

# Proof that the Center of Buoyancy is Equal to the Center of Hydrostatic Pressure (Part 3:) Submerged Circular Cylinder and Arbitrary Shaped Submerged Body

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## Summary

We recently proved that “the center of buoyancy is equal to the center of hydrostatic pressure” for floating bodies. This subject was an unsolved problem in physics and naval architecture, even though the buoyancy taught by Archimedes' principle can be obtained clearly by the surface integral of hydrostatic pressure. Then we thought that the reason why the vertical position of the center of pressure could not be determined was that the horizontal force would be zero due to equilibrium in the upright state.

As a breakthrough, we dared to create the left-right asymmetric pressure field by inclining the floating body with heel angle  $\theta$ . In that state, the forces and moments due to hydrostatic pressure were calculated correctly with respect to the tilted coordinate system fixed to the body. By doing so, we succeeded in determining the center of pressure. Then, by setting the heel angle  $\theta$  to zero in order to make it upright state, it could be proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy, *i.e.*, the centroid of the cross-sectional area under the water surface.

As mentioned above, we have already proved this problem for rectangular and arbitrarily shaped cross-sections, and published them on this *viXra.org* in English. Following that, in the 2<sup>nd</sup> report, separate proofs for a semi-submerged circular cylinder and a triangular prism were also published here. Thus, we have completed the proof for floating bodies, so in this 3<sup>rd</sup> report, we aim to prove for submerged bodies. We first prove for a submerged circular cylinder, and then apply Gauss's integral theorem to prove it clearly for an arbitrarily shaped submerged body.

**Keywords :** *Center of Buoyancy, Hydrostatic Pressure, Archimedes' Principle,  
 Submerged Circular Cylinder, Arbitrary shaped Submerged Body*

## 1. Introduction

It is a well-known fact in physics and naval architecture that the position of “*Center of Buoyancy*” acting on a ship is equal to the center of the volume of the geometric shape under the water surface.

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The buoyancy taught by Archimedes' principle<sup>(1)</sup> is clearly obtained by the surface integral of the hydrostatic pressure, but the position of the center of buoyancy is described in every textbook (on physics<sup>(2)</sup>, fluid dynamics<sup>(3),(4)</sup>, hydraulics<sup>(5)</sup>, naval architecture<sup>(6),(7),(8),(9),(10),(11)</sup> and nautical mechanics<sup>(12)</sup>, *etc.*) as the center of gravity where the volume under the water surface is replaced by water. There is no explanation that it is the center of pressure due to hydrostatic pressure<sup>(13),(14)</sup>.

Recently, Komatsu<sup>(15)</sup> raised the issue of “the center of buoyancy  $\neq$  the center of pressure?” at 2007 in Japan, and it was actively discussed by Seto<sup>(16),(17)</sup>, Suzuki<sup>(18)</sup>, Yoshimura & Yasukawa<sup>(19)</sup>, Komatsu<sup>(20)</sup>, Yabushita & Watanabe<sup>(21)</sup> and others in research committees and academic meetings of the Japan Society of Naval Architects and Ocean Engineers (hereinafter abbreviated as *JASNAOE*). At the same time, in Europe, the problem was studied in detail by M egel & Kliava<sup>(22),(23)</sup> in terms of potential energy. However, no one was able to solve this issue.

On the other hand, it is also an indisputable fact that the well-known center of buoyancy (*i.e.* the volume center of the underwater portion) is correct from the viewpoint of ship's hydrostatic stability<sup>(24)–(29)</sup> (that is to say, positioning of the metacenter by calculating the metacentric radius<sup>(30)–(33)</sup>  $\overline{BM}$ ).

In response to this unsolved problem, we considered that the reason why the vertical center of pressure could not be determined was because the horizontal forces equilibrated to zero in the upright state. To solve this problem, Hori<sup>(34),(35)</sup> dared to create the left-right asymmetric pressure field by inclining the ship with heel angle  $\theta$ , and attempted in 2018 to integrate the hydrostatic pressure acting on the ship surface. Then, the forces and moments acting on the inclined ship were calculated correctly with respect to a tilted coordinate system fixed to the ship. In this case, both orthogonal components of the force acting on the ship are not zero. Therefore, it was shown that the center of pressure at the inclined state can be determined. By setting the heel angle  $\theta$  to zero, we proved that the center of hydrostatic pressure coincides with the centroid of cross-sectional area under the water surface in the upright state, *i.e.*, the well-known center of buoyancy. First, a columnar ship with the rectangular cross-section<sup>(34)</sup> was proved and its proof is lectured<sup>(36),(37)</sup> to 2<sup>nd</sup> year students of the naval architectural engineering course<sup>(38),(39)</sup> in the “*Hydrostatics of Floating Bodies*” of the university where one of the authors<sup>(40)</sup> works. And then an arbitrary cross-sectional shape<sup>(35)</sup> was proved and published in the Journal “*NAVIGATION*” of Japan Institute of Navigation (hereinafter abbreviated as *JIN*).

In other way, as many researchers are studying this issue with various approaches<sup>(41)–(45)</sup>, the discussions have deepened in *JASNAOE*. To sublate these discussions, we have illustrated that “the center of buoyancy is equal to the center of pressure” for a semi-submerged circular cylinder<sup>1st half of (46)</sup> and a submerged circular cylinder<sup>(47)</sup> which does not change its shape under the water even if it is inclined, and for a triangular prisms<sup>(48)</sup>, using the same method<sup>(49)</sup>.

In order to put an end to the above discussions, we proved that “the center of buoyancy = the center of pressure” for a submerged body with arbitrary shape<sup>1st half of (50)</sup> using Gauss's integral theorem in 2021. Furthermore, it was published in the same journal “*NAVIGATION*” of *JIN* that it is easier to prove for a floating body with arbitrary shape<sup>2nd half of (50)</sup> than author's previous papers<sup>(35), 2nd half of (49)</sup> by using Gauss's theorem in the same way<sup>(51)</sup>.

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We subsequently summarized the proofs in English for the case of the rectangular cross-section<sup>(34), 1st half of (49)</sup>, which is the easiest to understand, and for the floating body of arbitrary cross-sectional shape<sup>2nd half of (50),(51)</sup> by applying Gauss's integral theorem. And we published them in *viXra.org*<sup>(52)</sup> and in the bulletin of our university, *Nagasaki Institute of Applied Science*<sup>(53)</sup>. Furthermore, we showed an extension to the center of buoyancy for a 3-D floating body. More recently, the authors have summarized the above as *new developments in the fundamental theory for hydrostatics of floating body* and published it on *viXra.org*<sup>(54)</sup>.

As mentioned above, we have already proved this problem for rectangular and arbitrarily shaped cross-sections, and published them on this *viXra.org*<sup>(52)</sup> in English. Following that, in the 2<sup>nd</sup> report<sup>(55)</sup>, separate proofs for a semi-submerged circular cylinder and a triangular prism were also published here. Thus, we have completed the proof for floating bodies, so in this 3<sup>rd</sup> report, we aim to prove for submerged bodies. We first prove for a submerged circular cylinder, and then apply Gauss's integral theorem to prove it clearly for an arbitrarily shaped submerged body.

