Proof of Collatz Conjecture Using Division Sequence Ⅳ

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Abstract

This paper is positioned as a sequel edition of [1]. First, as in [1], define "division sequence", "complete division sequence", "star conversion", and "extended star conversion". Next, we use Well-Founded Induction and Peirce's law to prove the Collatz conjecture. This proof uses the theorem proving system Idris.

Keywords

Collatz conjecture, Division sequence, Well-Founded Induction, Peirce's law, callCC, Idris

1 Introduction

1.1 Collatz Conjecture

The Collatz conjecture poses the question: "What happens if one repeats the operations of taking any positive integer n,

- Divide n by 2 if n is even, and
- Multiply n by 3 and then add 1 if n is odd

The Collatz conjecture affirms that "for any initial value, one always reaches 1 (and enters a loop of 1 to 4 to 2 to 1) in a finite number of operations."

We call "**(one) Collatz operation**" an operation of performing (3x+1) on an odd number and dividing by 2 as many times as one can.

The "**initial value**" is the number on which the Collatz operation is performed. This initial value is called the "**Collatz value**."

1.2 Division Sequence and Complete Division Sequence

Definition 1.1 A division sequence is the sequence given by arranging the numbers of division by 2 in each operation when the Collatz operation is continuously performed with a positive odd number, n, as the initial value.

For example, in the case of 9, the arrangement of numbers given by continuously performing 3x + 1, and dividing by 2 provides

9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1 (stops when 1 is reached).

Therefore, the division sequence of 9 is [2,1,1,2,3,4].

The division sequence of 1 is an empty list []. Further, [6] is a division sequence of 21, but [6,2] and [6,2,2] ... that repeat the loop of 1 to 4 to 2 to 1 are not division sequences.

When the division sequence is finite, it is equivalent to reaching 1 in a series of Collatz operations.

When the division sequence is infinite, it does not reach 1 in a series of Collatz operations.

It is equivalent to entering a loop other than 4-2-1 or increasing the Collatz value endlessly.

Definition 1.2 A complete division sequence is a division sequence of multiples of 3.

- 9[2,1,1,2,3,4] is a complete division sequence of 9.
- 7[1,1,2,3,4] is a division sequence of 7.

Definition 1.3 Supposing that only one element exists in the division sequence of n, no Collatz operation can be applied to n.

Theorem 1.1 When the Collatz operation is applied to x in the complete division sequence of x (two or more elements), (some) y and its division sequence are obtained.

Proof: This follows the Collatz operation and definition of a division sequence. □

Theorem 1.2 When the Collatz operation is applied to y in the division sequence of y (two or more elements), (some) y and its division sequence are obtained.

Proof: It is self-evident from the Collatz operation and definition of a division sequence. □

1.3 One Only Looks at Odd Numbers of Multiples of 3

There is no need to look at even numbers.

By continuing to divide all even numbers by 2, one of the odd numbers is achieved.

Therefore, it is only necessary to check "whether all odd numbers reach 1 by the Collatz operation."

One only needs to look at multiples of 3.

For a number x that is not divisible by 3, the Collatz inverse operation is defined as obtaining a positive integer by $(x \times 2^k - 1)/3$. Multiple numbers can be obtained using the Collatz reverse operation.
Here, we consider the Collatz reverse operation on x.
The remainder of dividing x by 9 is one of 1,2,4,5,7,8, i.e.:
$1 \times 2^6 \equiv 1$
$2 \times 2^5 \equiv 1$
$4 \times 2^4 \equiv 1$
$5 \times 2^1 \equiv 1$
$7 \times 2^2 \equiv 1$
$8 \times 2^3 \equiv 1 \pmod 9$
This indicates that multiplying any number by 2 appropriate number of times provides an even number with a reminder of 1 when divided by 9.
By subtracting 1 from this and dividing by 3, we get an odd number that is a multiple of 3.
Performing the Collatz reverse operation once from x provides an odd number y that is a multiple of 3.
If y reaches 1, then x, which was once given by the Collatz operation of y, also reaches 1. Therefore, the following can be stated.
Theorem 1.3 One only needs to check “whether an odd number that is a multiple of 3 reaches 1 by the Collatz operation.”

