A theorem on the Golden Section and Fibonacci numbers

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ABSTRACT

In the chapter 12° of his most significant book *LIBER ABACI*, Leonardo Pisano known as Fibonacci (Pisa 1170-1240 (?)) proposed a problem on the reproduction of rabbits **[*]**. So many scholars deduced that he arrives to his famous numerical sequence starting from this problem. In this article is explained a new hypothesis. The Fibonacci sequence was generated by the iteration of a theorem on the Golden Section, and it is presumably that was the great Italian mathematician to state and demonstrate it. The theorem allow us to proof a lot of properties of Fibonacci numbers.

[*] A man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced by the initial couple in a year supposing every couple each month produces a new pair that can reproduce itself from the second month?

CONTENTS

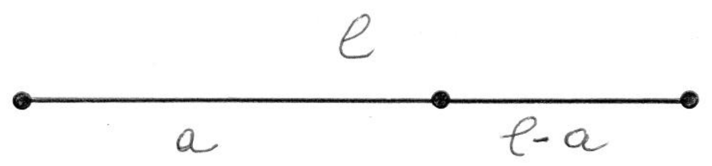
1. The Golden Section

Many math texts use the Greek letter φ to indicate the Golden Section and Φ to indicate the Golden Ratio. In this article, for such sizes, has been preferred to use the letters *a*, A; and u_n to indicate the generic number of the Fibonacci sequence.

Definition:

The golden section of a segment is the part of the segment mean proportional between the whole segment and the remaining part.

With reference to figure



we have : $l : a = a : l - a$

for the fundamental property of proportions it is :

$$a^2 = l \cdot (l - a)$$

whence : $a^2 + l \cdot a - l^2 = 0$

Solving we obtain the acceptable (positive) solution : $a = \left(\frac{\sqrt{5}-1}{2}\right) \cdot l$ (1)

Place $l = 1 \rightarrow a = 0{,}61803398875\ldots\ldots$ irrational value.

Theorem

If a is the Golden Section of l, then l is the Golden Section of l + a.

Demonstration :

From $l : a = a : (l - a)$ applying composing property we have: $(l + a) : l = (a + l - a) : a \rightarrow$ $(l + a) : l = l : a$ where *a* it's the remaining part $(l + a) - l$.

From the definition of the Golden Section we have :

$$(l + a) : l = l : (l + a) - l$$

i.e : $(l + a) : l = l : a$

from which the quadratic equation : $l^2 = (l + a) \cdot a \rightarrow l^2 - a \cdot l - a^2 = 0$

resolving we obtain the acceptable (positive) solution : $l = \left(\frac{\sqrt{5}+1}{2}\right) \cdot a$ (2)
where $\left(\frac{\sqrt{5}+1}{2}\right) = 1/a$ = A (Golden Ratio)

Replacing (1) in (2) we have :

$$l = \left(\frac{\sqrt{5}+1}{2}\right) \cdot \left(\frac{\sqrt{5}-1}{2}\right) \cdot l = \left(\frac{5-1}{4}\right) \cdot l = l \rightarrow 1 = 1 \rightarrow QED$$

So: $a \cdot 1/a = a \cdot$A = 1

The equation $l^2 - a \cdot l - a^2 = 0$ can be write $a^2 + l \cdot a - l^2 = 0$ from which the value of Golden Section.

Iterating the theorem we get the following algebraic succession :

$$l + 1a \rightarrow l$$

$$2l + 1a \rightarrow l + a$$

$$3l + 2a \rightarrow 2l + 1a$$

$$S(a) \qquad 5l + 3a \rightarrow 3l + 2a \quad (3)$$

$$8l + 5a \rightarrow 5l + 3a$$

$$13l + 8a \rightarrow 8l + 5a \quad (5)$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$u_{n+1} \cdot l + u_n \cdot a \rightarrow u_n \cdot l + u_{n-1} \cdot a$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

Check of the (3) :

$(5l + 3a) : (3l + 2a) = (3l + 2a) : [(5l + 3a) - (3l + 2a)]$
$(5l + 3a) : (3l + 2a) = (3l + 2a) : (2l + a)$
$(3l + 2a)^2 = (5l + 3a) \cdot (2l + a)$
$9l^2 + 12al + 4a^2 = 10l^2 + 5al + 6al + 3a^2$
$l^2 - al - a^2 = 0$ resolving equation

