THE SUNRISE MERIDIAN, ZENITH MERIDIAN, SUNSET MERIDIAN, APPARENT DAYTIME LENGTH RATIO, PUNCTUAL DAYTIME LENGTH RATIO, NUMERICAL EQUATION OF TIME.

ALEXIS ZAGANIDIS

Abstract. In the present article, we derive an analytic formula of the Apparent Daytime Length Ratio with respect to the Earth's orbit angle $\alpha(t)$ between the orbital position of the Earth's perigee and the orbital position of the Earth at a given t, with respect to the latitude, with respect to the Earth's axial tilt ϵ and with respect to the solar apparent diameter θ_{AS}. That formula is simplified a lot under the approximation of a punctual Sun ($\theta_{AS} = 0$) and we name it the Punctual Daytime Length Ratio. The first steps for the derivation of the Apparent Daytime Length Ratio is to consider the parametric curve on the Earth's surface satisfying a tangent sunlight with respect to the parametric angle θ, the Sunlight obliquity angle Ω (the equatorial declination of the Sun) and with respect to a solar apparent diameter θ_{AS}. The second part of the present article is about the precise numerical derivation of the Zenith Time Angle and the precise numerical derivation of the Equation Of Time in the following both cases : at the Meridian Plane S_{JS} parallel to the sunlight at the exact time of the June Solstice, and at the Meridian Plane S_{ME} parallel to the sunlight at the exact time of the March Equinox. For both Zenith Time angles and for both Equations Of Time, there is a tiny latitude dependency/variation of 17 seconds.

1. The Sunrise Meridian, Zenith Meridian, Sunset Meridian, Apparent Daytime Length Ratio, Punctual Daytime Length Ratio.

The parametric curve on the Earth's surface satisfying a tangent sunlight at a parameterized position $\vec{r}_{Tangent}\left(\theta;\Omega,\theta_{AS}\right)$ is the following with respect to the parametric angle θ, the Sunlight obliquity angle Ω (the equatorial declination of the Sun) and with respect to a solar apparent diameter θ_{AS} :

$$\vec{r}^{Tangent}\left(\theta,\Omega,\theta_{AS}\right) = \begin{pmatrix} Cos\left(\Omega\right) & 0 & -Sin\left(\Omega\right) \\ 0 & 1 & 0 \\ Sin\left(\Omega\right) & 0 & Cos\left(\Omega\right) \end{pmatrix} . \begin{pmatrix} -Sin\left(\frac{\theta_{AS}}{2}\right) \\ Cos\left(\frac{\theta_{AS}}{2}\right) Sin\left(\theta\right) \\ Cos\left(\frac{\theta_{AS}}{2}\right) Cos\left(\theta\right) \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} -Sin\left(\frac{\theta_{AS}}{2}\right) Cos\left(\Omega\right) - Cos\left(\frac{\theta_{AS}}{2}\right) Sin\left(\Omega\right) Sin\left(\theta\right) \\ Cos\left(\frac{\theta_{AS}}{2}\right) Cos\left(\theta\right) \\ Cos\left(\frac{\theta_{AS}}{2}\right) Cos\left(\Omega\right) Sin\left(\theta\right) - Sin\left(\frac{\theta_{AS}}{2}\right) Sin\left(\Omega\right) \end{pmatrix} \tag{2}$$

By using the third component of the parameterized position $\vec{r}_{Tangent}\left(\theta;\Omega,\theta_{AS}\right)$ on the Earth's surface, we can substitute the parameterized angle θ by the latitude angle λ :

Date: October 12, 2023.

$$\theta\left(\lambda;\Omega,\theta_{AS}\right)=ArcSin\left(\frac{Sin\left(\lambda\right)+Sin\left(\frac{\theta_{AS}}{2}\right)Sin\left(\Omega\right)}{Cos\left(\frac{\theta_{AS}}{2}\right)Cos\left(\Omega\right)}\right) \tag{3}$$