We would like to report all of you smart readers about the two proofs for submerged bodies.

**2. Positioning of the Center of Hydrostatic Pressure  $C_p$   
Acting on the Submerged Circular Cylinder**

In Chapter 2, we apply to the submerged circular cylinder<sup>(47), 1st half of (57)</sup> the same method developed for floating bodies (e.g., rectangular<sup>(34),(36),(37), 1st half of (49),(52),(53)</sup>, triangular<sup>(48), 2nd half of (55),(56)</sup>, semi-submerged circular cylinder<sup>1st half of (46),(55),(56)</sup> and an arbitrary shaped<sup>(35), 2nd half of (49)-(53)</sup> cross-section) by the authors already, in which the center of hydrostatic pressure is positioned by inclining the floating body laterally. It should be noted, however, in this submerged cylinder, as in the case of the semi-submerged cylinder, the cross-sectional shape does not change when it is laterally inclined.

Fig. 2.1 shows that the cross-section of a circular cylinder with radius  $R$  is submerged at water depth  $f$  to the top, and is inclined laterally by heel angle  $\theta$  to the starboard side. The origin  $o$  is placed at the center of circle, which depth is  $f + R$ . The coordinate system fixed in space with the  $z$ -axis pointing vertically downward is  $o-yz$ , and that fixed to the cylinder and tilted clockwise by  $\theta$  is  $o-\eta\zeta$ . The following analysis is performed for the latter  $o-\eta\zeta$  coordinate system, using the argument  $\phi$  measured counterclockwise from the  $\zeta$ -axis as a variable.

The water depth  $Z(\phi)$ , denoted by capital letter, from the still water surface is expressed as :

$$Z(\phi) = f + R + z(\phi) \dots\dots\dots(2.1)$$

Here, the small letter  $z(\phi)$  on the right side is the water depth measured from the origin  $o$  downward, and is obtained on the cylinder surface  $(\eta, \zeta) = (R\sin\phi, R\cos\phi)$ , denoted by  $c^{(\prime)}$ , as follows:

$$\begin{aligned} z(\phi) &= (\zeta + \eta \tan \theta) \cos \theta \\ &= R(\cos \phi \cos \theta + \sin \phi \sin \theta) \\ &= R \cos(\phi - \theta) \dots\dots\dots(2.2) \end{aligned}$$

The notation in the 3<sup>rd</sup> line of the above equation is also derived easily from Fig. 2.1. And, note that  $z(\phi)$  can also take negative values above the origin  $o$ .

Hydrostatic pressure  $p(\phi)$  at the cylinder surface can be written as follows, using  $p_0$  for the atmospheric pressure,  $\gamma$  for the specific weight of water, and Eq. (2.1) for the water depth  $Z(\phi)$ .

$$\begin{aligned}
 p(\phi) &= p_0 + \gamma Z(\phi) \\
 &= p_0 + \gamma f + \gamma(R + z(\phi)) \\
 &= p_0 + \gamma f + p'(\phi) \quad \dots\dots\dots(2.3)
 \end{aligned}$$

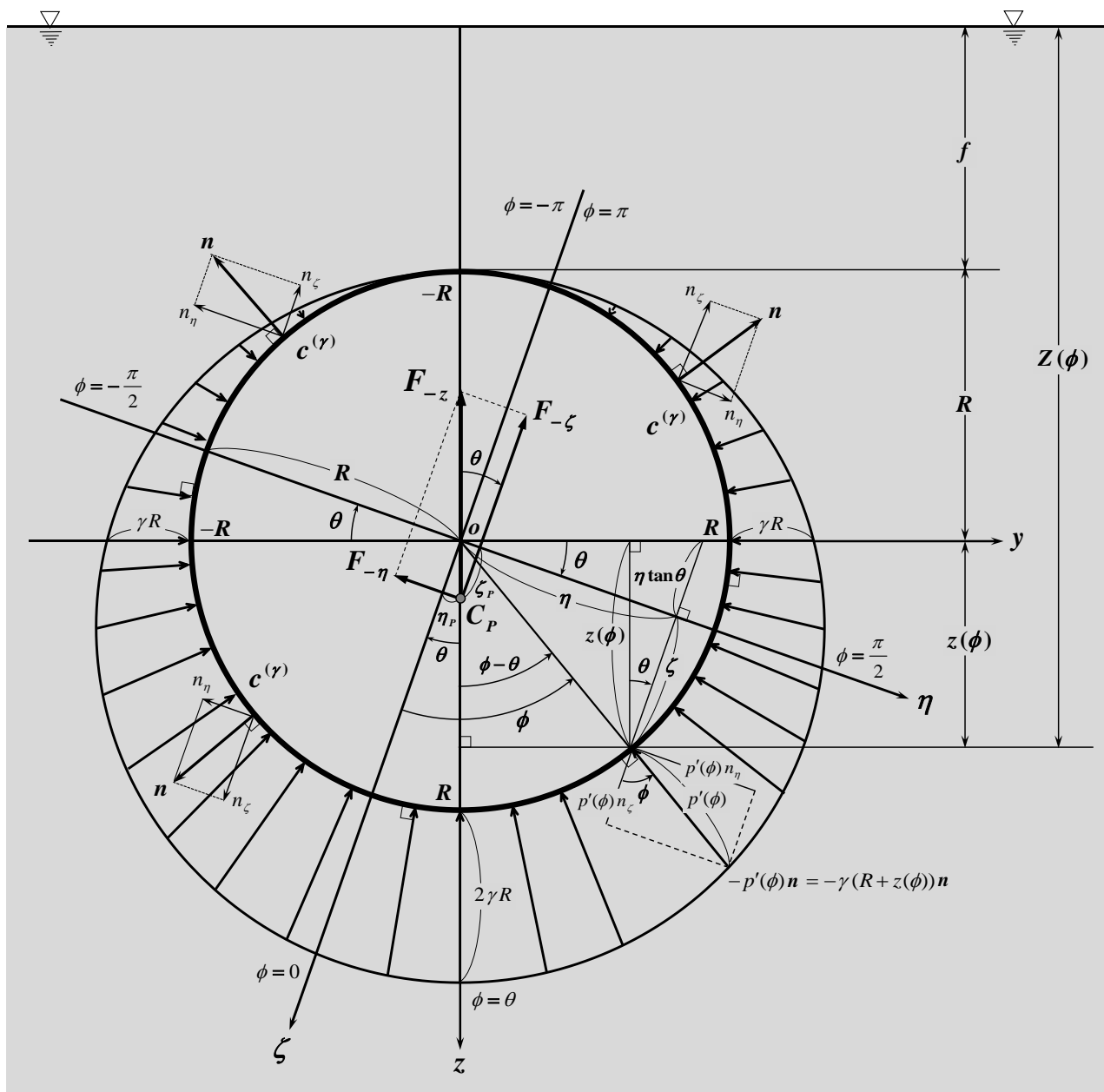


Fig. 2.1 Hydrostatic pressure  $p' = p - (p_0 + \gamma f)$  and the center of pressure  $C_P$  acting on the cross-section of an inclined submerged circular cylinder.