Check of the (5) :

$(13l + 8a) : (8l + 5a) = (8l + 5a) : [(13l + 8a) - (8l + 5a)]$
$(13l + 8a) : (8l + 5a) = (8l + 5a) : (5l + 3a)$
$(8l + 5a)^2 = (13l + 8a) \cdot (5l + 3a)$
$64l^2 + 80al + 25a^2 = 65l^2 + 39al + 40al + 24a^2$
$l^2 - al - a^2 = 0$ resolving equation

If in the sequence S(a) we consider the coefficients of the Golden Section we have the numerical sequence :

$$\{1;\ 1;\ 2;\ 3;\ 5;\ 8;\ 13;\ \ldots;\ u_{n-1};\ u_n;\ u_{n+1};\ldots\}$$

That is the sequence of Leonardo Pisano, known as Fibonacci.

It is likely that the Italian mathematician has come to the succession above by the study of the Golden Section and applying it the theorem proved above. Indeed, it is to be assumed that he was to state and prove it.

2. Properties of Fibonacci numbers

Taking into consideration the general relation: $u_{n+1}\cdot l + u_n\cdot a \rightarrow u_n\cdot l + u_{n-1}\cdot a$ from which the proportion :

$$(u_{n+1}\cdot l + u_n\cdot a) : (u_n\cdot l + u_{n-1}\cdot a) = (u_n\cdot l + u_{n-1}\cdot a) : [(u_{n+1}\cdot l + u_n\cdot a) - (u_n\cdot l + u_{n-1}\cdot a)] \rightarrow$$
$$(u_{n+1}\cdot l + u_n\cdot a) : (u_n\cdot l + u_{n-1}\cdot a) = (u_n\cdot l + u_{n-1}\cdot a) : [(u_{n+1} - u_n)\cdot l + (u_n - u_{n-1})] \rightarrow$$
$$(u_{n+1}\cdot l + u_n\cdot a) : (u_n\cdot l + u_{n-1}\cdot a) = (u_n\cdot l + u_{n-1}\cdot a) : (u_{n-1}\cdot l + u_{n-2}\cdot a)$$

Hence : $(u_n\cdot l + u_{n-1}\cdot a)^2 = (u_{n+1}\cdot l + u_n\cdot a)\cdot (u_{n-1}\cdot l + u_{n-2}\cdot a) \rightarrow \ldots \rightarrow$

$$(u_n^2 - u_{n+1}\cdot u_{n-1})l^2 + (u_n\cdot u_{n-1} - u_{n+1}\cdot u_{n-2})al + (u_{n-1}^2 - u_n\cdot u_{n-2})a^2$$

Comparing with the resolving equation: $l^2 - al - a^2 = 0$ we have :

$$\begin{cases} u_n^2 - u_{n+1}\cdot u_{n-1} = 1 \\ u_n\cdot u_{n-1} - u_{n+1}\cdot u_{n-2} = -1 \\ u_{n-1}^2 - u_n\cdot u_{n-2} = -1 \end{cases} \rightarrow$$

(1) $\begin{cases} a)\ u_n^2 - u_{n+1}\cdot u_{n-1} = 1 \\ b)\ u_{n+1}\cdot u_{n-2} - u_n\cdot u_{n-1} = 1 \\ c)\ u_n\cdot u_{n-2} - u_{n-1}^2 = 1 \end{cases}$ if we consider the Golden Ratio.

(2) $\begin{cases} u_n^2 - u_{n+1}\cdot u_{n-1} = -1 \\ u_{n+1}\cdot u_{n-2} - u_n\cdot u_{n-1} = -1 \\ u_n\cdot u_{n-2} - u_{n-1}^2 = -1 \end{cases}$ if we consider the Golden Section.