The Sunlight vector $\vec{r}_{Sun}$ in the equatorial coordinate system is the following with respect to the Earth's orbit angle $\alpha\left(t\right)$ between the orbital position of the Earth's perigee and the orbital position of the Earth at a given t, and with respect to the Earth's axial tilt ϵ :

$$\vec{r}^{Sun}\left(\theta\right)=\begin{pmatrix}Cos\left(\epsilon\right) & 0 & -Sin\left(\epsilon\right)\\ 0 & 1 & 0\\ Sin\left(\epsilon\right) & 0 & Cos\left(\epsilon\right)\end{pmatrix}\cdot\begin{pmatrix}Sin\left(\alpha\right)\\ -Cos\left(\alpha\right)\\ 0\end{pmatrix} \tag{4}$$

$$=\begin{pmatrix}Cos\left(\epsilon\right)Sin\left(\alpha\right)\\ -Cos\left(\alpha\right)\\ Sin\left(\epsilon\right)Sin\left(\alpha\right)\end{pmatrix} \tag{5}$$

Therefore, we can deduce :

$$\Omega\left(\alpha\right)=ArcSin\left(Sin\left(\epsilon\right)Sin\left(\alpha\right)\right) \tag{6}$$

$$\phi_{\Omega}=ArcTan\left(Cos\left(\epsilon\right)Tan\left(\alpha\right)\right)+\pi Round\left(\frac{\alpha}{\pi}\right) \tag{7}$$

(8)

From here, we can derive the expression of the Sunrise Meridian $\phi_{Sunrise}$, the Zenith Meridian ϕ_{Zenith}, the Sunset Meridian ϕ_{Sunset} and the Apparent Daytime Length Ratio $r_{Daytime}$:

(9)

$$\phi_{Sunrise}\left(\alpha\right)=\phi_{\Omega}-2\pi\left(1+\frac{1\ day}{1\ year}\right)\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right)+ArcTan\left(\vec{r}_1^{Tangent}\left(\alpha\right)/\vec{r}_2^{Tangent}\left(\alpha\right)\right)-\frac{\pi}{2}$$

$$\phi_{Zenith}\left(\alpha\right)=\phi_{\Omega}-2\pi\left(1+\frac{1\ day}{1\ year}\right)\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right) \tag{10}$$

(11)

$$\phi_{SunSet}\left(\alpha\right)=\phi_{\Omega}-2\pi\left(1+\frac{1\ day}{1\ year}\right)\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right)-ArcTan\left(\vec{r}_1^{Tangent}\left(\alpha\right)/\vec{r}_2^{Tangent}\left(\alpha\right)\right)+\frac{\pi}{2}$$

$$r_{Daytime}\left(\alpha\right)=\frac{1}{2}-\frac{1}{\pi}ArcTan\left(\vec{r}_1^{Tangent}\left(\alpha\right)/\vec{r}_2^{Tangent}\left(\alpha\right)\right) \tag{12}$$

α_0 is the angle between the March Equinox and the Earth's perigee :

(13)

$$\tilde{t}\left(\alpha\right)=\frac{1\ year}{1\ day}\frac{1}{2\pi}\left(2\ ArcTan\left(\sqrt{\frac{1-e}{1+e}}Tan\left(\frac{\alpha}{2}\right)\right)-\frac{e\sqrt{1-e^2}\ Sin\left(\alpha\right)}{1+e\ Cos\left(\alpha\right)}+2\pi\ Round\left(\frac{\alpha}{2\pi}\right)\right)$$

(14)

$$\alpha_0=\frac{\pi}{180^\circ}\left(180^\circ-102.9^\circ\right)$$

We can simplify the following expression :

(15)

$$-\ ArcTan\left(\vec{r}_1^{\,Tangent}\left(\alpha\right)/\vec{r}_2^{\,Tangent}\left(\alpha\right)\right)=$$

(16)

$$ArctTan\left(\frac{Sec\left(\frac{\theta_{AS}}{2}\right)Sin\left(\alpha\right)Sin\left(\epsilon\right)Sin\left(\lambda\right)+Tan\left(\frac{\theta_{AS}}{2}\right)}{\sqrt{1-Sec^2\left(\frac{\theta_{AS}}{2}\right)\left(Sin^2\left(\alpha\right)Sin^2\left(\epsilon\right)+Sin^2\left(\lambda\right)\right)-2\ Sec\left(\frac{\theta_{AS}}{2}\right)Tan\left(\frac{\theta_{AS}}{2}\right)Sin\left(\alpha\right)Sin\left(\epsilon\right)Sin\left(\lambda\right)}}\right)$$

(17)

$$\xrightarrow{\theta_{AS}\to 0} ArctTan\left(\frac{Sin\left(\alpha\right)Sin\left(\epsilon\right)Sin\left(\lambda\right)}{\sqrt{Cos^2\left(\lambda\right)-Sin^2\left(\alpha\right)Sin^2\left(\epsilon\right)}}\right)$$

The case $\theta_{AS}=0$ is the punctual daytime length with the Sun's Center above the horizon, the case $\theta_{AS}>0$ is the daytime length with the the Sun's Disk intersecting the horizon or above the horizon, and the case $\theta_{AS}<0$ is the daytime length with the entire Sun's Disk above the horizon.

