Inertial mass and equivalence principle in General Relativity

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Abstract. We formulate the equivalence principle in Newtonian mechanics and General Relativity. We distinguish seven formulations of the equivalence principle, but not all are equivalent. We summarize the methods used in General Relativity to calculate the inertial mass. His examination leads us to consider two total energy-momentum tensors: calculated from gravitational mass and using inertial mass. The first is the one that appears in the gravitational field equation, and the second is the one that allows us to determine the system's energy and, therefore, its inertial mass. We conclude that the theory of General Relativity does not explain the equality of inertial and gravitational mass, although it is a result derivable from Newtonian mechanics.

1.- Introduction

There needs to be more clarity in the scientific literature about the principle of equivalence, where we find different and even very different formulations. In this investigation, we define the equivalence principle in classical mechanics, which must be adapted to extend it to General Relativity. We formulate seven versions of the equivalence principle, not all equivalent to each other and expressing different physical realities.

The geodesic equation, the equation of motion of a test particle in a weak gravitational field, and the Einstein-Infeld-Hoffman equation of motion make it possible to formulate the weak equivalence principle in General Relativity accurately.

In General Relativity, we define the inertial mass by extending the result found in Special Relativity, that is to say, equating the mass to the total internal energy calculated at the centre of mass. In comparison, we find the gravitational mass from Newton's law at a great distance from the body that creates the field.

We find the energy of a system of interacting particles by the total energy-momentum tensor, of which we know several formulations, although different, give the same value of the inertial mass at a great distance from the system as long as we consider a space-time asymptotically flat.

Numerous authors have found that in General Relativity, the inertial mass is equal to the active gravitational mass; however, we find a defect in these reasonings because there is confusion with the energy-momentum tensor of matter since they mistakenly identify the energy-momentum tensor calculated with the gravitational mass with the one calculated with the inertial mass and therefore in the reasoning above accept from the beginning the equality of the two types of masses. Therefore these reasonings give an identity and not the equality between inertial and gravitational mass.

We adapt the gravitational field equation when, as Mach's principle requires, the proportionality between inertial and gravitational mass depends on time.

Based on assumptions consistent with Newtonian mechanics, we demonstrate that the inertial mass is proportional to the gravitational mass, resulting in a theoretical rather than experimental outcome.

2.- Inertial mass in Newtonian mechanics and Special Relativity

In Newtonian mechanics, inertia is a characteristic of matter that causes a physical object to resist any alteration in its state of motion, specifically changes in its speed. We determine the measure of the inertia of an object by its inertial mass, which is the ratio of the force exerted on the object to the acceleration the object experiences

$$m_i = \frac{F}{a}. \tag{1}$$

According to Newton's theory, the inertial mass of an object is directly proportional to its amount of matter (see appendix). This mass remains constant regardless of the object's position or direction of movement and is a scalar property.

In Special Relativity, the concept of inertial mass remains the same as in Newtonian physics (Sachs, 1987). It refers to the resistance that a body presents to changing its movement. However, we cannot calculate it using formula (1) because, in Special Relativity, force does not vary proportionally with acceleration.

The equation of motion in Special Relativity is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = \frac{m}{\sqrt{1-u^2/c^2}}\mathbf{a}_t + \frac{m}{\left(1-u^2/c^2\right)^{3/2}}\mathbf{a}_n = m_L\mathbf{a}_t + m_T\mathbf{a}_n, \tag{2}$$

m_L is the longitudinal mass; m_T is the transverse mass; $\mathbf{u}$ is the particle's velocity; and $\mathbf{a}_t$ and $\mathbf{a}_n$ are the tangential and normal accelerations. Expanding (2), we find the tensor expression

$$F_\alpha = m_{\alpha\beta}a_\beta; \quad \alpha,\beta = (1,2,3),$$

$m_{\alpha\beta}$ is the inertial mass tensor defined by

$$m_{\alpha\beta} = \frac{m}{\sqrt{1-u^2/c^2}}\delta_{\alpha\beta} + \frac{m}{\left(1-u^2/c^2\right)^{3/2}}\frac{u_\alpha u_\beta}{c^2}. \tag{3}$$

In Special Relativity, we measure the mass by (2), a formula that coincides with (1) for small speeds. m is independent of velocity. In (3), we observe that in Special Relativity, there are two inertias: the equivalent to Newtonian physics, characterized by inertial mass, and relativistic inertia related to speed, since as speed increases, it is necessary to apply a greater force to achieve the same acceleration.

In tensor notation the Newtonian inertia is $m\delta_{\alpha\beta}$ and the relativistic inertia $r_{\alpha\beta}$ is

$$r_{\alpha\beta} = m_{\alpha\beta} - m\delta_{\alpha\beta},$$

in the second-order approximation, we find

$$r_{\alpha\beta} \approx \frac{1}{2}m\frac{u^2}{c^2}\delta_{\alpha\beta} + m\frac{u_\alpha u_\beta}{c^2},$$

we verify that relativistic inertia depends on speed and inertial mass; we also check that relativistic inertia depends on the reference frame.

Another way of representing the relativistic inertia together with the inertial mass is by the relativistic inertial mass, defined by

$$m_r = \frac{m}{\sqrt{1-u^2/c^2}}.$$

3.- Active and passive gravitational mass

In Newton's law of gravitational attraction

$$F = G\frac{m_g m_g'}{r^2}$$

two masses appear; one is the gravitational mass m_g' of the body that creates the force, and the other is the gravitational mass m_g of the body on which the force acts, called active and passive gravitational mass, respectively. The active and passive gravitational mass are equal by the principle of action and reaction, which is strictly fulfilled in Newtonian mechanics. Passive gravitational mass is the mass that intervenes in the weight, which is the technique that helps us measure the body's gravitational mass.

We must point out that inertial and gravitational mass represent very different qualities; however, Newtonian physics considers them the same, causing unnecessary confusion between the two concepts.

In General Relativity, the idea of passive gravitational mass cannot be applied. Instead of gravitational forces, movements are the result of changes in the geometry of space-time.

4.- Equivalence principle in Newtonian physics

An experimental result of Newtonian physics is that in free fall towards the Earth's surface, all bodies acquire the same acceleration, regardless of the characteristics of the test body *, which is called the universality of free fall ** and we will call the *Galileo's equivalence principle*. This name will distinguish it from other formulations of the equivalence principle, which are not always equivalent to each other (Lehmkuhl, 2021).

According to Newtonian physics, the inertial mass of an object is proportional to its passive gravitational mass. This proportionality constant is universal and applies to all bodies. Including this constant of proportionality in the universal gravitational constant G, we get that the inertial and gravitational mass have the same value.

The experience also shows that the constant of proportionality, which we will call the coefficient of inertia ξ, is independent of time. Although we do not rule out the possibility that the dependence is with cosmic time $\xi = \xi(t)$, which means

$$\frac{d\xi(t)}{dt} << 1, \tag{4}$$

ξ is the same for all bodies. (4) tells us that the variability of the coefficient of inertia may be slow enough that we cannot measure it in terrestrial experiments. In Newtonian mechanics, $m_i = \xi(t) m_g$ is equivalent to assuming that G depends on time and the inertial mass is the same as the gravitational mass; we must apply this result with caution to General Relativity

There is another formulation of the equivalence principle that states the movement of a test body in a gravitational field is solely determined by its initial conditions, such as its velocity and position. This movement is not affected by the characteristics of the body, including its inertial or gravitational mass. This particular formulation, what we call the *general equivalence principle* is more general than the constancy of free fall acceleration. It is applicable not only in Newtonian physics but also in General Relativity.

The equivalence principle in Newtonian physics is based on empirical evidence. However, if we make several assumptions that are logical in Newtonian mechanics, we can theoretically deduce the proportionality of the inertial and gravitational masses, which becomes a theoretical and not an experimental result (see appendix).