2 Star Conversion

A star conversion is defined for a complete division sequence.
A complete division sequence of length, n, is copied to a complete division sequence of length, n or n+1.
The remainder, which is given by dividing the Collatz value x by 9 is
$x \equiv 3 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[6, a_1 - 4, a_2, a_3...]$ is described as A $[6, -4]$.
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[1, a_1 - 2, a_2, a_3...]$ is described as B $[1, -2]$.
$x \equiv 6 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[4, a_1 - 4, a_2, a_3...]$ is described as C $[4, -4]$.
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[3, a_1 - 2, a_2, a_3...]$ is described as D $[3, -2]$.
$x \equiv 0 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[2, a_1 - 4, a_2, a_3...]$ is described as E $[2, -4]$.
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[5, a_1 - 2, a_2, a_3...]$ is described as F $[5, -2]$.
Furthermore, the conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[a_1 + 6, a_2, a_3...]$ is described as G $[+6]$.
Conversions in which the elements of the division sequence are 0 or negative are prohibited.
If the original first term is 0 or negative, G [+6] is performed in advance.
Example
$117 \equiv 0 \pmod 9$, 117[5,1,2,3,4]
can be converted to E $[2, -4] \rightarrow 9[2, 5-4, 1, 2, 3, 4]$ and F $[5, -2] \rightarrow 309[5, 5-2, 1, 2, 3, 4]$.
Table 1 shows the functions corresponding to each star conversion.
The function represents a change in the Collatz value.

Table 1. Star conversion in mod 9.

When	star conversion 1	star conversion 2
$x \equiv 3 \bmod 9$	A $[6, -4]$ $y = 4x/3 - 7$	B $[1, -2]$ $y = x/6 - 1/2$
$x \equiv 6 \bmod 9$	C $[4, -4]$ $y = x/3 - 2$	D $[3, -2]$ $y = 2x/3 - 1$
$x \equiv 0 \bmod 9$	E $[2, -4]$ $y = x/12 - 3/4$	F $[5, -2]$ $y = 8x/3 - 3$
Always	G $[+6]$ $y = 64x + 21$	none

The star conversion A for 21[6] replaces [6]-A->[6, 2] with [6]-A->[6]. The Collatz value is 21, and it does not change.

3 Extended Star Conversion and Extended Complete Division Sequence

Definition 3.1 The extended star conversion is the conversion in which the star conversion is applied multiple times to the complete division sequence of x excluding the Collatz value x of 3, 9. Table 2 shows the extended star conversion.

Table 2. Extended star conversion and got smaller.

No.	when	Extended star conversion	After conversion	Got smaller
	0 mod 9			
1	9	None because of the base case		
2	72t+45	E[2, −4] $y = x/12 - 3/4$	6t+3	$72t + 45 > 6t + 3$
3	216t+81	DE[3, 0, −4] $y = x/18 - 3/2$	12t+3	$216t + 81 > 12t + 3$
4	216t+153	AE[6, −2, −4] $y = x/9 - 8$	24t+9	$216t + 153 > 24t + 9$
5	216t+225	FE[5, 0, −4] $y = 2x/9 - 5$	48t+45	$216t + 225 > 48t + 45$
6	108t+27	CF[4, 1, −2] $y = 8x/9 - 3$	96t+21	$108t + 27 > 96t + 21$
7	108t+63	BF[1, 3, −2] $y = 4x/9 - 1$	48t+27	$108t + 63 > 48t + 27$
8	108t+99	EF[2, 1, −2] $y = 2x/9 - 1$	24t+21	$108t + 99 > 24t + 21$
	6 mod 9			
9	18t+15	C[4, −4] $y = x/3 - 2$	6t+3	$18t + 15 > 6t + 3$
	3 mod 9			
10	3	None because of the base case		
11	36t+21	B[1, −2] $y = x/6 - 1/2$	6t+3	$36t + 21 > 6t + 3$
12	108t+39	DB[3, −1, −2] $y = x/9 - 4/3$	12t+3	$108t + 39 > 12t + 3$
13	108t+75	AB[6, −3, −2] $y = 2x/9 - 23/3$	24t+9	$108t + 75 > 24t + 9$
14	108t+111	FB[5, −1, −2] $y = 4x/9 - 13/3$	48t+45	$108t + 111 > 48t + 45$

The extended star conversion copies the initial value from $6t + 3$ to $6t' + 3$.

Definition 3.2 The extended complete division sequence is the division sequence obtained by performing the extended star conversion.

Elements of the extended complete division sequence can contain 0 or negative values.