In general, for four consecutive Fibonacci numbers result :

(3) $\begin{cases} u_n^2 - u_{n+1}\cdot u_{n-1} = \pm 1 \\ u_{n+1}\cdot u_{n-2} - u_n\cdot u_{n-1} = \pm 1 \\ u_n\cdot u_{n-2} - u_{n-1}^2 = \pm 1 \end{cases}$

Remembering :

$S(u) = \{1;\ 1;\ 2;\ 3;\ 5;\ 8;\ 13;\ 21;\ 34;\ 55;\ 89;\ \ldots;\ u_{n-2};\ u_{n-1};\ u_n;\ u_{n+1};\ldots\}$

Check :

a) 13^2 - 21·8 = 169 - 168 = 1
b) 21·5 - 13·8 = 105 - 104 = 1
c) 13·5 - 8^2 = 65 - 64 = 1

a) 55^2 - 89·34 = 3025 - 3026 = -1
b) 89·21 - 55·34 = 1869 - 1870 = -1
c) 55·21 – 34^2 = 1155 - 1156 = -1

If we choose a) b) of (1) and apply a linear combination for subtraction, we have :

$$a)\ u_n^2 - u_{n+1} \cdot u_{n-1} = 1$$
$$b)\ u_{n+1} \cdot u_{n-2} - u_n \cdot u_{n-1} = 1$$

$(u_n^2 - u_{n+1} \cdot u_{n-1}) - (u_{n+1} \cdot u_{n-2} - u_n \cdot u_{n-1}) = 0 \ \rightarrow$
$u_n^2 - u_{n+1} \cdot u_{n-1} - u_{n+1} \cdot u_{n-2} + u_n \cdot u_{n-1} = 0 \ \rightarrow$
$u_n^2 + \mathrm{u_{n-1}} \cdot (\mathrm{u_n} - \mathrm{u_{n+1}}) = u_{n+1} \cdot u_{n-2}$

Bearing in mind that $u_{n+1} = u_n + u_{n-1}$

$u_n^2 + \mathrm{u_{n-1}} \cdot (\mathrm{u_n} - \mathrm{u_n} - \mathrm{u_{n-1}}) = u_{n+1} \cdot u_{n-2} \ \rightarrow \ u_n^2 - u_{n-1}^2 = u_{n+1} \cdot u_{n-2}$

Check :

If we choose the four consecutive Fibonacci numbers: 3; 5; 8; 13 result: 64 - 25 = 13·3
If we choose 144; 233; 377; 610 result: 142129 - 54289 = 144·610

If we apply a linear combination for subtraction between a) c) of (1) we have :

$$a)\ u_n^2 - u_{n+1} \cdot u_{n-1} = 1$$
$$c)\ u_n \cdot u_{n-2} - u_{n-1}^2 = 1$$

$(u_n^2 - u_{n+1} \cdot u_{n-1}) - (u_n \cdot u_{n-2} - u_{n-1}^2) = 0 \rightarrow \ \ldots \rightarrow \ u_n^2 + u_{n-1}^2 = u_{n+1} \cdot u_{n-1} + u_n \cdot u_{n-2}$

And so on for many other properties.

We list some properties of Fibonacci numbers.

1) The sum of any ten consecutive Fibonacci numbers is always divisible by 11, and the quotient ranks seventh in the chosen sequence.
2) By adding more consecutive numbers of the sequence, from the first one and adding 1 you get another Fibonacci number which follows by two places the last number in the sequence.
3) Any number divided by the penultimate number preceding it in the sequence, you get quotient 2 and the remainder is the number preceding the divider.
4) The square of any number in the sequence is equal to the number preceding it, multiplied by the number that follows it, plus or minus 1.
5) The greatest common divisor of any two Fibonacci numbers is also a Fibonacci number.
6) The sum of the squares of two consecutive numbers of the sequence is equal to the number that occupies the place obtained from the sum of the places of the considered numbers.
7) The only square number in this sequence is 144.
8) The only cubic number in the sequence is 8.

We list some examples for each of them.

1) $u_n + u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4} + u_{n+5} + u_{n+6} + u_{n+7} + u_{n+8} + u_{n+9} = s/11 = u_{n+6}$

Examples :

$3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144 + 233 = 605/11 = 55 = u_7$
$8 + 13 + 21 + 34 + 55 + 89 + 144 + 233 + 377 + 610 = 1584/11 = 144 = u_7$
... ...

2) $u_1 + u_2 + u_3 + ... + u_n + 1 = u_{n+2}$

Examples :

$1 + 1 + 2 + 3 + 5 + 1 = 13 = u_7$
$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 1 = 55 = u_{10}$
... ...