If we assume that the gravitational force acting on the part of an object is not affected by force on another part of the same object, that is to say, that the force that acts on a system is the sum of the forces that act on its parts; if we also define that the total gravitational and inertial mass of a body is the sum of the masses of the parts that constitute it and finally if gravity and inertia are independent of the chemical composition and physical properties of the body, then we can use

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* We understand that a test particle has a small enough mass that its gravity does not affect the pre-existing gravity created by other bodies. As a result, it has minimal self-gravity and no intrinsic angular momentum or spin.

** The acceleration of a test body in free fall in terrestrial gravity is not constant but depends on the distance to the centre of the planet, according to

$$\mathbf{a} = -G\frac{m'_g}{r^2}\frac{\mathbf{r}}{r}.$$

However, if the particle travels only a short distance that is much less than r, we can observe the same experimental results that were discovered by Stevin in the 16th century and later confirmed by Galileo and Newton in the 17th century. In General Relativity, the acceleration of free fall is dependent on the velocity of the body, and therefore, it changes as the body moves (see below).

Newtonian mechanics to conclude that the inertial and gravitational mass are proportional to the amount of matter in the object; this satisfies the requirements of the equivalence principle (see appendix).

Since the two types of masses are proportional to the amount of matter, we can use the same unit for both masses. However, the unit kilogram in the International System defines the unit of inertial mass (Mana and Schlamminger, 2022); however, the old definition of the kilogram used from 1889 to 2019, gave the unit of gravitational mass.

5.- Equivalence Principle in General Relativity

According to Newtonian physics, the equivalence principle states that an observer in a freely falling reference frame will not detect locally any gravitational effect on the motion of a test particle *. This means that the motion of a free particle is inertial, what we will call *Einstein's equivalence principle in the first version* and is valid in both Newtonian mechanics and General Relativity.

Einstein's equivalence principle in the second version states that an accelerated reference frame is locally equivalent to a gravitational field.

The equivalence principle, both in the first and second versions, had a great heuristic value in the formation of General Relativity because the local equivalence of an accelerated frame and a gravitational field helped Einstein to glimpse some gravitational phenomena, such as the bending of a ray of light when it passes through the proximity of a body with strong gravity or the redshift of spectral lines depending on the position of the source in the gravitational field.

On the other hand, the equivalence principle suggested formulating gravity as a geometric theory; specifically, since in a free-falling reference frame, the movement of a test particle is inertial, the trajectory followed by the particle must be a generalization of the equation of motion in Special Relativity, that is, the trajectory is a geodesic. Conversely, if the equation of motion of a free test particle is a geodesic, then a freely falling reference frame is a locally inertial frame (see below). Although the issue is controversial, we believe the equivalence principle is not a foundation of General Relativity, as Einstein once expressed **. We can derive the gravitational field equation without considering the equivalence principle, as was done in the original works of Einstein and Hilbert (Einstein, 1952), (Hilbert, 1915).

However, Einstein's equivalence principle in the second version allows to deduce the «comma-goes-to-semicolon rule» that generalizes results of Special Relativity, substituting partial derivatives for total derivatives.

In General Relativity, three versions of the equivalence principle are distinguished: weak, medium-strong and very strong (Ciufolini and Wheeler, 1995, p. 90). The *weak equivalence principle* is the above; it refers exclusively to the movement of a test particle (it corresponds to Galileo's equivalence principle and Einstein's equivalence principle in its two versions). The *medium-strong equivalence principle* states that all physical laws (except gravitational ones) are locally equal to those of Special Relativity in a freely falling reference frame. Finally, the *very strong equivalence principle* states that in a freely falling frame, all physical laws, including gravitational ones, are locally identical to those of Special Relativity.

The difference between the medium-strong and the strong principle is evident in the Nordtvedt effect, according to which the gravitational self-energy of a system (that is, the potential energy of the interaction with its own gravitational field) would contribute differently to the mass inertial and gravitational, therefore breaking the equality between the two and breaching the principle of strong equivalence. However, the measurements carried out have yet to show any evidence of the Nordtvedt effect, which means that it does not exist or is extremely small.

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* The motion of the free particle is also independent of the velocity of the freely falling frame and its location in space and time.

** Einstein's identification of the equivalence principle and the covariance principle with the general principle of Relativity can be a confusing issue. We believe there are three distinct concepts involved (Segura, 2014, p. 47-152).

6.- Geodesics and the equivalence principle

Einstein's equivalence principle suggests that gravitation is an effect of the geometry of space-time; that is, there is no gravitational force on the test body, but rather its movement is caused by space-time geometry; therefore, we must consider that a particle moving in a gravitational field is a free particle (that is, not subjected to forces).

For the above reason, we extend to General Relativity the law of motion of Special Relativity, that is to say, that the movement of a free particle in a gravitational field is the extremal of the action

$$I = -mc\int ds = -mc\int \frac{ds}{dt}dt \quad \Rightarrow \quad L = -mc\frac{ds}{dt} = -mc^2\frac{d\tau}{dt} \tag{5}$$

ds is the space-time line element, L the Lagrangian of the free particle and τ the particle's proper time. Applying the Euler-Lagrange equation to (5), we obtain the trajectory of the particle, which is the equation of the geodesic line

$$\frac{d^2x^k}{d\tau^2} + \Gamma^k_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0, \tag{6}$$

since we assume that the geometry is riemannian, the affine connection coincides with the Crhistoffel symbols Γ^k_{ij} which have symmetric components and depend on the metric tensor and its first derivatives.

We find a situation similar to electromagnetic theory, which is constituted by Maxwell's equations and the Lorentz equation of motion. In gravitational theory, we have Einstein's field equations and the geodesic equation (6). However, as we will see later, by the non-linear nature of Einstein's equation, we can deduce from them the particle's motion equation without the need to impose the geodesic equation.

In the geodesic equation, only the active gravitational mass that produces the field appears, on which the metric tensor and the Christoffel symbols depend; however, the inertial and gravitational masses of the test body do not appear. That is to say, that equation (6) is compatible with the equivalence principle in the sense that the movement of a test body in a gravitational field only depends on its initial conditions. Note that from (6), we cannot affirm that the acceleration of the test particle is constant in its free movement in a gravitational field, since the coordinate acceleration dv/dt depends on the position and the speed.

In the second-order approximation with respect to the inverse of c, the equation of motion (6) in a weak field is (Segura, 2023a) and (Segura, 2013, p. 48-51)

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi - 4\frac{\partial\mathbf{A}}{\partial t} + 4\mathbf{v}\wedge(\nabla\wedge\mathbf{A}) - \nabla\left(\frac{2\phi^2}{c^2} + \frac{\psi}{c^2}\right) + 3\frac{\mathbf{v}}{c^2}\frac{\partial\phi}{\partial t} + 4\frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\nabla)\phi - \frac{v^2}{c^2}\nabla\phi, \tag{7}$$

ϕ is the Newtonian potential, $\mathbf{A}$ is the gravitomagnetic potential, and ψ is the inductive scalar potential (i.e., dependent on the velocity of the source) *. For the simplified case where we can neglect inductive effects and time dependence, (7) is

$$\frac{d\mathbf{v}}{dt} \approx -\nabla\phi - \nabla\frac{2\phi^2}{c^2} + 4\frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\nabla)\phi - \frac{v^2}{c^2}\nabla\phi,$$