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Here,  $p'(\phi)$  on the right-hand side of the above equation is the relative pressure to hydrostatic pressure  $p_0 + \gamma f$  at the top of the cylinder and is defined as follows :

$$\begin{aligned}
 p'(\phi) &\equiv p(\phi) - (p_0 + \gamma f) \\
 &= \gamma (R + z(\phi)) \dots\dots\dots(2.4)
 \end{aligned}$$

The pressure shown by vector in Fig. 2.1 is this  $p'(\phi)$ . It acts in the  $-\mathbf{n}$  direction perpendicular to the cylinder surface and is zero at the top of the cylinder.

And in the figure, the outward unit normal vector  $\mathbf{n}$ , standing on the cylinder surface, can be written using the argument  $\phi$ , as follows :

$$\begin{aligned}
 \mathbf{n} &= n_\eta \mathbf{j} + n_\zeta \mathbf{k} \\
 &= \sin \phi \mathbf{j} + \cos \phi \mathbf{k} \dots\dots\dots(2.5)
 \end{aligned}$$

Here,  $n_\eta$  and  $n_\zeta$  are the directional cosines in the  $\eta$  and  $\zeta$  coordinates fixed to the cylinder, and  $\mathbf{j}$  and  $\mathbf{k}$  are the basic vectors in the  $\eta$  and  $\zeta$  directions, similarly.

**2.1 Forces  $F_{-\eta}$  and  $F_{-\zeta}$  due to pressure in the  $-\eta$  and  $-\zeta$  directions acting on the surface of submerged cylinder**

The pressure  $p(\phi)$  acting on the cylinder surface can be expressed by Eqs. (2.2), (2.3) and (2.4) in the previous section, as follows :

$$\begin{aligned}
 p(\phi) &= p_0 + \gamma f + p'(\phi) \\
 &= p_0 + \gamma (f + R) + \gamma z(\phi) \\
 &= p_c + \gamma R \cos(\phi - \theta) \dots\dots\dots(2.6)
 \end{aligned}$$

In the above equation,  $p_c$  is defined as follows, and means the hydrostatic pressure on the y-axis, placed on water depth  $f + R$ , passing through the center  $O$  of circle.

$$p_c \equiv p_0 + \gamma (f + R) \dots\dots\dots(2.7)$$

The total force  $\mathbf{F}$  acting on the submerged cylinder is calculated by pressure integration over the cylinder surface  $c^{(\gamma)}$ , as follows :

$$\begin{aligned}
 \mathbf{F} &= -\oint_{c^{(\gamma)}} p(\phi) \mathbf{n} d\ell \\
 &= F_{-\eta} (-\mathbf{j}) + F_{-\zeta} (-\mathbf{k}) \dots\dots\dots(2.8)
 \end{aligned}$$

The  $-\eta$  directional component  $F_{-\eta}$  and the  $-\zeta$  directional component  $F_{-\zeta}$  of the above total force  $\mathbf{F}$  are obtained by integrating the  $-\eta$  and  $-\zeta$  components of the hydrostatic pressure  $p(\phi)$ , as follows :

$$\left. \begin{aligned}
 F_{-\eta} &= \oint_{c^{(\gamma)}} p(\phi) n_\eta d\ell \\
 F_{-\zeta} &= \oint_{c^{(\gamma)}} p(\phi) n_\zeta d\ell
 \end{aligned} \right\} \dots\dots\dots(2.9)$$

Here, on the cylinder surface  $c^{(\gamma)}$  ( $\sqrt{\eta^2 + \zeta^2} = R$ ), the line element is  $d\ell = R d\phi$ , and the directional cosines in the  $\eta$  and  $\zeta$  directions of the normal vector  $\mathbf{n}$  are  $n_\eta = \sin\phi$  and  $n_\zeta = \cos\phi$  according to Eq. (2.5), so that both  $F_{-\eta}$  and  $F_{-\zeta}$  can be written by integration with respect to the argument  $\phi$ .

Therefore,  $F_{-\eta}$  acting in the  $-\eta$  direction is expressed by using Eq. (2.6) for  $p(\phi)$ , as follows :

$$\begin{aligned} F_{-\eta} &= \int_{-\pi}^{\pi} \left\{ p_c + \gamma R \cos(\phi - \theta) \right\} \sin\phi \cdot R d\phi \\ &= p_c R \int_{-\pi}^{\pi} \sin\phi d\phi + \gamma R^2 \cos\theta \int_{-\pi}^{\pi} \sin\phi \cos\phi d\phi + \gamma R^2 \sin\theta \int_{-\pi}^{\pi} \sin^2\phi d\phi \\ &= p_c R \int_{-\pi}^{\pi} \sin\phi d\phi + \frac{1}{2} \gamma R^2 \cos\theta \int_{-\pi}^{\pi} \sin 2\phi d\phi \\ &\quad + \frac{1}{2} \gamma R^2 \sin\theta \left( \int_{-\pi}^{\pi} d\phi - \int_{-\pi}^{\pi} \cos 2\phi d\phi \right) \dots\dots\dots(2.10) \end{aligned}$$

Similarly,  $F_{-\zeta}$  acting in the  $-\zeta$  direction is expressed as :

$$\begin{aligned} F_{-\zeta} &= \int_{-\pi}^{\pi} \left\{ p_c + \gamma R \cos(\phi - \theta) \right\} \cos\phi \cdot R d\phi \\ &= p_c R \int_{-\pi}^{\pi} \cos\phi d\phi + \gamma R^2 \sin\theta \int_{-\pi}^{\pi} \sin\phi \cos\phi d\phi + \gamma R^2 \cos\theta \int_{-\pi}^{\pi} \cos^2\phi d\phi \\ &= p_c R \int_{-\pi}^{\pi} \cos\phi d\phi + \frac{1}{2} \gamma R^2 \sin\theta \int_{-\pi}^{\pi} \sin 2\phi d\phi \\ &\quad + \frac{1}{2} \gamma R^2 \cos\theta \left( \int_{-\pi}^{\pi} d\phi + \int_{-\pi}^{\pi} \cos 2\phi d\phi \right) \dots\dots\dots(2.11) \end{aligned}$$

In both Eqs. (2.10) and (2.11) above, after expansion and integration by terms, the integrals of 1st, 2nd and 4th terms in the 3rd and 4th lines are zero, and the integral value  $2\pi$  results from the 3rd term only. Thus, both  $F_{-\eta}$  and  $F_{-\zeta}$  are computed respectively, as follows :

$$\left. \begin{aligned} F_{-\eta} &= \gamma \cdot \pi R^2 \cdot \sin\theta \\ F_{-\zeta} &= \gamma \cdot \pi R^2 \cdot \cos\theta \end{aligned} \right\} \dots\dots\dots(2.12)$$

Therefore, it is indicated that the both forces acting on the submerged cylinder are not dependent of atmospheric pressure  $p_0$  or submerged depth  $f$ , since the term on  $p_c$  is vanished. In addition,  $F_{-\eta}$  and  $F_{-\zeta}$  are obtained as the sine and cosine components of the buoyancy  $\gamma \cdot \pi R^2$ , as shown by  $F_{-\zeta}$  of Eq. (2.13) in the next section, with respect to the heel angle  $\theta$ .