3) $u_n/u_{n-2} = 2$ *and* $r = u_{n-3}$

Examples :

$u_{14}/u_{12} = 377/144 = 2$ *and* $r = u_{11} = 89$
$u_{20}/u_{18} = 6765/2584 = 2$ *and* $r = u_{17} = 1597$
... ...

4) $u_n^2 = u_{n-1} \cdot u_{n+1} \pm 1$

This property is a) of (3)

Examples :

$u_5^2 = 21^2 = 441 = 13 \cdot 34 - 1 = u_4 \cdot u_6 - 1$
$u_{15}^2 = 610^2 = 372100 = 377 \cdot 987 + 1 = u_{14} \cdot u_{16} + 1$
... ...

5) $MCD(u_i;u_j) = u_k \in S(u)$

Examples :

$MCD(u_9;u_{12}) = MCD(34;144) = u_3 = 2$
$MCD(u_{15};u_{20}) = MCD(610;6765) = u_5 = 5$
$MCD(u_{18};u_{24}) = MCD(2584;46368) = u_6 = 8$
... ...

6) $u_{n-1}^2 + u_n^2 = u_{2n-1}$

Examples :

$u_4^2 + u_5^2 = 3^2 + 5^2 = 9 + 25 = 34 = u_9$

$u_7^2 + u_8^2 = 13^2 + 21^2 = 169 + 441 = 610 = u_{15}$
$u_{13}^2 + u_{14}^2 = 233^2 + 377^2 = 54289 + 142129 = 196418 = u_{27}$

… …

The properties 7) and 8) may be considered as conjectures.

3. Pythagorean tern

For four consecutive Fibonacci numbers u_{n-2}; u_{n-1}; u_n; u_{n+1} is valid the property exposed in 1948 by mathematician Charles Raine:

The product of extremes terms, twice the product of the mean terms, the sum of squares of the mean terms, form a Pythagorean tern, and the sum of squares of middles $(u_n^2 + u_{n-1}^2)$ *it is a Fibonacci number.*

i. e. : $$(u_{n-2} \cdot u_{n+1})^2 + (2 \cdot u_{n-1} \cdot u_n)^2 = (u_n^2 + u_{n-1}^2)^2$$

Here is the general proof of this important property :

$$u_{n-2}^2 \cdot u_{n+1}^2 + 4 \cdot u_{n-1}^2 \cdot u_n^2 = (u_{n-1}^2 + u_n^2)^2$$

$$u_{n-2}^2 \cdot u_{n+1}^2 = u_{n-1}^4 + u_n^4 + 2 \cdot u_{n-1}^2 \cdot u_n^2 - 4 \cdot u_{n-1}^2 \cdot u_n^2$$

$$u_{n-2}^2 \cdot u_{n+1}^2 = u_{n-1}^4 + u_n^4 - 2 \cdot u_{n-1}^2 \cdot u_n^2$$

Bearing in mind: $u_{n+1} = u_n + u_{n-1}$

$$(u_n + u_{n-1})^2 \cdot u_{n-2}^2 = (u_{n-1}^2 - u_n^2)^2$$

$$(u_n + u_{n-1})^2 \cdot u_{n-2}^2 = (u_n + u_{n-1})^2 \cdot (u_n - u_{n-1})^2$$

Bearing in mind : $u_n - u_{n-1} = u_{n-2}$

$u_{n-2}^2 = u_{n-2}^2 \rightarrow 1 = 1 \rightarrow$ QED

Checks :

{…; 5; 8; 13; 21; …} : $5 \cdot 21 = 105$; $2 \cdot 8 \cdot 13 = 208$; $8^2 + 13^2 = 233$
(105; 208; 233) : $105^2 + 208^2 = 233^2$ in fact: $11025 + 43264 = 54289$
And 233 is present in Fibonacci sequence.

{…; 13; 21; 34; 55; …} : $13 \cdot 55 = 715$; $1 \cdot 21 \cdot 34 = 1428$; $21^2 + 34^2 = 441 + 1156 = 1597$
(715; 1428; 1597) : $715^2 + 1428^2 = 1597^2$ in fact: $511225 + 2039184 = 1550409$
And 1597 is present in Fibonacci sequence.