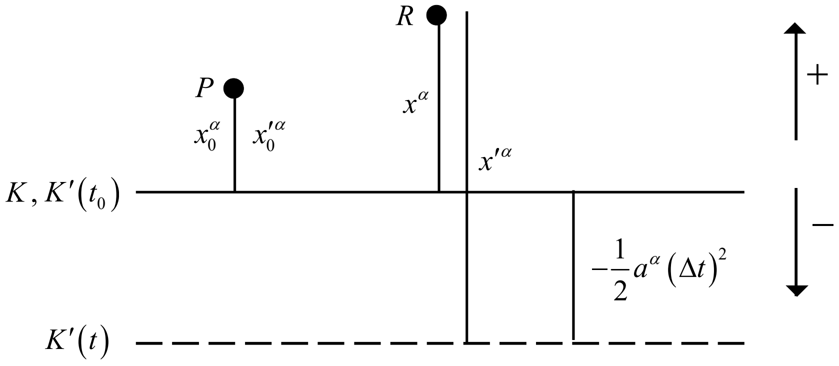
if we assume that the mass M creating the field is much greater than the gravitational mass of the test particle whose motion we are considering **, the equation of motion becomes

$$\frac{d\mathbf{v}}{dt} \approx -\frac{GM}{r^3}\mathbf{r} + \frac{4}{c^2}\frac{G^2M^2}{r^3} + \frac{4}{c^2}\frac{GM}{r^3}\mathbf{v}(\mathbf{v}\cdot\mathbf{r}) - \frac{GM}{r^3}\frac{v^2}{c^2}\mathbf{r} \tag{8}$$

the same formula that we derived from the Einstein-Infeld-Hoffman equation (see below). From (8), we verify that the acceleration of free fall depends not only on the position (as in Newtonian mechanics) but also on the velocity of the test particle. The first summand of (8) is the Newtonian term, and the rest are relativistic terms.

* (7) is a function of coordinate position and coordinate time; it is not a function of proper distance and proper time, which are measurable magnitudes (Segura, 2023).

** This condition means that we can neglect the motion of M.



Drawing 1.- K is a fixed reference frame in a gravitational field. K' is the freely falling frame, which occupies a position at t_0 and another place an infinitesimal moment later. P is the point around which we define the freely falling frame. Point R is infinitely close to P. In the drawing, we represent the coordinates of the points P and R. We also indicate the criterion of signs, upwards positive and downwards negative. Since the frame K' is accelerating downward, its acceleration is negative.

So, in General Relativity, the principle of weak equivalence must be formulated by saying that the movement of a test particle in a gravitational field only depends on its initial conditions and is independent of the characteristics of the body.

When considering equation of motion (6), it's important to note that Christoffel symbols may appear for three reasons. Firstly, if curvilinear coordinates are used instead of Cartesian ones. Secondly, if the movement is referred to a non-inertial reference frame. And finally, if there is a gravitational field present.

7.- Local inertial frame

Suppose a reference frame K concerning which a body of great gravitational mass M is at rest. Let K' be the reference frame freely falling, that is to say, that its movement is the same as that of a body in free fall; the three-dimensional acceleration of K' with respect to K is by (6)

$$\frac{d^2x^\alpha}{dt^2} = a^\alpha = -\Gamma^\alpha_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}, \tag{9}$$

we assume that the coordinates are Cartesian, that is, in the absence of gravity the Christoffel symbols are null. In drawing 1, the fixed point P relative to K is the place in whose infinitesimal environment we define the frame freely falling K' at the instant t_0. R is a point that occupies the position x^α relative to the frame K and the position x'^α relative to K' at the moment $t = t_0 + \Delta t$. From drawing 1, we find that the relationship between the coordinates of R relative to K and K' are

$$x'^\alpha - x_0'^\alpha = \left(x^\alpha - x_0^\alpha\right) - \frac{1}{2}\left(a^\alpha\right)_0 (\Delta t)^2 = \left(x^\alpha - x_0^\alpha\right) + \frac{1}{2}\left(\Gamma^\alpha_{ij}\right)_0 \frac{x^i - x_0^i}{\Delta t}\frac{x^j - x_0^j}{\Delta t}(\Delta t)^2 =$$
$$= \left(x^\alpha - x_0^\alpha\right) + \frac{1}{2}\left(\Gamma^\alpha_{ij}\right)_0\left(x^i - x_0^i\right)\left(x^j - x_0^j\right),$$

we have expressed the differentials that appear in (9) by finite differences; the subscript 0 means we measure the values at time t_0. Generalizing the previous expression to four dimensions so that the transformation also includes time

$$x'^k - x_0'^\alpha = \left(x^k - x_0^k\right) + \frac{1}{2}\left(\Gamma^k_{ij}\right)_0\left(x^i - x_0^i\right)\left(x^j - x_0^j\right), \tag{10}$$

(10) is the transformation of coordinates to a freely falling reference frame; we find $\left(\Gamma^0_{ij}\right)_0$ from the Lorentz transformation between the reference frames K and K', finding that the non-zero components are $\left(\Gamma^0_{0\alpha}\right)_0$ and $\left(\Gamma^\alpha_{00}\right)_0$. Now we verify that K' is a locally inertial reference frame. As shown in (Segura, 2015, p. 15-16), the metric tensor does not change with the transformation (10), but its derivative is zero in K' at point P and at time t_0. The Christoffel symbols are zero in K' in the neighbourhood of point P.

Then the equation of the geodesic (6) or equation of motion of a test particle that moves freely in Cartesian coordinates in a freely falling reference frame is

$$\frac{d^2x'^{\alpha}}{dt^2}=0$$

this tells us that the free particle's motion is inertial and therefore has a uniform and rectilinear motion. Cartesian speed is constant if the move is restricted to a zone infinitesimally close to *P*. Consequently, we conclude that the freely falling frame is a locally inertial reference frame, as long as we accept the universality of free fall.

The vanishing of Γ^k_{ij} in the infinitesimal neighbourhood of a point is not caused by the vanishing of the gravitational field, as shown by the fact that the curvature tensor does not vanish in *K'*. The Christoffel symbols are cancelled by the combined action of a non-inertial reference frame (the freely falling frame) and the Christoffel symbols associated with the gravitational field. The above shows that from the geodesic equation, we deduce the Einstein equivalence principle in the first version. As demonstrated in (Segura, 2014, p. 15-16), if the affine connection (which coincides with the Christoffel symbols in Riemann geometry) is symmetric at each point in space, we can define a locally inertial frame, which is a geometric property and not a physical property.

8.- Energy-momentum tensor in General Relativity

Test bodies moving freely in a gravitational field do not have the property of inertia. That is to say; they do not oppose resistance to the change its speed because the speed changes by following a geodesic movement.

If we define the tetra acceleration of a particle by

$$a^k=\frac{Du^k}{d\tau}=\frac{du^k}{d\tau}+\Gamma^k_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}$$

D is the covariant derivative, τ is the proper time of the particle and u^k is the tetra velocity; so when the particle is free in a gravitational field, its tetra acceleration is zero, which is the extension of the law of inertia to General Relativity. From here, we express the property of inertia in a gravitational field by saying that a free test body is opposed to having the non-zero tetra acceleration. Generalizing the equation of motion in Special Relativity, we arrive at

$$F^k=m_i\frac{Du^k}{d\tau}$$

F^k is the tetra force of non-gravitational origin that acts on the particle of inertial mass m_i.