### 2.2 Forces $F_{-y}$ and $F_{-z}$ converted in the $-y$ and $-z$ directions

In this section, let us find the horizontal and vertical components of the force acting on the submerged circular cylinder. By using  $F_{-\eta}$  and  $F_{-\zeta}$  obtained in Eq. (2.12) in the previous section, the horizontal component  $F_{-y}$  in the  $-y$  direction and the vertical component  $F_{-z}$  in the  $-z$  direction are converted as follows :

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$$\left. \begin{aligned}
 F_{-y} &= F_{-\eta} \cos \theta - F_{-\zeta} \sin \theta \\
 &= \gamma \cdot \pi R^2 (\sin \theta \cdot \cos \theta - \cos \theta \cdot \sin \theta) \\
 &= 0 \\
 F_{-z} &= F_{-\zeta} \cos \theta + F_{-\eta} \sin \theta \\
 &= \gamma \cdot \pi R^2 (\cos^2 \theta + \sin^2 \theta) \\
 &= \gamma \cdot \pi R^2 \quad (= \text{Buoyant Force})
 \end{aligned} \right\} \dots\dots\dots(2.13)$$

The above results show that the horizontal component  $F_{-y}$  does not act as the combined force due to pressure integration. The vertical component  $F_{-z}$  is the product of the specific weight  $\gamma$  of water and the area  $\pi R^2$  of submerged circle, and is indeed the buoyant force itself generated vertically upward, as Archimedes' principle<sup>(1)</sup> teaches.

**2.3 Moments  $M_\eta$  and  $M_\zeta$  due to pressure in the  $\eta$  and  $\zeta$  directions acting on the surface of submerged cylinder**

The clockwise moment  $M_\eta$  about the origin  $o$  due to the pressure  $p(\phi)$  in the  $-\eta$  direction acting on the cylinder surface  $c^{(\gamma)}$  and the counterclockwise moment  $M_\zeta$  due to the pressure in the  $-\zeta$  direction can be computed by multiplying the integrand in Eq. (2.9) by  $\zeta$  or  $\eta$  as the lever of moment respectively, as follows :

$$\left. \begin{aligned}
 M_\eta &= \oint_{c^{(\gamma)}} p(\phi) \zeta \cdot n_\eta d\ell \\
 M_\zeta &= \oint_{c^{(\gamma)}} p(\phi) \eta \cdot n_\zeta d\ell
 \end{aligned} \right\} \dots\dots\dots(2.14)$$

Here, if the above moments are expressed by contour integrals with respect to the argument  $\phi$ , as in Eqs. (2.10) and (2.11) for  $F_{-\eta}$  and  $F_{-\zeta}$  in the Section 2.1,  $M_\eta$  becomes as :

$$\begin{aligned}
 M_\eta &= \int_{-\pi}^{\pi} \{ p_c + \gamma R \cos(\phi - \theta) \} R \cos \phi \cdot \sin \phi \cdot R d\phi \\
 &= p_c R^2 \int_{-\pi}^{\pi} \sin \phi \cos \phi d\phi + \gamma R^3 \int_{-\pi}^{\pi} \cos(\phi - \theta) \sin \phi \cos \phi d\phi \quad \dots\dots\dots(2.15)
 \end{aligned}$$

And,  $M_\zeta$  becomes as :

$$\begin{aligned}
 M_\zeta &= \int_{-\pi}^{\pi} \{ p_c + \gamma R \cos(\phi - \theta) \} R \sin \phi \cdot \cos \phi \cdot R d\phi \\
 &= p_c R^2 \int_{-\pi}^{\pi} \sin \phi \cos \phi d\phi + \gamma R^3 \int_{-\pi}^{\pi} \cos(\phi - \theta) \sin \phi \cos \phi d\phi \quad \dots\dots\dots(2.16)
 \end{aligned}$$

The above results show that both equations for  $M_\eta$  and  $M_\zeta$  are equivalent. Hence, the total counterclockwise moment  $M_o$  around the origin  $o$  due to pressure is zero, as follows :

$$M_o = M_\zeta - M_\eta = 0 \quad \dots\dots\dots(2.17)$$

This is confirmed by the fact that the pressure acts perpendicular to the cylinder surface and is all

directed toward the center  $o$  of the circle, as in the case of the semi-submerged circular cylinder <sup>1st half of (46),(55),(56)</sup>.

Further expanding Eqs. (2.15) and (2.16) for both moments and proceeding with the calculation, we obtain as follows :

$$\begin{aligned}
 M_\eta &= M_\zeta \\
 &= \frac{1}{2} p_c R^2 \int_{-\pi}^{\pi} \sin 2\phi d\phi + \gamma R^3 \cos \theta \int_{-\pi}^{\pi} \sin \phi \cos^2 \phi d\phi \\
 &\quad + \gamma R^3 \sin \theta \int_{-\pi}^{\pi} \sin^2 \phi \cos \phi d\phi \\
 &= 2\gamma R^3 \sin \theta \int_0^{\pi} \sin^2 \phi \cos \phi d\phi \quad \dots\dots\dots(2.18)
 \end{aligned}$$

Here, in the above equation, the 1<sup>st</sup> and 2<sup>nd</sup> terms of 2<sup>nd</sup> and 3<sup>rd</sup> lines are zero, because both integrands are odd functions with respect to  $\phi$ . And, the 3<sup>rd</sup> term is an even function, so it is written with the integral interval folded in half.

Furthermore, by replacing the integral variable from  $\phi$  to  $\varphi$  by  $\varphi = \phi - \frac{\pi}{2}$ , the above equation is computed as follows :

$$\begin{aligned}
 M_\eta &= M_\zeta \\
 &= -2\gamma R^3 \sin \theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \varphi \sin \varphi d\varphi \\
 &= 0 \quad \dots\dots\dots(2.19)
 \end{aligned}$$

This is because the integrand in the above equation is an odd function with respect to  $\varphi$ , so the integral value is zero. This result shows that the  $\eta$  and  $\zeta$  components of the hydrostatic pressure do not cause moments around the center  $o$  of the circle in the case of the submerged cylinder, unlike the semi-submerged cylinder <sup>1st half of (46),(55),(56)</sup>.

**2.4 Positioning of the center of hydrostatic pressure  $C_p$  for the submerged circular cylinder**

To locate the center of pressure  $C_p$  in  $o-\eta\zeta$  coordinate system fixed to circular cylinder, the hydraulic method used in the author's previous papers<sup>(34),(35),(46)~(57)</sup> is applied. This method was used by Ohgushi<sup>(9)</sup> for an example problem of the rolling gate.

