say the map is* **degenerate***.*

Suppose $\Delta^\sharp$ is fibered in groupoids. Then, the canonical map

$$\mathcal{X} \rightleftarrows \Delta^\sharp \overset{can}{\mapsto} X$$

has an inverse projective resolution $\mathcal{X}$, which is a topological stack.

## 2.1 Stratification

Let $Strat^*_M$ be a $*$-stratified space with base manifold M. We model this by writing

$$\mathcal{X} \xrightarrow{*} Strat_M;$$

we may choose our group $\mathscr{W}$ appropriately, such that for every map $\alpha : u \to v \in \mathscr{W}$, we obtain the following data:

1. An induced norm, $\mu = inf(d(u,v))$
2. A collection of charts, $\mathcal{C} \supset \alpha \circ \alpha^{-1}$

Thus, we have, in $\mathscr{W}$, every map $\alpha$ corresponds to an exit path.

$$\alpha \models \mathcal{X} \to Strat_M$$

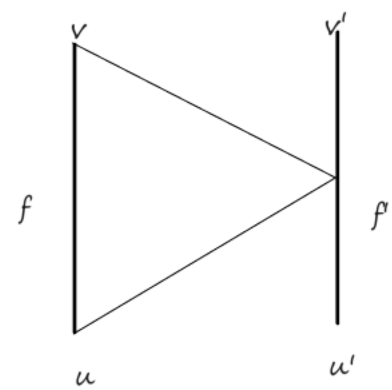
.

Let the right-hand side of the above expression be written as $\mathcal{EP}_0$, and denote all future maps $Strat_M \xrightarrow{*^n} Strat_M$ by $\mathcal{EP}_n$. We may prefer to treat the first exit path as the introduction of some local coordinate system over a variety $\mathcal{X}_v$ of $\mathcal{X}$. Thus, $\mathcal{EP}_n = End(LocSys(\mathcal{X}_v))\ \forall v$.

One of the most obviously ways to stratify a space is using the conical stratification, $S_\Delta$, by triangulating each pair of fibers. We do so by treating each fiber $f \in End(LocSys(\mathcal{X}_v))$ as an interval $[u, v]$, and then mapping the interval to a centralizer by

$$[u, v] \in f \mapsto \frac{u+v}{2} \in f'$$

.




However, we could also choose to take every object $o \simeq S^1$ in a topological space, and let every vector $\eta : o \to o'$ moving between sub-bundles be an exit path of the second kind. In this case, it is helpful to have some intuitive concept of a *portable object*, which is an idealized object with some pre-defined rules for transporting it across strata.

**Definition 4.** *A vector subbundle $v \in \mathcal{V}$ is a proper subset of a vector bundle $\mathcal{V}$.*

# 3 Geometric Bundles

Let $\mu : B \to E \to B$ be a microbundle [2], and let, for every fiber $i \in \mu$, there is a path $P : x \to ... \to x'$. Let there be a smooth and proper embedding $\mu \hookrightarrow \mathscr{H}^2$. We can now introduce the definition of a geometric bundle.

**Definition 5.** *A geometric bundle, $Bun_{Geom}$, is a microbundle with a non-zero image in a geometric space.*

Let us denote a section of $Bun_{Geom}$ by $\gamma_i$.

**Definition 6.** *The* **discretization** *of a space $S$ is a transformation of $S$ such that, for any given map between sections of a geometric bundle, $\gamma_i \to \gamma_j$ is an exit path.*

**Proposition 3.** *For any given epimorphism $\vec{k} : \gamma_i \to \gamma_j$, if the span of $\vec{k}$ is $\varepsilon$, then the underlying topological stack generates a discrete space.*

**Proposition 4.** *Let $\vec{k} \in \mathscr{G}$. Then, $\mathscr{G}$ is parallelizable.*

The primary effect afforded by geometric bundles is the ease by which visualizations of physical phenomena can be treated as geometric phenomenon. We are particularly keen to observe this fact in connection with the so-called "quasi-quanta" of Emmerson, et al. [3]

Let $\hat{q}$ be a quasi-quantum, and let there be a bijectio $\hat{q} \leftrightarrow I_q$, where q is a k-tuple corresponding to a real space $\mathbb{R}^{2,k}$ with a single ordinary time dimension and an additional *orthogonal* time dimension.

**Lemma 2.** *For every particle, $q$, there is a corresponding neighborhood $\mathcal{U}(p)$.*

We can construct this neighborhood by taking the fiber product $Bun_{Geom} \times T$ for any topological object, for instance $S^1$. As a matter of technicality, $\mathcal{U}(p)$ is a bundle of rays from a fixed point related to one another by an appropriate Lie action. If we normalize these rays to the metric of the underlying space, then we obtain a differential vector space, $\mathcal{V}_\partial$.

Much of the work of geometric bundles is to describe transport across the space $\mathcal{V}_\partial$, and geometric sub-bundles then formalize Lorentzian, Laplacian, and Eularian submanifolds of a principle space-time manifold. Observe the following partial equality:

$$\mathcal{V}_\partial \simeq LocSys(\mathfrak{X}, E)$$

where $\mathfrak{X}$ is a topological space, and E is a vector bundle smoothly fibered in groupoids, which we will recursively define to be any such bundle which is equivalent to an isotopy class of leaves of a foliation, $\mathcal{F}$. We are thus free to write

$$E \simeq \mathcal{F}/\theta$$

**Definition 7.** *A* **geometric subbundle**, *$\widetilde{Bun_{Geom}} \subset Bun_{Geom}$, is a portable, saturated subset of a geometric bundle.*

Let $\mathcal{I}^k$ be a collection of idempotents, and k be its rank, i.e., the maximal n for an n-cell in $\mathcal{I}$. Let $Aut(\mathcal{I}^k) \to St_{Top}$ be a proper injection, such that each $i \in \mathcal{I}^k$ is representable. Let $Man \circ (St_{Top} \circ \mathcal{I}^k)$ be non-degenerate. Then, there is some non-trivial orientable locus in $Man$ which has direct access to a $W$-group.

Suppose that for each point $p \in Man$, there is a completion, $\bar{p}$ which is the nth weight for a W-group supporting p. Then, there is a rank-one isomorphism $\hat{p} \leftrightarrow w_n$. Thus, the weak equivalences of the W-group correspond to $\theta$-isomorphisms of objects in the equivalence class of p. That is to say, equivalences are stronger for the geometric representation of an object, then for the underlying algebraic structures.

# 4 References

[1] B. Noohi, *Foundations of Topological Stacks I*, (2005)
[2] J. Milnor, *Microbundles Part I*, (1963)
[3] R.J. Buchanan, O. Hancock, P. Emmerson
*On the Mechanics of Quasi-quanta Realization*, (2023)