In General Relativity, we define the energy-momentum tensor complex Θ^k_i, which brings together the energy-momentum tensor of «matter» T^k_i (that is, matter and other fields different from the gravitational one) and the energy-momentum tensor of the gravitational field t^k_i produced by the particle system. Although there are various options for doing this decomposition, from all of them we derive a differential conservation equation

$$\frac{\partial\Theta^k_i}{\partial x^k}=0,$$

(Tolman, 1930), (Landau and Lifshitz, 1971, p, 304-311), (Møller, 1955, p. 333-345), (Møller, 1958) and (Weinberg, 1972, p. 165-171); in some formulations, the total energy-momentum tensor appears with the two contravariant indices

$$\begin{aligned}&\Theta^k_{i(T)}=\sqrt{g}\left(T^k_i+t^k_{i(T)}\right);\quad \Theta^{ik}_{(LL)}=g\left(T^{ik}+t^{ik}_{(LL)}\right);\\&\Theta^{ik}_{(W)}=T^{ik}+t^{ik}_{(W)};\quad \Theta^k_{i(M)}=\sqrt{g}\left(T^k_i+t^k_{i(M)}+S^k_i\right).\end{aligned}\tag{11}$$

S^k_i is a function with zero divergence, and its volume integral over the entire system is also zero. In all prescriptions, we can put Θ^k_i (o Θ^{ik}) as the divergence of a superpotential h^{ks}_i (or h^{iks}) that is a function of the metric tensor and its derivatives

$$\Theta^k_i=\frac{\partial h^{ks}_i}{\partial x^s},$$

the character of the indices of the superpotential can vary according to the chosen formulation.

We find the energy of the 0,0 component of the total energy-momentum tensor. Since superpotentials are antisymmetric, we derive by Gauss's integral theorem the total energy of the system

$$E = \int_V \Theta_0^0 dxdydz = \int_V \frac{\partial h_0^{0s}}{\partial x^s} dxdydz = \int_\Sigma h_0^{0\alpha} d\tilde{S}_\alpha \tag{12}$$

expression that is similar to Gauss's theorem of electrostatics; *x, y, z* represent orthogonal coordinates, *V* is the three-dimensional volume occupied by the system, Σ is the closed surface that encompasses *V*, and $d\tilde{S}_\alpha$ is defined by

$$d\tilde{S}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma} dS^{\beta\gamma} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(da^\beta db^\gamma - da^\gamma db^\beta\right) = \varepsilon_{\alpha\beta\gamma} da^\beta db^\gamma$$

moreover, the surface element is $dS_\alpha = \sqrt{\gamma} d\tilde{S}_\alpha$; $\varepsilon_{\alpha\beta\gamma}$ are the Levi-Civita symbols, da^β and db^β are the components of the vectors that limit the surface element, and γ is the determinant of the three-dimensional spatial metric tensor (Segura, 2015, p. 66-67). We note that the integration of (12) is not over the spatial volume dV but for the product of the differentials of the coordinates $d\Omega = dxdydz$, from which we deduce the three-dimensional volume element $dV = \gamma d\Omega$. We make this choice to apply Gauss's integral theorem and be able to deduce the integral theorem of conservation of energy; if we do the integral (12) for the three-dimensional volume, we will not find a law of conservation of energy and momentum.

Although there are several ways to define the total energy-momentum tensor, in all cases, we obtain the same total energy of a closed system as long as we use an asymptotically flat space-time (Florides, 2009). However, the energy calculation for limited portions of a system differs depending on the chosen prescription (Radinschi et al., 2020).

(Ohanian, 2010) observed a significant difference between Gauss's theorem of electrostatics and (12); while for the first, the result of the surface integral is the electric charge, for (12), it is the inertial mass (E/c^2) and not the gravitational mass as would be required by the similarity between both theorems; in section 12, we delve into this difference.

We can relate the various formalisms of the energy-momentum tensor complex. Starting from the Landau-Lifshitz (LL) energy-momentum tensor, we derive new energy-moment pseudo-tensors of the gravitational field through the following two procedures (Chandrasekhar, 1969)

$$\begin{aligned} \tau^{ik} &= t^{ik}_{(LL)} + g^{-\frac{\nu+2}{2}} \frac{\partial}{\partial x^l}\left(g^{\nu/2}\right) h^{ikl}_{(LL)} \\ \tau^k_j &= g_{ji} t^{ik}_{(LL)} + g^{-\frac{\nu+2}{2}} \frac{\partial}{\partial x^l}\left(g^{\nu/2} g_{ji}\right) h^{ikl}_{(LL)} \end{aligned} \tag{13}$$

$t^{ik}_{(LL)}$ and $h^{ikl}_{(LL)}$ is the pseudo-energy-momentum tensor of the gravitational field and the superpotential in the Landau-Lifshitz formalism, ν is an arbitrary integer; τ^{ik} and τ^k_j are the new pseudo-energy-momentum tensors of the gravitational field. From the first formula (13), we find complexes of the form

$$\Theta^{ik} = g^{\frac{\nu+2}{2}}\left(T^{ik} + \tau^{ik}\right),$$

from the second formula (19) we find

$$\Theta^k_i = g^{\frac{\nu+2}{2}}\left(T^k_i + \tau^k_i\right).$$

For the two groups of complexes, we find a differential conservation law. Equations (13) coincide with those (11) except for an indeterminate quantity V^k_i , whose divergence and volume integral are zero, at least for asymptotically flat coordinates. For example, for Tolman's formalism, whose expression we deduce from the second equation (13) by setting $\nu = -1$, we find the total energy-momentum tensor $\Theta^k_{i(T)} = \sqrt{g}\left(T^k_i + \tau^k_i + V^k_i\right)$.

We note that we cannot raise or lower the total energy-momentum tensor indices by multiplying it by the metric tensor. By doing this operation, we find new magnitudes that are no longer energy-momentum tensors since we do not obtain a conservation law from them. That is to say, Θ^{ik} and $g_{ji}\Theta^{jk}$ are different magnitudes.

Only the Landau-Lifshitz pseudo energy-momentum tensor $t^{ik}_{(LL)}$ is symmetric. The tensors derived from the first equation (13) are not.

We obtain the energy-momentum tensor of the gravitational field by utilizing the superpotential and field equation. For example, for the Landau-Lifshitz formalism, second equation (11), we find

$$t^{ik}_{(LL)} = \frac{1}{g}\frac{\partial h^{ikl}}{\partial x^l} + \frac{1}{\chi}\left(R^{ik} - \frac{1}{2}g^{ik}R\right),$$

we decompose the Einstein tensor into terms that only contain the first derivatives of the metric tensor and other terms that only contain the second derivatives, that is

$$t^{ik}_{(LL)} = \frac{1}{g}\frac{\partial h^{ikl}}{\partial x^l} + \frac{1}{\chi}\left(R^{ik} - \frac{1}{2}g^{ik}R\right)_1 + \frac{1}{\chi}\left(R^{ik} - \frac{1}{2}g^{ik}R\right)_2,$$

by definition, the superpotential corresponds to the terms that only contain the second derivatives of the metric tensor, then we find

$$\left(R^{ik} - \frac{1}{2}g^{ik}R\right)_2 = -\frac{\chi}{g}\frac{\partial h^{ikl}}{\partial x^l} \quad \Rightarrow \quad t^{ik}_{(LL)} = \frac{1}{\chi}\left(R^{ik} - \frac{1}{2}g^{ik}R\right)_1$$

and from here, we derive the components of the energy-momentum tensor of the gravitational field, which we can put in terms of the derivatives of the metric tensor or in terms of the Christoffel symbols (Landau and Lifshitz, 1971, p. 306). For a weak gravitational field the 0,0 component is

$$t^{00}_{(LL)} = -\frac{7}{8\pi G}\left(\nabla\phi\right)^2$$

(Chandrasekhar, 1969). We can apply the previous reasoning to any other formalism, from which we would obtain some of the expressions (13).