Since the forces  $F_{-\eta}$  and  $F_{-\zeta}$  due to the hydrostatic pressure obtained in Section 2.1 act on the center of pressure  $C_p (\eta_p, \zeta_p)$ , the moments  $M_\eta$  and  $M_\zeta$  due to the corresponding pressure obtained in Section 2.3 can be expressed respectively, as follows :

$$\left. \begin{aligned}
 M_\eta &= F_{-\eta} \zeta_p \\
 M_\zeta &= F_{-\zeta} \eta_p
 \end{aligned} \right\} \dots\dots\dots(2.20)$$

Therefore, the unknown coordinate  $(\eta_p, \zeta_p)$  of the center of hydrostatic pressure  $C_p$  can be



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determined by Eq. (2.20). Hence, the  $\eta$  - coordinate,  $\eta_p$ , can be determined by the combined force  $F_{-\zeta}$  and the moment  $M_\zeta$  due to the hydrostatic pressure in the  $-\zeta$  direction, and the  $\zeta$  - coordinate,  $\zeta_p$ , by  $F_{-\eta}$  and  $M_\eta$  in the  $-\eta$  direction, by using Eqs. (2.12) and (2.19) respectively, as follows :

$$\left. \begin{aligned} \eta_p &= \frac{M_\zeta}{F_{-\zeta}} = \frac{0}{\gamma \cdot \pi R^2 \cdot \cos \theta} = 0 \\ \zeta_p &= \frac{M_\eta}{F_{-\eta}} = \frac{0}{\gamma \cdot \pi R^2 \cdot \sin \theta} = 0 \end{aligned} \right\} \dots\dots\dots(2.21)$$

Let us consider the above equation. For the latter  $\zeta_p$  of vertical component, if we assume the upright state  $\theta=0$  from the beginning,  $\sin \theta$  in the denominator  $F_{-\eta}$  will be zero, so the fraction becomes indeterminate forms and  $\zeta_p$  cannot be determined. We were able to locate the vertical component  $\zeta_p$  of the center of pressure, because the submerged cylinder was laterally inclined along with its  $\eta\zeta$  - coordinate axes, even though the shape did not change when inclined.

On the other hand, for the former  $\eta_p$ , even if the heel angle is  $\theta=0$  from the beginning, the denominator  $F_{-\zeta}$  can take a finite value because of  $\cos \theta=1$ , and horizontal component  $\eta_p$  can be determined.

As a result, the center of pressure  $C_p$  of the submerged cylinder is obtained in the  $o-\eta\zeta$  coordinate system fixed and inclined to the cylinder, as follows. It is found that the  $C_p$  is located at the origin  $o$ , which is the center of the submerged circle.

$$C_p(\eta_p, \zeta_p) = (0, 0) \dots\dots\dots(2.22)$$

Therefore, its  $C_p$  is also located at the following position  $o$  in the space-fixed  $o-yz$  coordinate system and can be determined without any need for coordinate transformation, unlike the semi-submerged case <sup>1st half of (46),(55),(56)</sup>.

$$C_p(y_p, z_p) = (0, 0) \dots\dots\dots(2.23)$$

The both Eqs. (2.22) and (2.23) above is correctly indicates the figure centroid of the submerged circle. Hence, we were able to prove that the center of hydrostatic pressure is equal to the well-known center of buoyancy even for submerged bodies, as in the case of floating bodies<sup>(34)~(37),(46),(48)~(56)</sup> published by authors already.

### 3. Positioning of the Center of Hydrostatic Pressure Acting on an Inclined Submerged Body with Arbitrary Shape

In Chapter 3, we apply the same method as used in the previous Chapter 2, in which the submerged circular cylinder is inclined laterally, to the submerged body with the arbitrary shape <sup>1st half of (50),(51)</sup>. It is then proved that the center of hydrostatic pressure is equal to well-known center of buoyancy by using Gauss's integral theorem, which has already been applied to floating bodies with arbitrary shape by the authors <sup>2nd half of (50)~(53)</sup>.

Fig. 3.1 shows the cross-section of an arbitrary shaped body submerged at water depth  $f$  to the top, inclined laterally about its top by heel angle  $\theta$  to the starboard side. The origin  $o$  is placed at the top of submerged body. The coordinate system fixed in space with the  $z$ -axis pointing vertically downward is  $o-yz$ , and that fixed to the body and tilted clockwise by  $\theta$  is  $o-\eta\zeta$ . The following analysis is performed for the latter inclined  $o-\eta\zeta$  coordinate system.

The water depth  $Z$ , denoted by capital letter, on the surface  $(\eta, \zeta)$  of submerged body from the still water line is expressed as :

$$\begin{aligned} Z &= f + z(\eta, \zeta) \\ &= f + (\zeta + \eta \tan \theta) \cos \theta \\ &= f + (\zeta \cos \theta + \eta \sin \theta) \dots\dots\dots(3.1) \end{aligned}$$

Here, in the above equation, the small letter  $Z$  on the right-hand represents the water depth measured from the top  $o$ .

Hydrostatic pressure  $p$  at the body surface, denoted by  $c^{(\gamma)}$ , can be written as follows, by using  $p_0$  for the atmospheric pressure,  $\gamma$  for the specific weight of water, and the 1<sup>st</sup> line of Eq. (3.1) for the water depth  $Z$ .

$$\begin{aligned} p &= p_0 + \gamma Z \\ &= p_0 + \gamma f + \gamma z(\eta, \zeta) \\ &\equiv p_0 + \gamma f + p'(\eta, \zeta) \dots\dots\dots(3.2) \end{aligned}$$

Where,  $p'$  on the right-hand side of the above 3<sup>rd</sup> line is the relative pressure to hydrostatic pressure  $p_0 + \gamma f$  at the top of submerged body and is defined by using Eq. (3.1), as follows :

$$\begin{aligned} p'(\eta, \zeta) &\equiv p - (p_0 + \gamma f) \\ &= \gamma z(\eta, \zeta) \\ &= \gamma (\zeta \cos \theta + \eta \sin \theta) \dots\dots\dots(3.3) \end{aligned}$$

Here, the pressure shown by vector in Fig. 3.1 is the above relative pressure  $p'$ . It acts in the  $-\mathbf{n}$  direction perpendicular to the body surface and is zero at the top of body.

Then,  $\mathbf{n}$  is the outward unit normal vector standing on the body surface, and as in the case of circular cylinder in the previous chapter, is written as follows :

$$\mathbf{n} = n_\eta \mathbf{j} + n_\zeta \mathbf{k} \dots\dots\dots(3.4)$$

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Here,  $n_\eta$  and  $n_\zeta$  are the directional cosines in the  $\eta$  and  $\zeta$  coordinates fixed to the body, and  $\mathbf{j}$  and  $\mathbf{k}$  are the basic vectors in the  $\eta$  and  $\zeta$  directions, similarly.