To conclude, the Sunrise/Sunset Meridian are slightly approximate since the Sunlight obliquity angle Ω change slightly between the Azimuth Meridian and the Sunrise/Sunset Meridian. Moreover, the Azimuth Meridian is very slightly approximate between the June Solstice and the December Solstice since the solar zenith angle is not constant in the neighborhood of the Azimuth Meridian, especially at the Mach/September Equinoxes.

2. Numerical Equation of Time.

To derive the Equation of time and the latitude/time dependency of the Solar Zenith angle, we can choose the Meridian Plane S_{JS} parallel to the sunlight at the exact time of the June Solstice.

The time/latitude dependency formula of the Solar Zenith angle for the Meridian Plane S_{JS} :

$$\theta_s=\frac{180^\circ}{\pi}$$

(18)

$$VectorAngle\left(\begin{pmatrix}Cos\left(\epsilon\right) & 0 & Sin\left(\epsilon\right)\\ 0 & 1 & 0\\ -Sin\left(\epsilon\right) & 0 & Cos\left(\epsilon\right)\end{pmatrix}.\begin{pmatrix}Cos\left(\frac{\pi}{180^\circ}\lambda\right)Cos\left(\omega\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right)\right)\\ Cos\left(\frac{\pi}{180^\circ}\lambda\right)Sin\left(\omega\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right)\right)\\ Sin\left(\frac{\pi}{180^\circ}\lambda\right)\end{pmatrix},\begin{pmatrix}Cos\left(\alpha\right)\\ Sin\left(\alpha\right)\\ 0\end{pmatrix}\right)$$

(19)

$$\omega = 2\pi \left(1 + \frac{1\ year}{1\ day}\right)$$

(20)

$$\tilde{t}\left(\alpha\right) = \frac{1\ year}{1\ day}\frac{1}{2\pi}\left(2\ ArcTan\left(\sqrt{\frac{1-e}{1+e}}Tan\left(\frac{\alpha}{2}\right)\right) - \frac{e\sqrt{1-e^2}\ Sin\left(\alpha\right)}{1+e\ Cos\left(\alpha\right)} + 2\pi\ Round\left(\frac{\alpha}{2\pi}\right)\right)$$

(21)

$$\alpha_0 = \frac{\pi}{180°}\left(180° - 102.9° + 90°\right)$$

Alternatively, to derive the Equation of time and the latitude/time dependency of the Solar Zenith angle, we can choose the Meridian Plane S_{ME} parallel to the sunlight at the exact time of the March Equinox.

The time/latitude dependency formula of the Solar Zenith angle for the Meridian Plane S_{ME} :

$$\theta_s = \frac{180°}{\pi}$$

(22)

$$VectorAngle\left(\begin{pmatrix}1 & 0 & 0\\ 0 & Cos\left(\epsilon\right) & Sin\left(\epsilon\right)\\ 0 & -Sin\left(\epsilon\right) & Cos\left(\epsilon\right)\end{pmatrix}.\begin{pmatrix}Cos\left(\frac{\pi}{180°}\lambda\right)Cos\left(\omega\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right)\right)\\ Cos\left(\frac{\pi}{180°}\lambda\right)Sin\left(\omega\left(\tilde{t}\left(\alpha+\alpha_0\right)-\tilde{t}\left(\alpha_0\right)\right)\right)\\ Sin\left(\frac{\pi}{180°}\lambda\right)\end{pmatrix}, \begin{pmatrix}Cos\left(\alpha\right)\\ Sin\left(\alpha\right)\\ 0\end{pmatrix}\right)$$