9.- Inertial and Gravitational Mass in General Relativity

In Special Relativity we show that the inertial mass of interacting particles is the system's total energy measured at its centre of mass (where the total linear momentum is zero) divided by the speed of light squared. Generalizing this result to General Relativity, we find the inertial mass as a function of the 0,0 component of the total energy-momentum tensor

$$m_i = \frac{1}{c^2}\int \Theta^0_0 dxdydz.$$

According to the formalism used, the energy density is Θ^0_0 or Θ^{00}. Computing (12) for a closed system in asymptotically flat spacetime, for which the metric at a great distance from the source where we make the surface integral of (12) is

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 - \left(1 + \frac{2GM}{c^2 r}\right)\left(dx^2 + dy^2 + dz^2\right),$$

M is the gravitational mass of the system, we find

$$m_i = M$$

then the inertial mass of the system is identical to its active gravitational mass, which is not the same as Galileo's equivalence principle, which states that the inertial mass is equal to the passive gravitational mass; as we said this magnitude does not appear in General Relativity (Vollick, 2021). We find the previous result with any formulations of the total energy-momentum tensor.

In the Tolman-Einstein formulation, the energy-momentum tensor of the gravitational field is the canonical one, that is, the one found through the Lagrangian field theory (Segura, 2014, p. 62-67). When the field is static, we find

$$\begin{aligned} E &= \int\left(\sqrt{g}T^0_0 + \sqrt{g}t^0_0\right)dxdydz = \int\sqrt{g}\left(T^0_0 - T^1_1 - T^2_2 - T^3_3\right)dxdydz = \\ &= \int\sqrt{g_{00}}\sqrt{\gamma}\left(T^0_0 - T^1_1 - T^2_2 - T^3_3\right)dxdydz = \int\sqrt{g_{00}}\left(T^0_0 - T^1_1 - T^2_2 - T^3_3\right)dV \end{aligned} \tag{14}$$

γ is the determinant of the three-dimensional metric tensor, and dV is the three-dimensional volume element. Interestingly, in (14), the components of the gravitational energy-momentum tensor do not appear explicitly but exclusively the components of the matter energy-momentum tensor.

For a perfect fluid

$$T^{ik}=\left(\rho_0+p/c^2+\frac{\rho_0 U}{c^2}\right)u^i u^k-g^{ik}p \tag{15}$$

ρ_0 in the volume density of matter at rest, p is the pressure, U is the internal energy per unit mass, $\rho_0 U$ is the internal energy per unit volume as it is considered in thermodynamics and is measured in the proper frame, and u^k is the tetra velocity of an elementary portion of the fluid.

Calculating the tetra velocity, we obtain by (15) the components of the energy-momentum tensor of matter in the desired order in the centre of mass frame

$$\begin{aligned}
T^{00}&=\rho_0 c^2\left(1+\frac{v^2}{c^2}-\frac{2\phi}{c^2}+\frac{U}{c^2}\right)+O\left(c^{-2}\right)\\
T^{0\alpha}&=\rho_0 c\left(1+\frac{v^2}{c^2}-\frac{2\phi}{c^2}+\frac{U}{c^2}+\frac{1}{c^2}\frac{p}{\rho_0}\right)+O\left(c^{-3}\right)\\
T^{\alpha\beta}&=\rho_0 v^\alpha v^\beta+p\delta_{\alpha\beta}+\rho_0 v^\alpha v^\beta\left(\frac{v^2}{c^2}-\frac{2\phi}{c^2}+\frac{p}{c^2\rho_0}+\frac{U}{c^2}\right)+\frac{1}{c^2}2p\phi\delta_{\alpha\beta}+O\left(c^{-4}\right).
\end{aligned} \tag{16}$$

v is the three-dimensional velocity of a volume element in the centre of mass frame, and ϕ is the gravitational self-potential; the previous components allow us to determine the inertial mass of the system in the approximation second-order of the inverse of c. Substituting (16) into (14)

$$m_i=\frac{1}{c^2}\int\left(1+\frac{\phi}{c^2}\right)\left[\rho_0 c^2\left(1+\frac{v^2}{c^2}+\frac{U}{c^2}\right)+\rho_0 v^2+3p\right]dV=\int\left[\rho_0\left(1+\frac{2v^2}{c^2}+\frac{U}{c^2}+\frac{\phi}{c^2}\right)+3\frac{p}{c^2}\right]dV,$$

then the inertial mass density per unit three-dimensional volume is

$$\rho=\rho_0\left(1+\frac{2v^2}{c^2}+\frac{U}{c^2}+\frac{\phi}{c^2}\right)+3\frac{p}{c^2},$$

for cosmological calculations, we assume that the velocity relative to the centre of mass is small relative to *c,* the potential energy and internal energy are also negligible, then the inertial mass density is

$$\rho\approx\rho_0+3\frac{p}{c^2}.$$

According to the virial theorem, the pressure from the continuous medium is comparable to its potential self-energy. However, we cannot ignore the pressure since we assume that, in addition to matter, electromagnetic radiation also contributes to the gravitational field. We know this to be true from the presence of microwave background radiation and dark energy.

From the above, we conclude that all forms of internal energy contribute to the inertial mass of a system, including the energy of the gravitational field and potential energy. As we have assumed the equality between inertial and gravitational mass, we admit that all forms of energy also contribute to the gravitational mass; this leads us to conclude that in General Relativity the principle of strong equivalence is applicable.

The active gravitational mass m_g is defined in General Relativity by the asymptotic value or Newtonian limit of the metric tensor

$$g_{00}=1-\frac{2Gm_g}{c^2 r}.$$

which determines the orbital motions at large distances if the velocities are small.

Contrary to other authors, we understand that the General Relativity does not explain the identity between inertial and gravitational mass. Note that (12) is the system's energy as long as the energy-momentum tensor is a function of the inertial mass since the energy is equivalent to the inertial mass. The tensor T_{ik} that appears in Einstein's field equations is function of the gravitational masses that create the field, and we can only find the energy from T_{ik} if we previously assumed the inertial and gravitational masses to be identical.

This matter becomes essential when we assume (as we will do later) that the proportionality between inertial and gravitational mass varies with cosmic time.

10.- Einstein–Infeld–Hoffmann theory

In a celebrated article written in 1938, Einstein, Infeld and Hoffmann presented a long and cumbersome investigation through which they found, from the gravitational field equation, the equation of motion of a system of particles, understood as singularities. The method assumes small velocities and weak fields. It uses approximations to obtain the motion equation in first approximation (Newtonian) and second approximation.

The developed method demonstrated that we can deduce the equation of motion from a nonlinear field theory, such as General Relativity. Therefore, the motion equation is unnecessary to impose as a complementary law as in electromagnetic theory or as we have done before with the geodesic equation.

The method is based on applying the gravitational field equation in the outer case

$$R_{ik}=0 \quad \Rightarrow \quad R_{ik}-\frac{1}{2}\eta_{ik}\eta^{pq}R_{pq}=0 \tag{17}$$

η_{ik} is the metric Minkowski tensor. We decompose the previous equation in two terms

$$\Phi_{ik}+2\Lambda_{ik}=2\left(R_{ik}-\frac{1}{2}\eta_{ik}\eta^{pq}R_{pq}\right)$$

in Λ_{ik} there are all the nonlinear terms of the metric tensor and some linear terms, and Φ_{ik} only has linear terms (although not all of them) and has a particular property that we will now see.

We assume that the system is composed of *N*-point particles. We choose the particle *s* and a closed surface Σ that encompasses only the particle *s* and that no other particles are found on the same surface, then inside Σ (17) is fulfilled. For the choice of Λ_{ik} it is found that

$$\int_{\Sigma}\Lambda_{i\alpha}dS_{\alpha}=0 \tag{18}$$

and it is also independent of the chosen surface Σ.

If the integral (18) does not depend on the shape of the surface, it cannot depend on the field potentials because these depend on the coordinates of the surface, which, as we have said, is arbitrary; therefore, (18) can only depend on the coordinates of the singularity *s* and its time derivatives. Or put another way, (18) is the equation of motion of the particle *s*.

