A new maximum interval between any number and the nearest prime number and related conjectures

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Abstract

In this short paper we prove that for $n \geq 2953652287$ it exists some prime number between n and $n + \log(n)$, improving the best known proved bounds for the maximum interval between any number and the nearest prime number, as well as the maximum difference between two consecutive prime numbers (prime gap). We note that this result proves some open conjectures on prime gaps and maximum intervals between any number and the nearest prime number.

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1 Introduction

The purpose of this paper is establishing a new maximum interval between any positive real number and the nearest prime number, as well as improving the best bounds known on the distance between two consecutive prime numbers (prime gaps).

Some of the most recent results on the subject can be found, for instance, in a recent paper of Axler [2], where there are cited some of the best known explicit bounds for the n_{th} prime number p_n. For this paper, we will use one of the best explicit bounds for the prime counting function $\pi(x)$, which counts the number of prime numbers not exceeding x, published in a paper of Dussart [4]. Concretely, we have that, for $x \geq 2953652287$,

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.334}{\log^2 x}\right) \tag{1}$$

As corollaries of the Main Theorem proved on Section 2 of this paper, they can be proved some of the most known open conjectures on prime gaps and maximum intervals between any number and the nearest prime number. At Section 3 it is provided a non-exhaustive list of the most representative and generalist conjectures on the existence of prime numbers in bounded intervals, as well as a representative and strong open conjecture on the maximum difference between two consecutive prime numbers, all of which can be proved using the Main Theorem.

2 Proof of the Main Theorem

From (1), we have that, for $x \geq 2953652287$,

$$\frac{x}{\log x}(f'(x)) < \pi(x) < \frac{x}{\log x}(g'(x)) \tag{2}$$

Where

$$1 < f'(x) < g'(x)$$

are positive decreasing functions, such that $\lim_{n\to\infty} \frac{g'(x)}{f'(x)} = 1$ and $h'(x) = g'(x) - f'(x)$ is a positive decreasing function.

The main theorem proved in this paper is the following:

Theorem. $\pi(n+\log n)-\pi(n)>1$ *for* $n\geq 2953652287$

Proof.

For convenience, along the proof it is used the fact that $\log(n)=2\log(\sqrt{n})$, and both terms of this equality are used interchangeably.

Applying (2), we have that

$$\frac{\sqrt{n}}{\log\sqrt{n}}(f'(\sqrt{n}))<\pi(\sqrt{n})<\frac{\sqrt{n}}{\log\sqrt{n}}(g'(\sqrt{n})) \tag{3}$$

Also, applying (2), we have that

$$\pi(n+2\log(\sqrt{n})>\frac{n+2\log(\sqrt{n}}{\log(n+2\log(\sqrt{n})}f'(n+2\log(\sqrt{n}))$$

As we have that

$$\frac{n+2\log(\sqrt{n}}{\log(n+2\log(\sqrt{n})}f'(n+2\log(\sqrt{n}))=\frac{\sqrt{n}\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)}{\log\left(\sqrt{n}\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\right)}f'\left(n+2\log(\sqrt{n})\right)$$

Then we have that

$$\pi(n+2\log(\sqrt{n})>\frac{\sqrt{n}\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)}{\log\left(\sqrt{n}\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\right)}f'\left(n+2\log(\sqrt{n})\right) \tag{4}$$

Substituting with (3), we have that

$$\frac{\sqrt{n}\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)}{\log\left(\sqrt{n}\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\right)}f'\left(n+2\log(\sqrt{n})\right)=$$

$$\pi(\sqrt{n})\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\left(\frac{1}{2}\right)\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right)$$

And thus, at the end we have that

$$\pi(n+\log n)>\pi(\sqrt{n})\left(\sqrt{n}+\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\left(\frac{1}{2}\right)\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) \tag{5}$$

Other hand, we have applying (2) that

$$\pi(n)<\frac{n}{\log n}(g'(n)) \tag{6}$$

As we have that

$$\frac{n}{\log n}(g'(n))=\frac{(\sqrt{n})\sqrt{n}}{\log((\sqrt{n})\sqrt{n})}(g'(n))$$

Then we have that

$$\pi(n)<\frac{(\sqrt{n})\sqrt{n}}{\log((\sqrt{n})\sqrt{n})}(g'(n)) \tag{7}$$

Substituting with (3), we have that

$$\frac{(\sqrt{n})\sqrt{n}}{\log((\sqrt{n})\sqrt{n})}(g'(n)) = \pi(\sqrt{n})(\sqrt{n})\left(\frac{1}{2}\right)\left(\frac{g'(n)}{g'(\sqrt{n})}\right) \tag{8}$$