4. The Golden Ratio

From the first proportion $l : a = a : l - a \rightarrow a^2 = l^2 - al \rightarrow a^2 + al - l^2 = 0 \rightarrow$

$$a = \frac{\sqrt{5}-1}{2} = 0{,}6180398875\ \ldots$$

and the Golden Ratio $\quad A = 1/a = 1{,}6180398875\ \ldots$

We note that *a* and *A* have the same decimal part.

From the resolving equation $a^2 + al - l^2 = 0 \rightarrow l^2 - al = a^2 \rightarrow l^2/a^2 - l/a = 1$; put $l = 1$ we have :

$$A^2 = A + 1$$

If we consider the relation $l + a \rightarrow l$ and the relative proportion $l + a : l = l : a$ the resolving equation is $l^2 - al - a^2 = 0 \rightarrow l^2/a^2 - l/a = 1$; put $l = 1$ we have :

$$A^2 = A + 1$$

Check :

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{\sqrt{5}+1}{2} + 1 \rightarrow \frac{2\sqrt{5}+6}{4} = \frac{\sqrt{5}+3}{2} \rightarrow \frac{\sqrt{5}+3}{2} = \frac{\sqrt{5}+3}{2} \rightarrow \text{QET}$$

Summary :

$A = A$
$A^2 = A + 1$
$A^3 = A^2 \cdot A = (A+1) \cdot A = A^2 + A$
$A^4 = A^3 \cdot A = (A^2 + A) \cdot A = A^3 + A^2$
$A^5 = A^4 \cdot A = (A^3 + A^2) \cdot A = A^4 + A^3$
$A^6 = A^5 \cdot A = (A^4 + A^3) \cdot A = A^5 + A^4$
$A^7 = A^6 \cdot A = (A^5 + A^4) \cdot A = A^6 + A^5$
$A^8 = A^7 \cdot A = (A^6 + A^5) \cdot A = A^7 + A^6$
… …
$A^n = A^{n-1} \cdot A = (A^{n-2} + A^{n-3}) \cdot A = A^{n-1} + A^{n-2}$
$A^{n+1} = A^n \cdot A = (A^{n-1} + A^{n-2}) \cdot A = A^n + A^{n-1}$

Or :

$A = A$
$A^2 = A + 1$
$A^3 = A^2 \cdot A = (A+1) \cdot A = A^2 + A = (A+1) + A = 2A + 1$

Iterating the procedure we have :

$A^4 = A^3 \cdot A = (2A+1) \cdot A = 2A^2 + A = 2(A+1) + A = 3A + 2$
$A^5 = A^4 \cdot A = (3A+2) \cdot A = \ldots = 5A + 3$
$A^6 = A^5 \cdot A = (5A+3) \cdot A = \ldots = 8A + 5$
$A^7 = \ldots = 8A^2 + 5A = 8(A+1) + 5A = 13A + 8$
$A^8 = \ldots = \ldots = 21A+13$
$A^n = \ldots = u_n \cdot A + u_{n-1}$
$A^{n+1} = \ldots = u_{n+1} \cdot A + u_n$

Summarizing :

$$S(A) \qquad \begin{aligned} &A = A = u_1 A \\ &A^2 = A + 1 = u_2 \cdot A + u_1 \\ &A^3 = 2A + 1 = u_3 \cdot A + u_2 \\ &A^4 = 3A + 2 = u_4 \cdot A + u_3 \\ &A^5 = 5A + 3 = u_5 \cdot A + u_4 \\ &A^6 = 8A + 5 = u_6 \cdot A + u_5 \\ &A^7 = 13A + 8 = u_7 \cdot A + u_6 \\ &A^8 = 21A + 13 = u_8 \cdot A + u_7 \\ &\ \dots\ \dots \\ &A^n = u_n \cdot A + u_{n-1} \\ &A^{n+1} = u_{n+1} \cdot A + u_n \end{aligned}$$

Bearing in mind: $S(u) = \{1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; \dots; u_{n-2}; u_{n-1}; u_n; u_{n+1}; \dots\}$

The coefficients of the Golden Ratio *A* give again the Fibonacci sequence.