Developing (18), we find the equation of motion of each of the *N* point particles that make up the system in the approximation of small velocities and weak field. If we assume that the system is made up of two particles of active gravitational masses, m_1 and m_2, the Einstein-Infeld-Hoffmann equation is

$$\begin{aligned}\mathbf{a}_1=\frac{Gm_2}{r^2}+\frac{1}{c^2}\left\{5\frac{G^2m_1m_2}{r^3}+4\frac{G^2m_2^2}{r^3}+\frac{Gm_2}{r^2}\left[\frac{3}{2}\left(\frac{\mathbf{r}\cdot\mathbf{v}_2}{r}\right)^2-v_1^2+4\left(\mathbf{v}_1\cdot\mathbf{v}_2\right)-2v_2^2\right]\right\}\frac{\mathbf{r}}{r}+\\ +\frac{1}{c^2}\frac{Gm_2}{r^2}\left[4\left(\frac{\mathbf{r}\cdot\mathbf{v}_1}{r}\right)-3\left(\frac{\mathbf{r}\cdot\mathbf{v}_2}{r}\right)\right]\left(\mathbf{v}_1-\mathbf{v}_2\right)\end{aligned} \tag{19}$$

$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$; for another method of obtaining (19) see (Papapetrou, 1951). In (19), we verify that the acceleration of particle 1 depends not only on its speed but also on its active gravitational mass, as seen in the second summand. Then (19) does not comply with the general equivalence principle unless we assume that particle 1 is a test particle, then its mass is small enough not to alter the gravitational field produced by particle 2; that is to say that in (19) we can neglect the second summand.

The second summand of (19) arises as a consequence of the self-force, that is to say, of the force that particle 1 exerts on particle 2, producing acceleration that in turn influences particle 1

$$5\frac{G^2m_1m_2}{r^3}=5\frac{Gm_2}{r}\frac{Gm_1}{r^2}\approx 5\frac{Gm_2}{r}a_2\,.$$

In the equations of motion of order greater than the second, we again find that the movement of a point particle depends on its mass as a consequence of the self-force. We eliminate these terms assuming m_1 is small enough (Blanchet, 2001).

Assuming that particle 1 is a test mass, we cancel the terms containing m_1 and the velocity of particle 2; then, from (19), we calculate equation (8) that we found from the geodesic equation.

Then from the gravitational field equation, we deduce the general equivalence principle: the motion of a test particle in a gravitational field only depends on its initial conditions and not on its mass.

11.- Other definitions of mass

Various proposals exist to define the inertial mass of a system of particles that interact gravitationally. Here, we present some of these proposals without evaluating their validity.

1.- We have previously defined the inertial mass by generalizing the relationship between mass and energy of Special Relativity, so the inertial mass is

$$m_i = \frac{E}{c^2}$$

E is the total energy at the center of mass of the isolated system *. Inertial mass can also be defined by the differential equation of mass conservation (applicable in isentropic processes)

$$D_k\left(\rho_0 u^k\right) = \frac{1}{\sqrt{g}}\partial_k\left(\sqrt{g}\rho_0 u^k\right) \;\Rightarrow\; \partial_k\left(\sqrt{g}\rho_0 u^k\right) = 0$$

ρ_0 is the volume density of mass at rest, and u^k is the tetra velocity of a portion of the volume occupied by matter. Applying Gauss's integral theorem and integrating over a surface that encompasses the entire system, we find the conserved magnitude

$$M = \frac{1}{c}\int_V \sqrt{g}\rho_0 u^0 dxdydz. \tag{20}$$

Expanding (20)

$$M = \frac{1}{c}\int_V \sqrt{g}\rho_0 u^0 dxdydz \approx \frac{1}{c}\int \sqrt{g_{00}}\rho_0 \frac{c}{\sqrt{1+\frac{2\phi}{c^2}-\frac{v^2}{c^2}}}dV \approx \int \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}dV = \int \rho_0 dV_0, \tag{21}$$

$dV = \sqrt{\gamma}dxdydz$ is the volume occupied by a portion of proper density ρ_0 measured from the frame with respect to which the portion of volume moves with speed v; dV_0 is the element of proper volume, the one measured by the observer who moves with the matter. By (21), we verify that M is the inertial rest mass, where we do not consider the various types of energy associated with matter, such as gravitational potential energy, thermodynamic internal energy, kinetic energy or elastic energy.

2.- The so-called ADM mass is frequently used in the literature (Arnowitt, Dreser and Misner, 2004)

$$m_i = \frac{1}{2\chi}\int\left(\frac{\partial\sum h_{\gamma\gamma}}{\partial x^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial x^\beta}\right)dS_\alpha$$

$g_{\alpha\beta} = -1 + h_{\alpha\beta}$; we deduce this expression from the Weinberg and Tolman-Einstein formulations, so they do not add anything new to the calculation of the inertial mass.

3.- If we consider a spherically symmetric, asymptotically flat and static spacetime, we find from the field equation that the *r,r* component of the metric tensor in spherical coordinates is

$$g_{rr} = \left[1 - \frac{2Gm(r)}{c^2 r}\right]$$

where

$$m(r) = 4\pi\int_0^r \rho(r')r'^2 dr'$$

from where we define the ρ-mass

$$m_\rho = 4\pi\int_0^\infty \rho(r)r^2 dr$$

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* The energy E includes all forms of internal energy. We calculate this energy when the system is isolated; that is to say, the system's kinetic energy as a whole does not contribute to the energy E (because we do the calculations in the centre of mass), and the potential energy of external fields does not contribute.

which coincides with the gravitational mass of the system.
4.- (Bonnor, 1992) provides another definition of the inertial mass of an perfect fluid. Starting from the conservation of the energy-momentum tensor of matter

$$D_k T^{ik} = D_k\left[\left(\rho_0 + p/c^2\right)u^i u^k - g^{ik} p\right] = \\ = D_k\left[\left(\rho_0 + p/c^2\right)u^k\right]u^i + \left(\rho_0 + p/c^2\right)u^k D_k u^i - g^{ik}\partial_k p = 0 \tag{22}$$

multiplying (22) by u_i

$$D_k\left[\left(\rho_0 + p/c^2\right)u^k\right]c^2 = u^k \partial_k p, \tag{23}$$

we have considered

$$u^i u_i = c^2; \quad u_i u^k D_k u^i = 0,$$

substituting (23) into (22)

$$\left(\rho_0 + p/c^2\right)a^i = \left(g^{ik} - \frac{u^i u^k}{c^2}\right)\partial_k p, \tag{24}$$

we define the tetra acceleration by

$$u^k D_k u^i = \frac{dx^k}{d\tau}\frac{Du^i}{dx^k} = \frac{Du^i}{d\tau} = a^i.$$

From equation (24), we derive the Newtonian equation of motion when we neglect the second-order terms in the inverse of c. Bonnor established a parallelism between (24) and the second law of dynamics applied to a fluid and deduced that the volume density of the inertial mass is

$$\rho = \rho_0 + p/c^2,$$

therefore the inertial mass of a volume of the perfect fluid is

$$m_i = \int\left(\rho_0 + p/c^2\right)dV, \tag{25}$$

ρ_0 is the density of the rest mass of an elementary portion of the perfect fluid, where we include the thermodynamic internal energy. (25) is quite different from other expressions. Also, it does not have the potential energy nor the kinetic energy associated with each of the elementary portions of the system.
5.- (Rosen and Cooperstock, 1992) modify Bonnor's previous result, arguing that it does not include the energy of the gravitational field, which, although not necessary for small bodies, is significant for bodies of astronomical size. For a perfect fluid, Rosen and Cooperstock find

$$m_i = \int\sqrt{g}\left(\rho_0 + 3p/c^2\right)dxdydz = \int\sqrt{g_{00}}\left(\rho_0 + 3p/c^2\right)dV$$

expression differs from (25) because of the coefficient 3 and because it contains the gravitational self-energy, coinciding with the formula we found in Tolman's theory.
6.- (Denisov, Logunov and Chugreev, 1986) determine the inertial mass of a particle from its energies: particle's proper energy, kinetic energy, internal energy and potential energy.