**3.1 Components  $F_{-\eta}$  and  $F_{-\zeta}$  of the total force due to hydrostatic pressure in the  $-\eta$  and  $-\zeta$  directions acting on the submerged body**

The total force  $\mathbf{F}$  acting on the submerged body is calculated by pressure integration over the body surface  $c^{(\gamma)}$  with the line element  $d\ell$ , as follows :

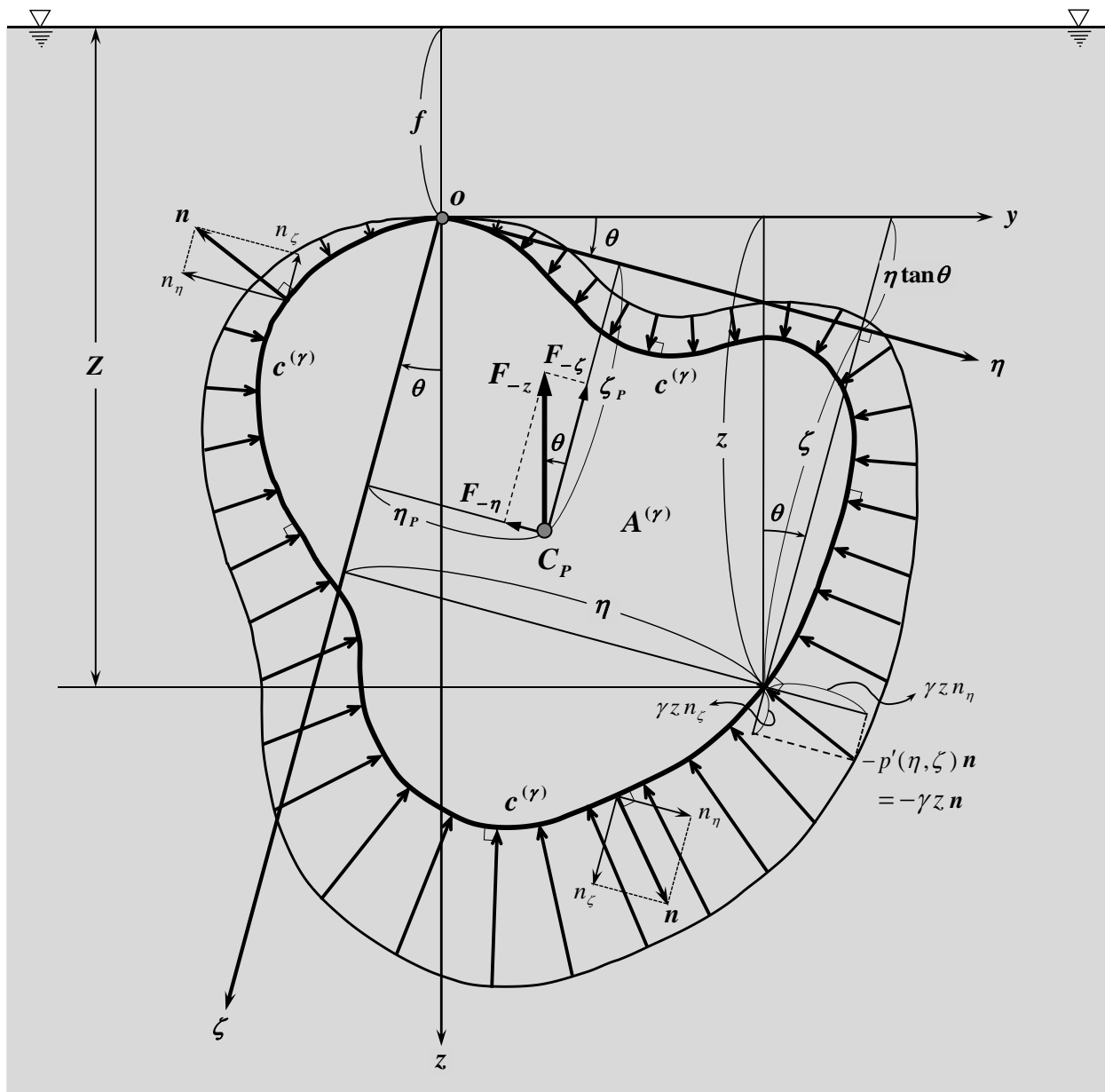


Fig. 3.1 Hydrostatic pressure  $p' = p - (p_0 + \gamma f)$  and the center of pressure  $C_P$  acting on the cross-section of an inclined submerged body with arbitrary shape.

$$\begin{aligned} \mathbf{F} &= -\oint_{c^{(\gamma)}} p(\eta, \zeta) \mathbf{n} d\ell \\ &= F_{-\eta}(-\mathbf{j}) + F_{-\zeta}(-\mathbf{k}) \quad \dots\dots\dots(3.5) \end{aligned}$$

The  $-\eta$  directional component  $F_{-\eta}$  and the  $-\zeta$  directional component  $F_{-\zeta}$  of the above total force  $\mathbf{F}$  can be obtained by integrating the  $-\eta$  and  $-\zeta$  components of the hydrostatic pressure  $p(\eta, \zeta)$  in Eq. (3.2), as follows :

$$\left. \begin{aligned} F_{-\eta} &= \oint_{c^{(\gamma)}} p(\eta, \zeta) n_{\eta} d\ell \\ &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + p'(\eta, \zeta)\} n_{\eta} d\ell \\ F_{-\zeta} &= \oint_{c^{(\gamma)}} p(\eta, \zeta) n_{\zeta} d\ell \\ &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + p'(\eta, \zeta)\} n_{\zeta} d\ell \end{aligned} \right\} \dots\dots\dots(3.6)$$

In proceeding with the calculation, the 3<sup>rd</sup> line of Eq. (3.3) is used for the relative pressure  $p'$  and the termwise integration is performed respectively. Then,  $F_{-\eta}$  is calculated as :

$$\begin{aligned} F_{-\eta} &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + \gamma(\zeta \cos\theta + \eta \sin\theta)\} n_{\eta} d\ell \\ &= \oint_{c^{(\gamma)}} (p_0 + \gamma f) n_{\eta} d\ell + \gamma \cos\theta \oint_{c^{(\gamma)}} \zeta n_{\eta} d\ell + \gamma \sin\theta \oint_{c^{(\gamma)}} \eta n_{\eta} d\ell \quad \dots\dots\dots(3.7) \end{aligned}$$

Similarly,  $F_{-\zeta}$  is calculated as :

$$\begin{aligned} F_{-\zeta} &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + \gamma(\eta \sin\theta + \zeta \cos\theta)\} n_{\zeta} d\ell \\ &= \oint_{c^{(\gamma)}} (p_0 + \gamma f) n_{\zeta} d\ell + \gamma \sin\theta \oint_{c^{(\gamma)}} \eta n_{\zeta} d\ell + \gamma \cos\theta \oint_{c^{(\gamma)}} \zeta n_{\zeta} d\ell \quad \dots\dots\dots(3.8) \end{aligned}$$

Let us now apply the following two-dimensional ( $\eta\zeta$ -plane) Gauss's integral theorem to the above contour integral in order to convert the line integral into an areal integral, same as the case of an arbitrary shaped floating body <sup>2nd half of (50)-(53)</sup>.

$$\left. \begin{aligned} \oint_c u(\eta, \zeta) n_{\eta} d\ell &= \iint_A \frac{\partial u}{\partial \eta} dA \\ \oint_c v(\eta, \zeta) n_{\zeta} d\ell &= \iint_A \frac{\partial v}{\partial \zeta} dA \end{aligned} \right\} \dots\dots\dots(3.9)$$

In the above theorem,  $n_{\eta}$  and  $n_{\zeta}$  are the directional cosines of the outward unit normal vector  $\mathbf{n}$  in  $\eta$  and  $\zeta$  directions, as shown in Eq. (3.4) and Fig. 3.1.