(23)

$$\omega = 2\pi \left(1 + \frac{1\ year}{1\ day}\right)$$

(24)

$$\tilde{t}\left(\alpha\right) = \frac{1\ year}{1\ day}\frac{1}{2\pi}\left(2\ ArcTan\left(\sqrt{\frac{1-e}{1+e}}Tan\left(\frac{\alpha}{2}\right)\right) - \frac{e\sqrt{1-e^2}\ Sin\left(\alpha\right)}{1+e\ Cos\left(\alpha\right)} + 2\pi\ Round\left(\frac{\alpha}{2\pi}\right)\right)$$

(25)

$$\alpha_0 = \frac{\pi}{180°}\left(180° - 102.9°\right)$$

The 365 minima of the solar Zenith angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (the 365 Solar Noons) can be calculated with a Mathematica FindRoot function with the starting points as the angle α_i of the i-th solar day. Again, the angle α_i of the i-th solar day can be can be calculated with the following Mathematica FindRoot function with the starting points as the angle $2\pi\frac{1\ day}{1\ year}$:

$$\tilde{t}\left(\alpha_i + \alpha_0\right) - \tilde{t}\left(\alpha_0\right) - i = 0 \tag{26}$$

Therefore, the equation of time reduces to $\Delta t = \tilde{t}\left(\alpha_i^{max} + \alpha_0\right) - \tilde{t}\left(\alpha_0\right) - i$.

To conclude, the is a very slight time difference between the solar noon with different latitudes on the same Meridian between the June Solstice and the December

Solstice peaking at the Mach Equinox and the September Equinox. In other words, at every time on the Earth's surface, the sunrise, solar noon and the sunset are 3 lines merging all together at one point in the North/South hemisphere. The punctual sunrise/sunset line is a circular line. The solar noon line almost fit a Meridian and fit exactly a Meridian at the June/December solstice, while the punctual sunrise/sunset line fit exactly a Meridian at the Mach/September Equinox.

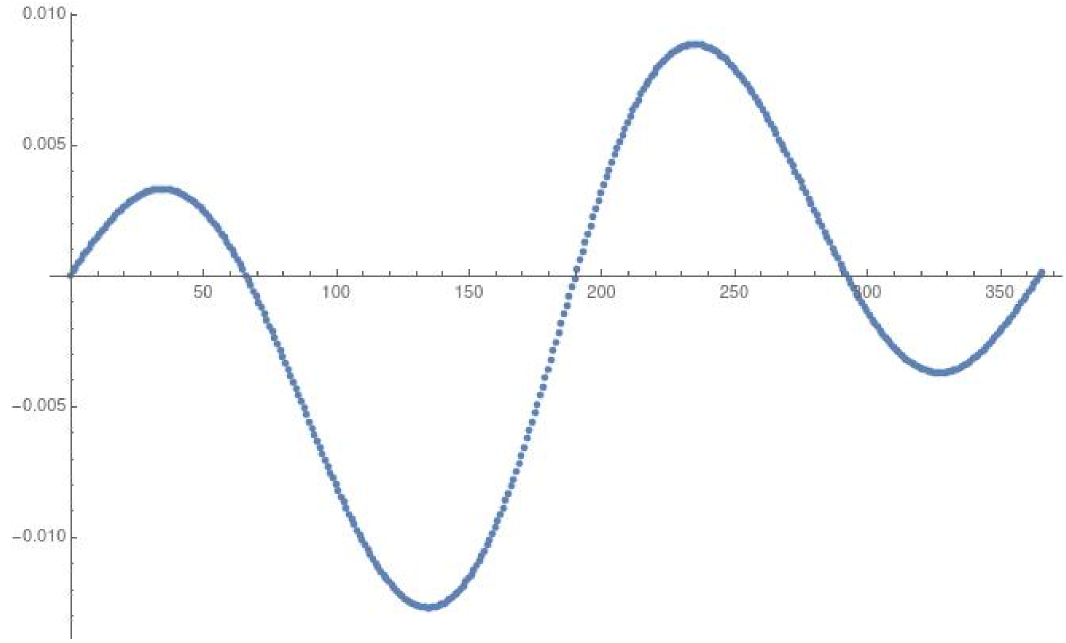

FIGURE 1. The Equation of Time Δt in day unit throughout the year with the Mach Equinox Origin and with the Equator Origin.