$$m_i = \frac{1}{c^2}\int\rho_0\left(1 + \frac{U}{c^2} + \frac{1}{2}\frac{v^2}{c^2} + \frac{1}{2}\frac{\phi}{c^2}\right)dV$$

and for an extensive body, add the gravitational potential energy of the external field; the energies are calculated in the centre of mass frame.
7.- By including kinetic, potential, and internal energy to the conserved mass (21), we can derive inertial mass

$$m_i = M + \int\rho_0\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{1}{2}\frac{\phi}{c^2} + \frac{U}{c^2}\right)dV = \int\rho_0\left(1 + \frac{v^2}{c^2} + \frac{1}{2}\frac{\phi}{c^2} + \frac{U}{c^2}\right)dV =$$

$$= \int\rho_0\left(1 + \frac{v^2}{c^2} + \frac{1}{2}\frac{\phi}{c^2} + \frac{U}{c^2}\right)\sqrt{\gamma}d\Omega = \int\rho_0\left(1 + \frac{v^2}{c^2} + \frac{1}{2}\frac{\phi}{c^2} + \frac{U}{c^2}\right)\left(1 - \frac{3\phi}{c^2}\right)d\Omega = \int\rho_0\left(1 + \frac{v^2}{c^2} - \frac{5}{2}\frac{\phi}{c^2} + \frac{U}{c^2}\right)d\Omega$$

which is the same result obtained by Chandrasekhar using the weak field approximation (Chandrasekhar, 1965), (Chandrasekhar, 1969).

Let's consider a system of particles subject to their mutual gravitational attraction, which results in pressure. If the system is homogeneous, meaning its density remains constant throughout, then if p is the pressure of the matter

$$-\frac{1}{2}\int \rho_0 \phi dV = 3\int p dV$$

(Papapetrou, 1951, appendix II). By the virial theorem, we find a relationship between the average values of the total kinetic and potential energy of the particle system

$$2\langle E_c \rangle = -\langle E_p \rangle$$

both expressions allow us to simplify the energy formulas found previously.

Another issue to consider in the energy formulas above is that the expression

$$\int \frac{1}{2}\rho_0 \phi \, dV$$

is the total potential self-energy of the particle system; however

$$\frac{1}{2}\rho_0 \phi \, dV$$

is the potential energy of the mass $\rho_0 \, dV$ only if the density does not depend on position.

12.- Criticism of the proof of equality of the inertial and gravitational mass

The gravitational mass of a body is an entirely different concept from its inertial mass; however, in Newtonian mechanics, they share the property that their magnitudes are proportional to the amount of matter in the body; this allows using the same unit of measurement for both masses and also affirming that their magnitudes are proportional.

The proportionality coefficient between both magnitudes, or inertia coefficient, is a universal constant and seems not to depend on time. However, we cannot rule out that the inertia coefficient depends weakly on time, so it can vary in periods of cosmic time, as follows from Mach's principle.

Therefore, we formulate two problems: why phenomena as disparate as inertia and gravitation have proportional magnitudes and verify if the coefficient of inertia depends on time.

There is agreement that General Relativity shows that the inertial mass is equal to the gravitational mass and not only proportional as required by the Gallielian equivalence principle; that both types of masses are the same means that in General Relativity G is not indeterminate, which does not occur in Newtonian mechanics.

In the gravitational field equation

$$R_{ik} - \frac{1}{2} g_{ik} R = -\chi T_{ik} \tag{26}$$

the energy-momentum tensor of matter T_{ik} encompasses the sources of gravity. But the source of the gravitational field is the gravitational mass, not the inertial mass, so in (26) T_{ik} is calculated with the gravitational mass.

Energy is related to inertial mass, so if we want to calculate the energy of a system from the total energy-momentum tensor, we have to use the energy-momentum tensor of matter calculated from the inertial mass

Therefore, we must distinguish two energy-momentum tensors of matter, the one formed with the gravitational mass $T_{ik(g)}$ and the one that uses the inertial mass $T_{ik(i)}$; the first appears in the field equation (26), and we use the second to find the energy of the system.

Accepting these premises, the deductions that identify inertial and gravitational mass in General Relativity are not such, since they are reduced to an identity, because to determine the total energy these demonstrations use the gravitational energy-momentum tensor.

13.- Mach's principle

Mach's principle states that the inertia of a body results from the gravitational action of the Universe as a whole. That is, inertia (and therefore inertial mass) is an acquired property, and due to the Universe's dynamic nature it must change over time *. Since we assume that gravitational

* Even for a static Universe the coefficient of inertia could vary.

mass is an innate characteristic of the body, from Mach's principle we arrive at the relation

$$m_i = \xi(t) m_g \tag{27}$$

$\xi(t)$ is the coefficient of inertia, which must vary very little with time, and we will only be able to notice its temporal dependence for cosmic intervals of time.

We find the energy with the energy-momentum tensor of matter deduced in Special Relativity and that we extend to General Relativity, that is, the tensor that we have previously called $T_{ik(i)}$. Generalizing (27) we have the relation

$$T_{ik(g)} = \frac{1}{\xi(t)} T_{ik(i)}, \tag{28}$$

which is the energy-momentum tensor that has to appear in (26).

We derive the energy-momentum tensor of the gravitational field from (26), therefore using $T_{ik(g)}$; then the total energy-momentum tensor is

$$\Theta^k_{r(g)} = \frac{1}{\xi(t)} \Theta^k_{r(i)} = \frac{\partial h_r^{ks}}{\partial x^s}, \tag{29}$$

applying Tolman's formula (14) we find

$$m_i = \frac{1}{c^2} \int \Theta^0_{0(i)} dxdydz = \frac{\xi(t)}{c^2} \int \sqrt{g_{00}} \left(T^0_{0(g)} - T^1_{1(g)} - T^2_{2(g)} - T^3_{3(g)} \right) dV$$

therefore the inertial mass of the system is $\xi(t)$ times that found with $T_{ik(g)}$

At present $\xi(t) = 1$ by definition, a value that does not hold in the past or the future. Therefore, according to Mach's principle, the inertial and gravitational mass will always be proportional; however, the proportionality coefficient will vary with cosmic time. The development of Mach's principle will have to explain why inertial and gravitational mass are proportional and will have to determine the time dependence of the coefficient of inertia.

In Newtonian mechanics the variation of the inertial mass with time is equivalent to assuming that the inertial and gravitational mass are always identical, but that the gravitational constant varies with time (De Sabbata, 1980) *. Indeed, from Newton's laws we find

$$G_0 \frac{m'_g m_g}{r^2} = m_i a \quad \Rightarrow \quad a = \frac{G_0}{\xi(t)} \frac{m'_g}{r^2} = G(t) \frac{m'_g}{r^2} \quad \Rightarrow \quad G(t) = \frac{G_0}{\xi(t)},$$

G_0 is the usual value of the universal gravitational constant. To extend the previous result to General Relativity, (28) should be similar to doing the transformation

$$T_{ik(i)} = T_{ik(g)} \;\; ; \;\; G_0 = G(t)\xi(t),$$

then the gravitational field equation is

$$R_{ik} - \frac{1}{2} g_{ik} R = -\frac{8\pi G_0}{c^4 \xi(t)} T_{ik} = -\frac{8\pi G(t)}{c^4} T_{ik} = -\chi(t) T_{ik} \tag{30}$$

T_{ik} is indistinctly $T_{ik(g)}$ or $T_{ik(i)}$.