Then,  $F_{-\eta}$  in Eq. (3.7) can be converted to an areal integral and written as :

$$\begin{aligned} F_{-\eta} &= \iint_{A^{(\gamma)}} \frac{\partial(p_0 + \gamma f)}{\partial \eta} dA + \gamma \cos\theta \iint_{A^{(\gamma)}} \frac{\partial \zeta}{\partial \eta} dA + \gamma \sin\theta \iint_{A^{(\gamma)}} \frac{\partial \eta}{\partial \eta} dA \\ &= \gamma \sin\theta \iint_{A^{(\gamma)}} dA = \gamma A^{(\gamma)} \sin\theta \quad \dots\dots\dots(3.10) \end{aligned}$$

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Similarly,  $F_{-\zeta}$  in Eq. (3.8) can be written as :

$$\begin{aligned}
 F_{-\zeta} &= \iint_{A^{(\gamma)}} \frac{\partial(p_0 + \gamma f)}{\partial \zeta} dA + \gamma \sin \theta \iint_{A^{(\gamma)}} \frac{\partial \eta}{\partial \zeta} dA + \gamma \cos \theta \iint_{A^{(\gamma)}} \frac{\partial \zeta}{\partial \zeta} dA \\
 &= \gamma \cos \theta \iint_{A^{(\gamma)}} dA = \gamma A^{(\gamma)} \cos \theta \quad \dots\dots\dots(3.11)
 \end{aligned}$$

From the results above, it can be seen that both forces are determined by the cross-sectional area  $A^{(\gamma)}$  of the submerged body and the lateral inclination angle  $\theta$ , and do not depend on the atmospheric pressure  $p_0$  and the submerged depth  $f$ . The reason is why the integrands of 1<sup>st</sup> and 2<sup>nd</sup> terms in 1<sup>st</sup> line of both Eqs. (3.10) and (3.11) become zero and vanished.

In addition,  $F_{-\eta}$  and  $F_{-\zeta}$  are obtained as  $-\eta$  and  $-\zeta$  directional components of the buoyancy  $\gamma A^{(\gamma)}$ , as shown by  $F_{-z}$  of Eq. (3.12) in the next section, respectively.

**3.2 Forces  $F_{-y}$  and  $F_{-z}$  converted in the  $-y$  and  $-z$  directions**

In this section, let us find the horizontal and vertical components of the force acting on the submerged body. By using  $F_{-\eta}$  and  $F_{-\zeta}$  obtained in Eqs. (3.10) and (3.11) in the previous section, the horizontal component  $F_{-y}$  in the  $-y$  direction and the vertical component  $F_{-z}$  in the  $-z$  direction are converted as follows :

$$\left. \begin{aligned}
 F_{-y} &= F_{-\eta} \cos \theta - F_{-\zeta} \sin \theta \\
 &= \gamma A^{(\gamma)} (\sin \theta \cdot \cos \theta - \cos \theta \cdot \sin \theta) \\
 &= 0 \\
 F_{-z} &= F_{-\zeta} \cos \theta + F_{-\eta} \sin \theta \\
 &= \gamma A^{(\gamma)} (\cos^2 \theta + \sin^2 \theta) \\
 &= \gamma A^{(\gamma)} (= \text{Buoyant Force})
 \end{aligned} \right\} \dots\dots\dots(3.12)$$

The above results show that the horizontal component  $F_{-y}$  does not act as the combined force due to pressure integration, even when the pressure field is left-right asymmetric. The vertical component  $F_{-z}$  is the product of the specific weight  $\gamma$  of water and the cross-sectional area  $A^{(\gamma)}$  of submerged body, and is the buoyant force itself generated vertically upward, as Archimedes' principle<sup>(1)</sup> teaches. This situation is similar to Eq. (2.13) for the submerged circular cylinder in Section 2.2.

**3.3 Moments  $M_\eta$  and  $M_\zeta$  due to hydrostatic pressure in the  $\eta$  and  $\zeta$  directions acting on the submerged body**

In this section, we shall calculate the total counterclockwise moment  $M_o$  around the origin  $o$  due to hydrostatic pressure acting on the surface of the submerged body. It can be calculated by superimposing the clockwise moment  $M_\eta$  due to the pressure component in the  $-\eta$  direction and the counterclockwise moment  $M_\zeta$  due to in the  $-\zeta$  direction, as follows :

$$M_o = -M_\eta + M_\zeta \quad \dots\dots\dots(3.13)$$

Here,  $M_\eta$  and  $M_\zeta$  can be calculated by multiplying the integrand in Eq. (3.6) by  $\zeta$  or  $\eta$  as the lever of moment respectively, in the following form :

$$\left. \begin{aligned} M_\eta &= \oint_{c^{(\gamma)}} p(\eta, \zeta) \zeta \cdot n_\eta d\ell \\ &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + p'(\eta, \zeta)\} \zeta \cdot n_\eta d\ell \\ M_\zeta &= \oint_{c^{(\gamma)}} p(\eta, \zeta) \eta \cdot n_\zeta d\ell \\ &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + p'(\eta, \zeta)\} \eta \cdot n_\zeta d\ell \end{aligned} \right\} \dots\dots\dots(3.14)$$

Now, as in the case of forces  $F_{-\eta}$  and  $F_{-\zeta}$  in Eqs. (3.7) and (3.8),  $M_\eta$  can be expressed as the superposition of the contour integrals along  $c^{(\gamma)}$  by using Eq. (3.3) for the relative pressure  $p'$ , as follows :

$$\begin{aligned} M_\eta &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + \gamma(\zeta \cos\theta + \eta \sin\theta)\} \zeta n_\eta d\ell \\ &= (p_0 + \gamma f) \oint_{c^{(\gamma)}} \zeta n_\eta d\ell + \gamma \cos\theta \oint_{c^{(\gamma)}} \zeta^2 n_\eta d\ell + \gamma \sin\theta \oint_{c^{(\gamma)}} \eta \zeta n_\eta d\ell \quad \dots\dots\dots(3.15) \end{aligned}$$

Similarly,  $M_\zeta$  can be expressed as :

$$\begin{aligned} M_\zeta &= \oint_{c^{(\gamma)}} \{p_0 + \gamma f + \gamma(\eta \sin\theta + \zeta \cos\theta)\} \eta n_\zeta d\ell \\ &= (p_0 + \gamma f) \oint_{c^{(\gamma)}} \eta n_\zeta d\ell + \gamma \sin\theta \oint_{c^{(\gamma)}} \eta^2 n_\zeta d\ell + \gamma \cos\theta \oint_{c^{(\gamma)}} \zeta \eta n_\zeta d\ell \quad \dots\dots\dots(3.16) \end{aligned}$$

Therefore, we can apply Gauss's integral theorem in Eq. (3.9) to the above contour integrals, as in the case of forces  $F_{-\eta}$  and  $F_{-\zeta}$  in Section 3.1, and convert them into areal integrals.