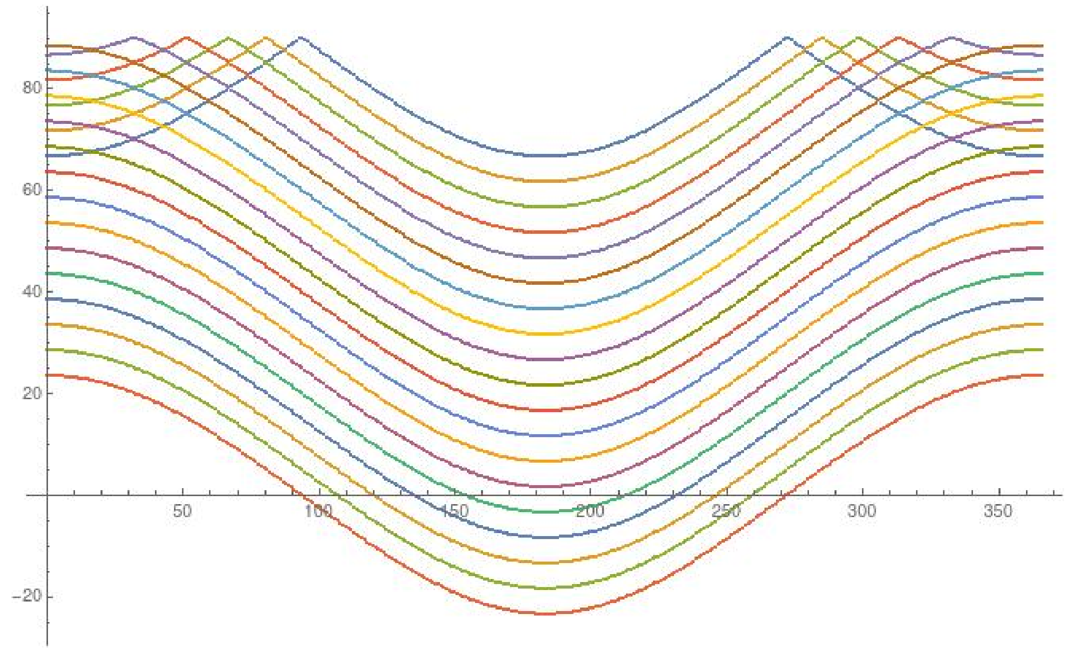

FIGURE 2. The Minima of the Solar Zenith Angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (Solar Noon) throughout the year with the Mach Equinox Origin and with the dynamic Latitude parameter λ.

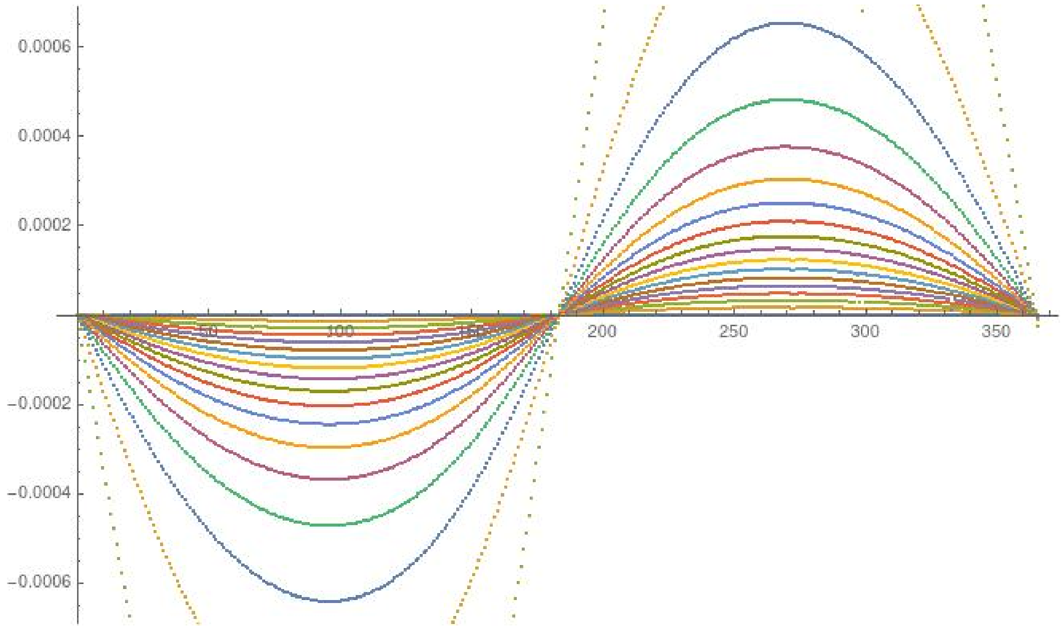

FIGURE 3. The Time difference in day unit throughout the year of the Zoomed Minima of the Solar Zenith Angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (Solar Noon) with respect to the ones at the Equator, with the Mach Equinox Origin and with the dynamic Latitude parameter λ.