4 DivSeq and allDivSeq

4.1 DivSeq

divSeq t is a function that returns a complete division sequence of 6t + 3 using 6t + 3 as the Collatz value. Using CoList, we can take both finite and infinite lengths.

```
divSeq : Nat -> CoList Integer
divSeq n = divSeq' (S (S (S (n+n+n+n+n+n)))) (S (S (S (n+n+n+n+n+n)))) where
  divSeq' : Nat -> Nat -> CoList Integer
  divSeq' n     Z     = □
  divSeq' Z     (S k) = □
  divSeq' (S n) (S k) with (parity n)
    divSeq' (S (S (j + j))) (S k) | Odd  = divSeq' (S j) k
    divSeq' (S (j + j))     (S k) | Even =
      map toIntegerNat
        (unfoldr (¥b => if b <= 1  then Nothing
                                  else Just (countEven (b*3+1 ) (b*3+1 ) 0) ) (S (j + j)))
```

4.2 AllDivSeq

allDivSeq t is a function that returns a complete division sequence of 6t + 3 and all extended complete division sequences of 6t + 3 using 6t + 3 as the Collatz value.

4.3 FirstLimited and AllLimited

FirstLimited is a predicate that indicates that the complete division sequence of 6t + 3 in allDivSeq t has a finite length.
AllLimited is a predicate that indicates that all division sequences in allDivSeq t have finite lengths.
Theorem 4.1 allDivSeq of 3, 9 is FirstLimited.
Proof: This follows as the complete division sequence of 3, 9 has a finite length. □
Based on this, the program predicates IsFirstLimited10 and IsFirstLimited01.
The proof to the Collatz conjecture in this study is shown by the fact that the finite lengths of all complete division sequences are reduced to IsFirstLimited10 and IsFirstLimited01.

5 Well-Founded Induction Method

In the proof, the well-founded induction library wfInd is used.
(Old source code is used for illustration purposes)

```
makeLimitedDivSeq : (d : Nat) -> (n : Nat)
  -> ((z : Nat) -> (FirstLimited $ allDivSeq d z -> AllLimited $ allDivSeq d z))
    -> FirstLimited $ allDivSeq d n
makeLimitedDivSeq d n firstToAll =
  [★1]wfInd {P=(¥z=>FirstLimited $ allDivSeq d z)} {rel=LT'} (step d firstToAll) n where
  [★2]step : (d : Nat) -> ((z : Nat) -> (FirstLimited $ allDivSeq d z -> AllLimited $ allDivSeq d z))
    -> (x : Nat) -> ((y : Nat) -> [★3]LT' y x -> FirstLimited $ allDivSeq d y)
      -> FirstLimited $ allDivSeq d x
  step _ _          Z      _  = IsFirstLimited10                                    -- 6*<0>+3 = 3
  step _ firstToAll (S x) rs with (mod3 x)
    -- 0 mod 9
    step _ firstToAll (S (Z     + Z     + Z))     rs | ThreeZero = IsFirstLimited01  -- 6*<1>+3 = 9
    step _ firstToAll (S ((S j) + (S j) + (S j))) rs | ThreeZero = ?rhs2
    -- 6 mod 9
    step _ firstToAll (S (S (j + j + j)))     rs | ThreeOne
      = [★4](IsFirstLimited09 j . firstToAll j) (rs j $ lteToLt' $ lte18t15 j)
    -- 3 mod 9
        ...
```

S is a function that gives +1 to a natural number. Z is 0.

- [★1]wfInd is an original function of the well-founded induction method. Incorporate the function “step” into the function “wfInd”.
- The [★2] step is a user-implemented function. A function rs with the type (y: Nat → LT’: y x → P y) can be used.
- [★3] LT’ y x means y < x.
- [★4] In each case division,

the proof of LT’ y x is passed to the function rs to give P y.
firstToAll and IsFirstLimited** are applied here to give P x.

6 Proof of Final Theorem

6.1 Base: wfCollatz

Theorem 6.1 *wfCollatz: ((z: Nat) -> FirstL z -> Cont (AllL z) (FirstL z))*
-> (n: Nat) -> Cont (FirstL n) (FirstL n)

Machine proof: We prove this using the well-founded induction method and case division.
For the number of each case division:

- The number is reduced using rs;
- The predicate is converted to (Cont (AllL z) (FirstL z)) using ftoa;
- Finally, the predicate and numbers are undone using IsFirstLxx.

WfCollatz is a precursor to Peirce's law ((P→Q)→P).

6.2 Proof of Final Theorem: limitedDivSeq

The proof uses Peirce's law. In the world of programs, it is called callCC.

This is used to prove the final theorem.
Theorem 6.2 *limitedDivSeq: (n: Nat) -> FirstL n*
Machine proof:

```
limitedDivSeq n = runCont callCC id where
  callCC = MkCont (\k => runCont ((wfCollatz (\_, a => MkCont (\q => q a))) n) k)
```

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References

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