Test $A^4 = 3A + 2 = u_4 \cdot A + u_3$

$$\left(\frac{\sqrt{5}+1}{2}\right)^4 = 3 \cdot \left(\frac{\sqrt{5}+1}{2}\right) + 2 \to \dots \to \left(\frac{3+\sqrt{5}}{2}\right)^2 = \frac{3\sqrt{5}+7}{2} \to \frac{6\sqrt{5}+14}{4} = \frac{3\sqrt{5}+7}{2} \to \frac{3\sqrt{5}+7}{2} = \frac{3\sqrt{5}+7}{2} \to \text{QET}$$

Test $A^5 = 5A + 3 = u_5 \cdot A + u_4$

$$A^5 = A^3 \cdot A^2 = \left(\frac{\sqrt{5}+1}{2}\right)^3 \cdot \left(\frac{\sqrt{5}+1}{2}\right)^2 = 5 \cdot \left(\frac{\sqrt{5}+1}{2}\right) + 3 \to \dots \to \left(\sqrt{5}+2\right) \cdot \left(\frac{3+\sqrt{5}}{2}\right) = \frac{5\sqrt{5}+11}{2} \to \dots \to \frac{5\sqrt{5}+11}{2} = \frac{5\sqrt{5}+11}{2} \to \text{QET}$$

In general we can write :

$$A^n = \left(\frac{\sqrt{5}+1}{2}\right)^n = \left(\frac{\sqrt{5}+1}{2}\right) \cdot u_n + u_{n-1} = \frac{u_n \cdot \sqrt{5} + u_n + 2u_{n-1}}{2}$$

Examples :

$$A^5 = \frac{u_5 \cdot \sqrt{5} + u_5 + 2u_4}{2} = \frac{5\sqrt{5}+11}{2}$$
$$A^6 = \frac{u_6 \cdot \sqrt{5} + u_6 + 2u_5}{2} = 4\sqrt{5} + 9$$
$$A^7 = \frac{u_7 \cdot \sqrt{5} + u_7 + 2u_6}{2} = \frac{13\sqrt{5}+29}{2}$$
$$A^8 = \frac{u_8 \cdot \sqrt{5} + u_8 + 2u_7}{2} = \frac{21\sqrt{5}+47}{2}$$
$$A^9 = \frac{u_9 \cdot \sqrt{5} + u_9 + 2u_8}{2} = 17\sqrt{5} + 38$$

… …
… …
… …

5. The property of Keplero

$A = A^3/A^2 = (2A+1)/(A+1) \approx u_3/u_2 = 2$
$A = A^4/A^3 = (3A+2)/(2A+1) \approx u_4/u_3 = 3/2$
$A = A^5/A^4 = (5A+3)/(3A+2) \approx u_5/u_4 = 5/3$
$A = A^6/A^5 = (8A+5)/(5A+3) \approx u_6/u_5 = 8/5$
$A = A^7/A^6 = (13A+8)/(8A+5) \approx u_7/u_6 = 13/8$
$A = A^8/A^7 = (21A+13)/(13A+8) \approx u_8/u_7 = 21/13$
… …

$A = A^{n+1}/A^n = (u_{n+1}\cdot A+u_n)/(u_n\cdot A+u_{n-1}) \approx u_{n+1}/ u_n$

In other words: *The ratio of two consecutive Fibonacci numbers is approximately the Golden Ratio*.

This property was noted by Johannes Keplero (1571 – 1630).
This method allow us to replace the procedure of continuous fractions.

By theorem previously illustrated, now we are able to confirm this important property of Fibonacci numbers.

One test

$$\frac{A^8}{A^7} = A:\ \frac{21A+13}{13A+8} = \frac{21\cdot\left(\frac{\sqrt{5}+1}{2}\right)+13}{13\cdot\left(\frac{\sqrt{5}+1}{2}\right)+8} = \ldots = \frac{21\cdot\sqrt{5}+47}{13\sqrt{5}+29} = \frac{(21\cdot\sqrt{5}+47)\cdot(13\sqrt{5}-29)}{845-841} = \frac{1365-609\cdot\sqrt{5}+611\cdot\sqrt{5}-1363}{4}$$
$$= \frac{2\cdot\sqrt{5}+2}{4} = \frac{\sqrt{5}+1}{2} = A \ \rightarrow \text{ QET}$$