If we accept that *G* varies with time, the equation of motion of a free particle would be the geodesic line derived from the action (5), as in General Relativity. However, if we assume that the inertial mass varies with time, then the action of a free particle would be by (5)

$$I = -c \int m(t) ds \tag{31}$$

obtaining an equation of motion different from the geodesic, and that depends on the inertial mass, which would violate the principle of weak equivalence. Since the variation of $m(t)$ is minimal, the equation of motion would still be the geodesic for practical purposes.

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* To study the time dependence of *G*, it is preferred to use a dimensionless constant that contains *G*, such as the gravitational fine-structure constant α_g, instead of the dimensional constant *G*

$$\alpha_g = \frac{G m_p^2}{\hbar c}$$

m_p is the mass of the proton (Moss et al., 2010).

If we perform the conformal transformation

$$\bar{g}_{ik} = m(t)^2 g_{ik} \quad \Rightarrow \quad ds = \frac{1}{m(t)} d\bar{s} \quad \Rightarrow \quad I = -c\int d\bar{s}$$

we derive the geodesic equation. That is, we recover the geodesic equation and the weak equivalence principle under a redefinition of the measurement units (which have to change over time) (Brans and Dicke, 1961).

If the inertial mass varies with time, the law of conservation of energy and momentum ceases to exist. From the Bianchi identities applied to the field equation (30), we find

$$D_k\left[\chi(t)T^{ik}\right] = 0$$

which does not match $D_k T^{ik} = 0$. In addition, we find

$$\frac{\partial \Theta^k_{r(i)}}{\partial x^k} \neq 0,$$

therefore, the total energy-momentum of the system is not conserved; this is because we are considering the variation of the inertial mass and, consequently, the variation of the energy of the matter that generates the gravitational field.

However, from the field equation, we obtain the conservation law

$$\frac{\partial \Theta^k_{r(g)}}{\partial x^k} = 0$$

which is formally identical to the law obtained in (16), although it does not correspond to the energy and momentum conservation of the system.

(30) should not be understood as the equation of a scalar-tensor field of the Jordan-Brans-Dicke type (Brans, 2014); in (30), G is not a scalar potential that we can derive from the field equation, but we deduce its temporal variation from the gravitoelectromagnetic potentials derived from the integration of the field equation of General Relativity (Segura, 2018).

14.- Appendix.- Demonstration of the equality of inertial and gravitational mass in Newtonian mechanics

To simplify the reasoning, we make an idealized demonstration, but the results that we will find will be physically accurate. We accept the hypotheses set out in section 4.

We measure force by a dynamometer, which gives measurements in arbitrary units. Newton's law of gravitation states that the force between two point bodies is inversely proportional to the square of the distance between the bodies. We also know that the force of gravitation depends on a characteristic of the bodies that we call gravitational mass (active or passive, depending on the case).

We choose an arbitrary body as a standard of gravitational mass, which has a specific chemical composition and a defined physical state, to which we give the value 1, regardless of the units in which it is expressed *. To find the gravitational mass of another body with the same chemical and physical characteristics as the standard, we use the volume of the body since the gravitational mass is proportional to the volume for bodies of the same characteristics.

Since the force acting on a point body and its gravitational mass are additive, the force on a body with the same physical properties as the standard is proportional to its gravitational mass m_g

$$F \propto \frac{m_g}{r^2}. \tag{32}$$

Suppose we have another body with chemical and physical properties different from the body chosen as the standard. In that case, we find its gravitational mass assuming that (32) applies to any substance and using a balance that compares the standard mass with the body's gravitational mass.

* This happened in 1889 when the international prototype of the kilogram was defined based on a platinum and iridium weight, to which the gravitational mass of 1 kilogram was assigned.

We define that the quantity of matter is a property measured by the gravitational mass in the sense that both quantities are equal or proportional. Therefore the gravitational mass of a body is equal or proportional to the amount of matter.

Based on our reasoning, we can extend the standard of gravitational mass to include inertial mass; this means that the inertial mass of the standard is equal to unity or proportional to unity with a coefficient of proportionality ξ. By applying the additivity of force and masses and Newton's second law of mechanics, we conclude that the gravitational and inertial mass are equal (or proportional with the proportionality factor ξ) for bodies with the same characteristics as the mass standard.

The main problem of all this simple reasoning is to relate the inertial and gravitational masses of a body with different characteristics to the mass standard. We find the inertial mass of this body using the second law of mechanics or the conservation of linear momentum; we also find its gravitational mass with a balance, but we cannot be sure that both masses are equal or proportional.

Let us assume that the inertial mass is not affected by chemical and physical characteristics, as seems to follow from the second law of mechanics. We can reason as if all bodies have the same characteristics. If the inertial mass does not depend on the characteristics of the body, then 1 unit of the inertial mass of body A with the same properties as the mass standard is entirely equivalent to 1 unit of a different material B. Since A has, by definition, 1 unit of gravitational mass, body B will have the same gravitational mass.

We conclude that, regardless of their characteristics, the inertial and gravitational masses coincide (or are proportional) for all bodies. With which the laws of Newtonian mechanics demonstrate the weak equivalence principle.

From the previous deduction, we verify that the inertial mass, like the gravitational mass, is proportional to the amount of matter; therefore, both masses are proportional.

Although the assumptions we made in section 4 are logical and acceptable in Newtonian mechanics, they must be verified experimentally. For this, experiments have been carried out to verify if the equality of inertial and gravitational mass holds for any substance. As is known, these experiments results favour the weak equivalence principle.

15.- Conclusions

We have divided this work into two parts. First, we analyze the various formulations of the equivalence principle, defining seven different principles. We distinguish between them what we have called the general equivalence principle, which states that the movement of a test particle in a gravitational field only depends on its initial conditions. This formulation is applicable in Newtonian mechanics and General Relativity.

The equation of motion in a weak gravitational field to the second approximation, the geodesic equation, and the Einstein-Infeld-Hoffmann equation confirm the general equivalence principle.

We have shown that the existence of locally inertial systems is a geometric property related to the symmetry of the affine connection, which coincides with the Christoffel symbols in a Riemannian space-time.

In the second part, we examine the techniques used in General Relativity to define and calculate the inertial mass of a system of interacting bodies, which is the total internal energy of the isolated system and referred to the center of mass, divided by the speed of light squared. For example, we have applied Tolman's formula (14) to a perfect fluid to find the inertial mass in the second approximation, verifying that all forms of energy participate in the inertial mass. Although the inertial mass of a particle is invariable, the inertial mass of a particle system is variable, as is its energy content.

We criticize the calculations in General Relativity to demonstrate that the inertial mass is the equal as the gravitational mass *. We understand that there are two energy-momentum tensors of

* This result refers to the equality of the inertial mass and the active gravitational mass, which is different from what we have called Galileo's equivalence principle, which indicates that the inertial mass is equal to the passive gravitational mass. In Newtonian mechanics, the two types of gravitational masses are equal as a consequence of the validity of the principle of action and reaction; this principle cannot be extended beyond

matter, the one that uses the gravitational mass (and is the one that appears in the field equation) and the energy-momentum tensor calculated with the inertial mass, which allows us to find the energy and, therefore the inertial mass of the particle system. From here, we conclude that General Relativity does not explain the equality or proportionality of the inertial and gravitational masses.

We end this section with a new formulation of the field equation where it is taken into account that the inertial and gravitational mass have a time-dependent proportionality, as required by Mach's principle. With this approach, we conclude that there is no law of conservation of the energy-momentum of matter since the inertial mass varies with cosmic time.

In the appendix, we present elementary reasoning using Newtonian mechanics, from which we theoretically deduce that the inertial and gravitational mass are proportional as long as we accept several logical hypotheses in the Newtonian mechanics.

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Newtonian mechanics, where in addition to the linear momentum of the interacting bodies, we must consider the linear momentum associated with the field. Therefore, the statement that inertial mass equals active gravitational mass is another formulation of the equivalence principle.

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