And thus, we have that

$$\pi(n) < \pi(\sqrt{n})(\sqrt{n})\left(\frac{1}{2}\right)\left(\frac{g'(n)}{g'(\sqrt{n})}\right) \tag{9}$$

Substracting (9) to (5), we have that

$$\pi(n+\log n) - \pi(n) > \pi(\sqrt{n})\left(\sqrt{n} + \frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\left(\frac{1}{2}\right)\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) -$$

$$\pi(\sqrt{n})(\sqrt{n})\left(\frac{1}{2}\right)\left(\frac{g'(n)}{g'(\sqrt{n})}\right)$$

Operating, we have that

$$\pi(n+\log n) - \pi(n) > \left(\pi(\sqrt{n})(\sqrt{n})\left(\frac{1}{2}\right)\right)\left(\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) - \left(\frac{g'(n)}{g'(\sqrt{n})}\right)\right) +$$

$$\pi(\sqrt{n})\left(\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\left(\frac{1}{2}\right)\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) \tag{10}$$

Focusing on the first summand of the RHS of (10), we notice that

$$\left(\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) - \left(\frac{g'(n)}{g'(\sqrt{n})}\right)\right) > 0 \quad \forall n \geq 2953652287$$

As a result, the product of terms (which are all positive) forming this summand is always positive.

Focusing on the second summand of the RHS of (10), we can substitute with (3) and cancel terms, to obtain that

$$\pi(\sqrt{n})\left(\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\left(\frac{1}{2}\right)\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) =$$

$$\left(\frac{\sqrt{n}}{\log\sqrt{n}}(f'(\sqrt{n}))\right)\left(\frac{2\log(\sqrt{n})}{\sqrt{n}}\right)\left(\frac{1}{2}\right)\left(\frac{\log(n)}{\log(n+\log(n))}\right)\left(\frac{f'\left(n+2\log(\sqrt{n})\right)}{f'\left(\sqrt{n}\right)}\right) =$$

$$\left(\frac{\log(n)}{\log(n+\log(n))}\right) f'\left(n+2\log(\sqrt{n})\right)$$

Therefore, at the end, we have that

$$\pi(n+\log n) - \pi(n) > \left(\frac{\log(n)}{\log(n+\log(n))}\right) f'\left(n+\log n\right)$$

As $\left(\frac{\log(n)}{\log(n+\log(n))}\right) f'\left(n+\log n\right) > 1 \; \forall n \geq 2953652287$, we finally get that $\pi(n+\log n) - \pi(n) > 1$ for $n \geq 2953652287$, concluding the proof.

3 Non-exhaustive list of open conjectures that can be proved using the Main Theorem

It follows a list of some relevant open conjectures that can be proved using the Main Theorem, ordered by the tightness of the bounds proposed. Each subsequent conjecture implies that the precedent conjecture listed holds.

Note that each of them can be proved just checking that they hold for $n < 2953652287$ and afterwards applying the Main Theorem for $n \geq 2953652287$.

3.1 Legendre's Conjecture

Legendre's Conjecture [7] states that for every natural number n, it exists at least a prime number p such that $n^2 < p < (n+1)^2$.

3.2 Brocard's Conjecture

Brocard's Conjecture [6] states that, if p_n and p_{n+1} are two consecutive prime numbers greater than two, then between p_n^2 and p_{n+1}^2 exist at least four prime numbers.

3.3 Andrica's Conjecture

Andrica's Conjecture [1] states that for every pair of consecutive prime numbers p_n and p_{n+1}, we have that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$

3.4 Oppermann's Conjecture

Oppermann's Conjecture [5] states that for every positive integer n, we have that

$$n^2 - n < p_{\pi(n^2-n)+1} < n^2 < p_{\pi(n^2)+1} < n^2 + n$$

3.5 Cramér's Conjecture

Cramér's Conjecture [3] states that $p_{n+1} - p_n = O((\log p_n)^2)$.

4 Final Remarks and acknowledgements

The Main Theorem sets the strongest possible boundary for intervals between any number and the nearest prime number. Maybe slight refinements can be made, but the order of magnitude is the ”best possible” that can be attained. This can be easily checked, for instance, evaluating the two summands on the RHS of (10) and noting that the first summand tends to 0 and the second summand tends to 1 when n tends to infinity.

I want to specially thank my caring wife Elena for supporting me throughout this marvellous journey of free-time researching and learning during this last eight years. And ”*I praise you, Father, Lord of Heaven and Earth, because you have hidden these things from the wise and learned, and revealed them to little children*” (Matthew 11, 25).

References

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