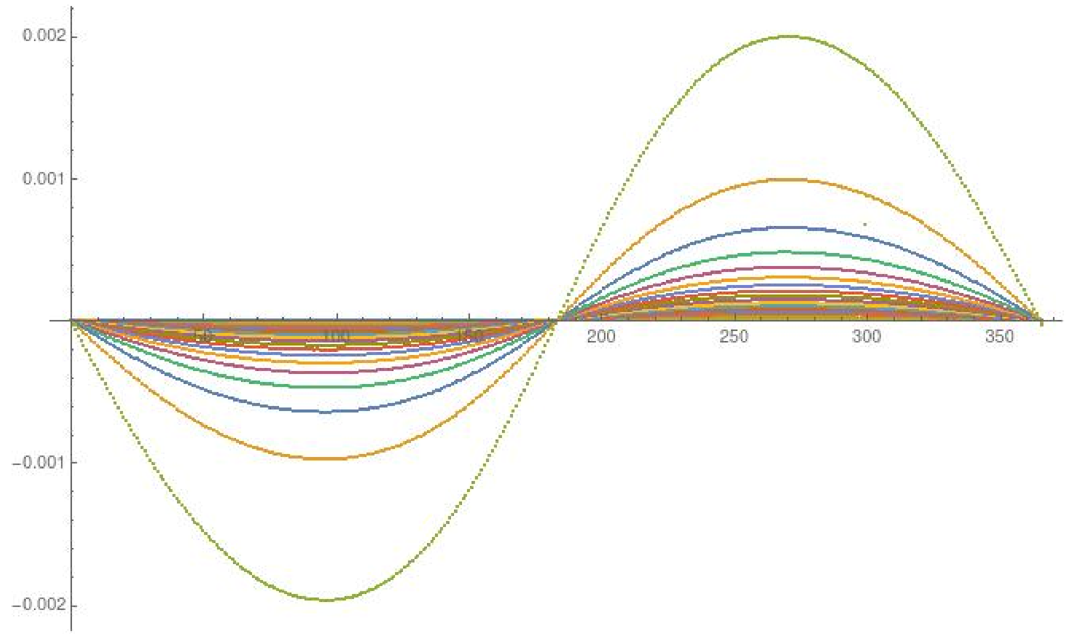

FIGURE 4. The Time difference in day unit throughout the year of the Minima of the Solar Zenith Angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (Solar Noon) with respect to the ones at the Equator, with the Mach Equinox Origin and with the dynamic Latitude parameter λ.