Or

$$\frac{A^8}{A^7} = \frac{u_8\cdot A+u_7}{u_7\cdot A+u_6} = \frac{(u_7+u_6)\cdot A+u_7}{u_7\cdot A+u_6} = \frac{u_7\cdot A+u_6\cdot A+u_7}{u_7\cdot A+u_6} = \frac{u_7\cdot(A+1)+u_6\cdot A}{u_7\cdot A+u_6} = \frac{u_7\cdot A^2+u_6\cdot A}{u_7\cdot A+u_6} = \frac{(u_7\cdot A+u_6)\cdot A}{u_7\cdot A+u_6} = A \ \rightarrow \text{ QET}$$

General test :

$$\frac{A^{n+1}}{A^n} = A:\ \frac{u_{n+1}\cdot A+u_n}{u_n\cdot A+u_{n-1}} = \frac{(u_n+u_{n-1})\cdot A+u_n}{u_n\cdot A+u_{n-1}} = \frac{u_n\cdot A+u_{n-1}\cdot A+u_n}{u_n\cdot A+u_{n-1}} =$$

$$\frac{u_n\cdot(A+1))+u_{n-1}\cdot A}{u_n\cdot A+u_{n-1}} = \frac{u_n\cdot A^2+u_{n-1}\cdot A}{u_n\cdot A+u_{n-1}} = \frac{A\cdot(u_n\cdot A+u_{n-1})}{u_n\cdot A+u_{n-1}} = A \ \rightarrow \text{ QET}$$

General proof

A generic Fibonacci number can be write : $u_n = \frac{A^n-(1-A)^n}{\sqrt{5}}$

But $1 - A = 1 - \left(\frac{\sqrt{5}+1}{2}\right) = \frac{2-\sqrt{5}-1}{2} = \frac{1-\sqrt{5}}{2} = -a$

So: $u_n = \frac{A^n-(1-A)^n}{\sqrt{5}} = \frac{A-(-a)^n}{\sqrt{5}}$

Applying the formula we have the following list :

$u_1 = \frac{A+a}{\sqrt{5}} = \left(\frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2}\right) \cdot \frac{1}{\sqrt{5}} = \ldots = \frac{\sqrt{5}}{\sqrt{5}} = 1$

$u_2 = \frac{A^2-a^2}{\sqrt{5}} = \ldots = 1$

$u_3 = \frac{A^3+a^3}{\sqrt{5}} = \ldots = 2$

$u_4 = \frac{A^4-a^4}{\sqrt{5}} = \ldots = 3$

$u_5 = \frac{A^5+a^5}{\sqrt{5}} = \ldots = 5$

$u_6 = \frac{A^6-a^6}{\sqrt{5}} = \ldots = 8$

… …

$u_n = \frac{A^n-(-a)^n}{\sqrt{5}} = \frac{A^n-a^n}{\sqrt{5}}$ (if n $\in E$) or $\frac{A^n+a^n}{\sqrt{5}}$ (if n $\in O$)

$u_{n+1} = \frac{A^{n+1}-(-a)^{n+1}}{\sqrt{5}} = \frac{A^{n+1}+a^{n+1}}{\sqrt{5}}$ (if n $\in E$) or $\frac{A^{n+1}-a^{n+1}}{\sqrt{5}}$ (if n $\in O$)

Bearing in mind that 1 – A < 1 $\rightarrow$

$$\lim_{n\to+\infty}(1-A)^n = 0$$

Hence :

$$u_n = \frac{A^n-(1-A)^n}{\sqrt{5}} = \approx \frac{A^n}{\sqrt{5}}$$

$$u_{n+1} = \frac{A^{n+1}-(1-A)^{n+1}}{\sqrt{5}} = \approx \frac{A^{n+1}}{\sqrt{5}}$$

So :

$\frac{u_{n+1}}{u_n} = \approx \frac{A^{n+1}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{A^n} = \approx \frac{A^{n+1}}{A^n} = A \rightarrow$ QED

Test

If we indicate with x the ratio of two consecutive Fibonacci numbers:

$$\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} = x \rightarrow \frac{u_{n-1}}{u_n} = \frac{1}{x}$$

Then :

$$\frac{u_{n+1}}{u_n} = \frac{u_n + u_{n-1}}{u_n} = 1 + \frac{u_{n-1}}{u_n} = 1 + \frac{1}{x}$$

Hence :

$$x = 1 + \frac{1}{x} \rightarrow x^2 - x - 1 = 0$$

So : $x = \frac{\sqrt{5}+1}{2} = A \;\; \rightarrow$ QET

6. Golden Section series

If we consider the geometrical series :

$$\sum_{0}^{+\infty} a^n = 1 + a + a^2 + a^3 + \cdots + a^n + \cdots$$

The sum is $s(a) = \frac{1}{1-a} = \frac{1}{a^2} = \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^2} = \cdots = \frac{2}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2} = A^2$

So : $A^2 = \frac{1}{a^2} \rightarrow A = \frac{1}{a}$ we find the relation between *a* and *A*.

