The Dirac Comb and its Fourier Transform

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Abstract

We determine the Fourier Transform of the Dirac comb.

Definition 1 *Denoting with $\delta(t)$ the Dirac delta function, we call* **Dirac comb** *the following bilateral series:*

$$\sum_{n=-\infty}^{+\infty} \delta(t - t_n) \tag{1}$$

where the sequence $\{t_n\}_{n\in\mathbb{Z}}$ is assigned arbitrarily. If t is time, the quantity

$$u(t) = u_0 \sum_{n=-\infty}^{+\infty} \delta(t - t_n) \tag{2}$$

is called **unit impulse train**. *If the sequence $\{t_n\}_{n\in\mathbb{Z}}$ is a random variable, (1) and (2) are respectively a* **random Dirac comb** *and a* **pulse train random**.

The Dirac comb is said to be **periodic** *if exists $T_c > 0 \mid t_n = nT_c, \ \forall n \in \mathbb{Z}$. The quantity T_c is the comb period.*

Let us consider the particular case of a periodic comb and therefore, of a periodic train:

$$u(t) = u_0 \sum_{n=-\infty}^{+\infty} \delta(t - nT_c) \tag{3}$$

Theorem 2 *The periodic Dirac comb or what is the same as a periodic train, is invariant in form under the Fourier transform:*

$$u(t) = u_0 \sum_{n=-\infty}^{+\infty} \delta(t - nT_c) \underset{FT}{\longrightarrow} U(f) = \frac{u_0}{T_c} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_c}\right) \tag{4}$$

that is, the Fourier transform is still a periodic, period Dirac comb $1/T_c$.

Dimostrazione. The Fourier transform of $u(t)$ is:

$$U(f) = \int_{-\infty}^{+\infty} u(t) e^{-j2\pi ft} dt = u_0 \sum_{n=-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} \delta(t - nT_c) e^{-j2\pi ft} dt}_{=e^{-j2\pi nT_c f}}$$

$$U(f) = u_0 \sum_{n=-\infty}^{+\infty} e^{-j2\pi nT_c f} \tag{5}$$

That is, a bilateral series of imaginary exponentials. To explain this series we observe that since $u(t)$ is periodic with period T_c, we can develop it into a Fourier series:

$$u(t) = \sum_{n=-\infty}^{+\infty} c_n e^{-j\frac{2\pi n}{T_c}t}, \quad c_n = \frac{1}{T_c} \int_{-T_c/2}^{T_c/2} u(t) e^{j\frac{2\pi n}{T_c}t} dt$$

Observing that $t \in \left[-\frac{T_c}{2}, \frac{T_c}{2}\right] \Longrightarrow u(t) = u_0 \delta(t)$, we have

$$c_n = \frac{u_0}{T_c} \underbrace{\int_{-T_c/2}^{T_c/2} \delta(t)\, e^{j\frac{2\pi n}{T_c}t} dt}_{=1} = \frac{u_0}{T_c}, \quad \forall n \in \mathbb{Z}$$

It follows

$$u(t) = \frac{u_0}{T_c} \sum_{n=-\infty}^{+\infty} e^{-j\frac{2\pi n}{T_c}t} \tag{6}$$

which compared with (4) returns:

$$\sum_{n=-\infty}^{+\infty} e^{-j\frac{2\pi n}{T_c}t} = T_c \sum_{n=-\infty}^{+\infty} \delta(t - nT_c) \tag{7}$$

That is, a bilateral series of imaginary exponentials is less than a multiplicative factor, a bilateral series of Dirac deltas. By carrying out the appropriate substitutions in (7), we have

$$\sum_{n=-\infty}^{+\infty} e^{-j2\pi n T_c f} = \frac{1}{T_c} \sum_{n=-\infty}^{+\infty} \delta\left(t - \frac{n}{T_c}\right) \tag{8}$$

hence the assertion. ■

References

[1] Sullivan D.M. *Signals and Systems for Electrical Engineers I.* 2018

[2] Martins E. R. *Essentials of Signals and Systems.* 2023