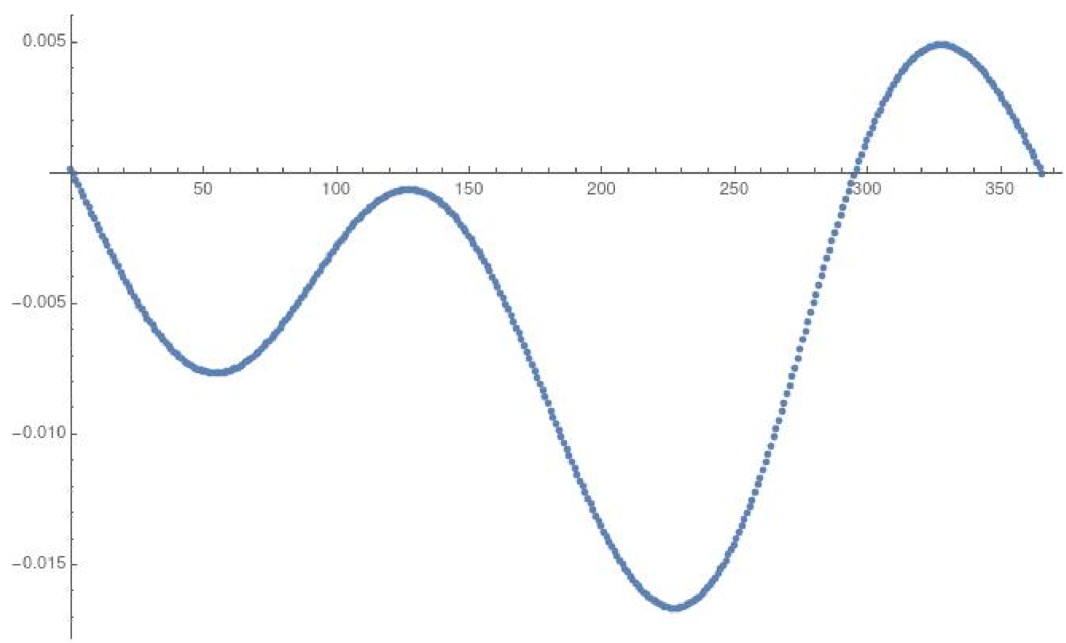

FIGURE 5. The Equation of Time Δt in day unit throughout the year with the Mach Equinox Origin and with the Equator Origin.

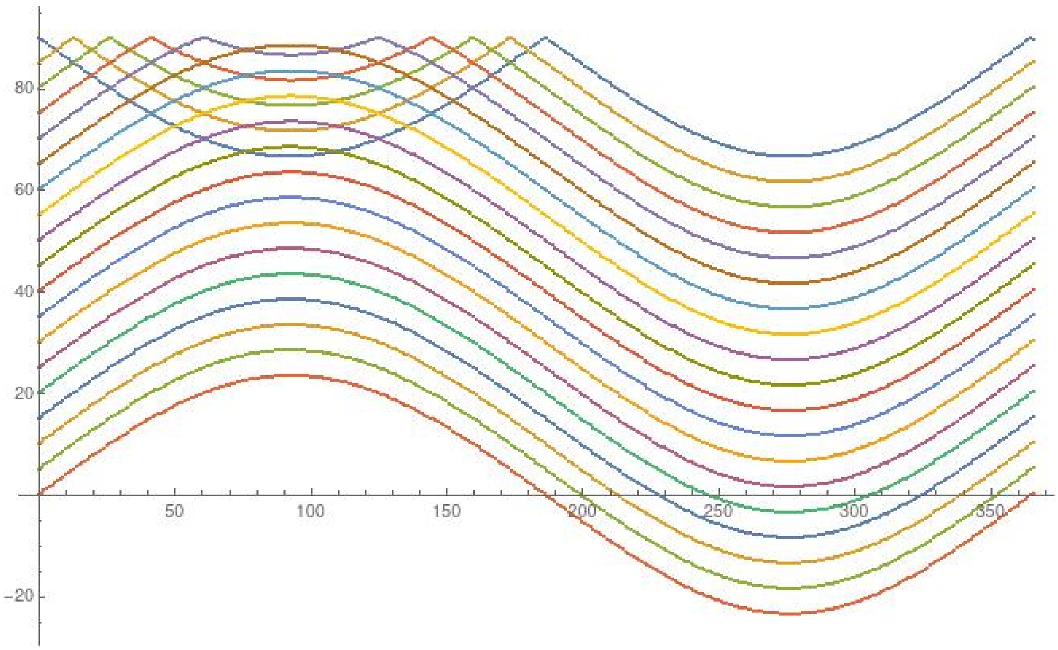

FIGURE 6. The Minima of the Solar Zenith Angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (Solar Noon) throughout the year of with the Mach Equinox Origin and with the dynamic Latitude parameter λ.