7. Historical notes

Note 1

Leonardo Pisano, known as Fibonacci (Pisa, b. 1170-1240 (?)), introduced in Europe the zero and the Hindu-Arabic numeral system and so he started the development of arithmetic as we know it today, when, in 1202, he published his most famous book *Liber Abaci*. In the incipit of this book he writes: "*The nine Indian figures are: 9 8 7 6 5 4 3 2 1. With these nine figures, and with the sign 0, that the Arabs call Zefiro, any number may be written, as shown below*". Interestingly, he indicates zero as a sign and not as a digit (i.e. a number) and calls it Zephyr, that for the Arabs indicates the concept of nothing. Yet, despite this defectiveness, he has the distinction of having introduced in our continent the Hindu numeral system, also called Hindu-Arabic. Fibonacci was the son of a spice merchant, Guglielmo Bonacci. At the end of the twelfth century, the heart of his business was in North Africa in the town of Bugia (the present-day Algerian town of Behaia). So it was that his son Leonardo, said "di Bonacci", hence Fibonacci, following his father in his trade, learnt Arabic and developed a passion for the study of mathematics. He examined Islamic and Greek texts and travelled in Egypt, Syria, Asia Minor and Greece. He was one of the first great European mathematicians. With him began the revival of the study of mathematics in Europe. In his long stay in North Africa he was in contact with the Islamic mathematics culture, which had in the Persian Musa al- Khwàrizmì his most distinguished exponent along with the Egyptian Abu Kamil. Arabic mathematics was closely related to that of India, but it is to be assumed that at the time they were not aware that, beyond the Indus River, and with a bold logical step, they had raised the zero to a *"real number"*. So, while the Arabs were still speaking about a numbering based on nine digits and

the symbol of zephyr (zero), in India they were already referring to a positional numbering system based on ten digits, including zero.
Some historians argue that Fibonacci was not the first to have the cognition of Hindu-Arabic numeral system, and that before him, it had been known by the French Benedictine monk Gerberto de Aurillac (950-1003), who, however, didn't publish anything about it. It is therefore Fibonacci that has officially the historical merit of having brought in the European mathematics culture the Hindu-Arabic numeral system. Fibonacci conducted interesting studies on the *golden section,* and probably stated and proved the theorem mentioned above. As already mentioned, it's likely that it was from this theorem he deduced his famous sequence in which each number is the sum of the previous two: $u_n = u_{n-1} + u_{n-2}$.

S(u) = {1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; 144; 233; 377; 610; 987; 1597; 2584; 4181; 6765; 10946; 17711; 28657; 46368; 75025; 121393; 196418; 317811; 514229; 832040; 1346269; 2178309; 3524578; …}

Note 2

The denomination Golden Section was used for the first time by German mathematician Martin Ohm (1792 – 1872) in his book *Die Reine Elementar Mathematik*, for to pay homage to the aesthetic characteristic of this particular division of a segment. Euclid, in VI book of *Elementi* called it "*divisione del segmento in estrema e media ragione*" ("*division of a segment in extreme and mean ratio*"). Before him was known by the Pythagoreans, who did not study it in depth because was an irrational number. The Golden Ratio was used by famous Greek architect Phidias in the construction of the Parthenon. For this reason some scholars say that in ancient Greece it was called *Phidias Section*. After a long period of oblivion it was discovered again by Leonardo Pisano (Fibonacci). During Renaissance the Italian mathematician Luca Pacioli (1445 (?) – 1517), in his book *Summa de Arithmetica, Geometria, Proportioni e Proportionalità* (1496) called it *divina proportione*. After him Keplero defined it *Sectio Divina*. The architect Le Corbusier (1887 – 1965) confirmed that Golden Section and Golden Ratio was present in the single parts or in the whole of many art works and in the finest architectures of all times.

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