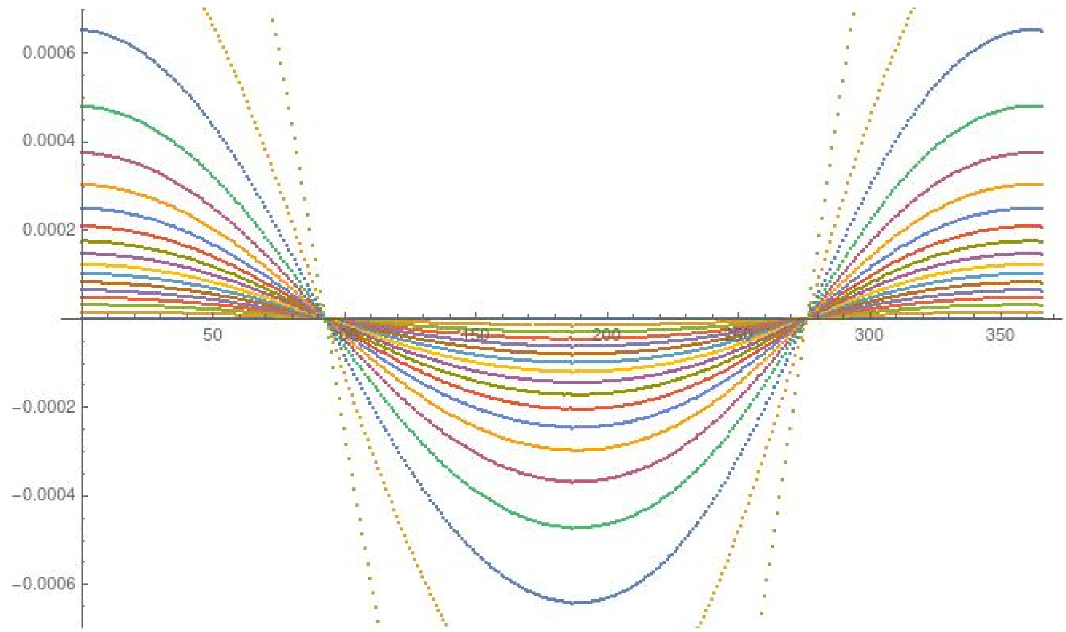

FIGURE 7. The Time difference in day unit throughout the year of the Zoomed Minima of the Solar Zenith Angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (Solar Noon) with respect to the ones at the Equator, with the Mach Equinox Origin and with the dynamic Latitude parameter λ.

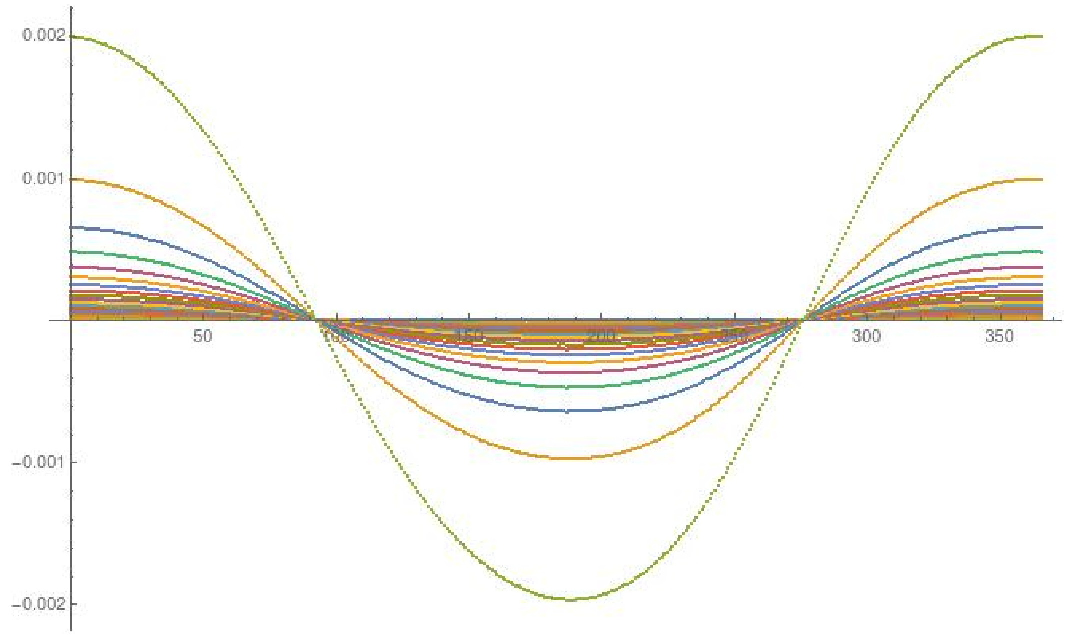

FIGURE 8. The Time difference in day unit throughout the year of the Minima of the Solar Zenith Angle $\theta_{s,i}^{max} = \theta_s\left(\alpha_i^{max}\right)$ (Solar Noon) with respect to the ones at the Equator, with the Mach Equinox Origin and with the dynamic Latitude parameter λ.

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