Then, the clockwise moment  $M_\eta$  in Eq. (3.15) can be converted to an areal integral and written as follows, and is consequently obtained in proportion to the areal moment about the  $\eta$ -axis.

$$\begin{aligned} M_\eta &= (p_0 + \gamma f) \iint_{A^{(\gamma)}} \frac{\partial \zeta}{\partial \eta} dA + \gamma \cos\theta \iint_{A^{(\gamma)}} \frac{\partial \zeta^2}{\partial \eta} dA + \gamma \sin\theta \iint_{A^{(\gamma)}} \frac{\partial(\eta \zeta)}{\partial \eta} dA \\ &= \gamma \sin\theta \iint_{A^{(\gamma)}} \zeta dA \quad \dots\dots\dots(3.17) \end{aligned}$$

On the other hand, the counterclockwise moment  $M_\zeta$  in Eq. (3.16) can be written as follows, and is consequently obtained in proportion to the areal moment about the  $\zeta$ -axis.

$$\begin{aligned} M_\zeta &= (p_0 + \gamma f) \iint_{A^{(\gamma)}} \frac{\partial \eta}{\partial \zeta} dA + \gamma \sin\theta \iint_{A^{(\gamma)}} \frac{\partial \eta^2}{\partial \zeta} dA + \gamma \cos\theta \iint_{A^{(\gamma)}} \frac{\partial(\eta \zeta)}{\partial \zeta} dA \\ &= \gamma \cos\theta \iint_{A^{(\gamma)}} \eta dA \quad \dots\dots\dots(3.18) \end{aligned}$$

From the both results above, it can be found that both moments do not depend on the atmospheric

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pressure  $p_0$  and the submerged depth  $f$ , as in the case of forces  $F_{-\eta}$  and  $F_{-\zeta}$  in Eqs. (3.10) and (3.11). The reason is why the integrands of 1<sup>st</sup> and 2<sup>nd</sup> terms in 1<sup>st</sup> line of both Eqs. (3.17) and (3.18) become zero and vanished.

**3.4 Positioning of the center of hydrostatic pressure  $C_p$   
for the floating body with an arbitrary form**

For the positioning of the center of pressure of the submerged body, we will use the hydraulic method by Ohgushi<sup>(9)</sup>, as in the case of the submerged circular cylinder in the previous Chapter 2.

Since the forces  $F_{-\eta}$  and  $F_{-\zeta}$  due to the hydrostatic pressure obtained in Section 3.1 act on the center of pressure  $C_p (\eta_p, \zeta_p)$ , the clockwise moment  $M_\eta$  and the counterclockwise moment  $M_\zeta$  due to the corresponding pressure obtained in Section 3.3 can be expressed exactly same as Eq. (2.20) in the previous chapter, as follows :

$$\left. \begin{aligned} M_\eta &= F_{-\eta} \zeta_p \\ M_\zeta &= F_{-\zeta} \eta_p \end{aligned} \right\} \dots\dots\dots (3.19)$$

Then, the total counterclockwise moment  $M_o$  around the origin  $o$  in Eq. (3.13) can be calculated as :

$$M_o = -F_{-\eta} \zeta_p + F_{-\zeta} \eta_p \dots\dots\dots (3.20)$$

On the other hand, the moment  $M_{C_p}$  around the point  $C_p$ , at which  $F_{-\eta}$  and  $F_{-\zeta}$  act, is computed as follows, and becomes zero.

$$M_{C_p} = -F_{-\eta} \times 0 + F_{-\zeta} \times 0 = 0 \dots\dots\dots (3.21)$$

This correctly indicates that  $C_p$  is the center of hydrostatic pressure due to the forces  $F_{-\eta}$  and  $F_{-\zeta}$ .

Therefore, the unknown coordinate  $(\eta_p, \zeta_p)$  of this center of pressure  $C_p$  can be determined by Eq. (3.18). First, the  $\eta$ -coordinate,  $\eta_p$ , can be determined by using Eq. (3.11) for  $F_{-\zeta}$  and Eq. (3.18) for  $M_\zeta$ , as follows :

$$\begin{aligned} \eta_p &= \frac{M_\zeta}{F_{-\zeta}} = \frac{\gamma \cos \theta \iint_{A^{(\gamma)}} \eta \, dA}{\gamma A^{(\gamma)} \cos \theta} \\ &= \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} \eta \, dA \quad (= \eta_G) \dots\dots\dots (3.22) \end{aligned}$$

Next, the  $\zeta$ -coordinate,  $\zeta_p$ , can be determined by using Eq. (3.10) for  $F_{-\eta}$  and Eq. (3.17) for  $M_\eta$ , as follows :

$$\begin{aligned} \zeta_p &= \frac{M_\eta}{F_{-\eta}} = \frac{\gamma \sin \theta \iint_{A^{(\gamma)}} \zeta \, dA}{\gamma A^{(\gamma)} \sin \theta} \\ &= \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} \zeta \, dA \quad (= \zeta_G) \dots\dots\dots (3.23) \end{aligned}$$

As a result, since the  $\eta_p$  and  $\zeta_p$  are obtained with the form in which the areal moment about the  $\zeta$ - and  $\eta$ -axis is divided by the cross-sectional area  $A^{(\gamma)}$  respectively, it can be seen that they are the  $\eta_G$  and  $\zeta_G$  of the figure centroid position for submerged body geometrically. And, in the both equations, the specific weight  $\gamma$  of water and the heel angle  $\theta$  have been cancelled out in the denominator and numerator respectively, so that the both results are independent of  $\theta$ . Furthermore, the geometric shape of a submerged body does not change when it is laterally inclined, unlike the case of a floating body. Therefore, this shows that the center of pressure ( $\eta_p, \zeta_p$ ) of the submerged body in the inclined state always coincides with the centroid ( $\eta_G, \zeta_G$ ) of the cross-sectional area  $A^{(\gamma)}$ , i.e., the well-known center of buoyancy, regardless of whether it is laterally inclined or not.

Considering the above,  $\zeta_p$  of vertical component can be obtained by offsetting the zero factor  $\sin \theta$  at the heel angle  $\theta \rightarrow 0$  with the denominator and numerator, as shown in Eq. (3.23). Here, if we start the calculation as the upright state  $\theta = 0$ , both the denominator  $F_{-\eta}$  and the numerator  $M_\eta$  are in equilibrium and become zero, so the fraction becomes indeterminate forms and  $\zeta_p$  cannot be determined. This is the reason why we were able to determine the position of the center of pressure in the  $\zeta$  direction as  $\zeta_p = \zeta_G$  by inclining the submerged body laterally.

On the other hand, in the calculation of  $\eta_p$  in Eq. (3.22), even if the heel angle is  $\theta = 0$  from the beginning, the denominator  $F_{-\zeta}$  takes a finite value as the cosine component of the buoyancy. Therefore, the horizontal component  $\eta_p$  can be determined as  $\eta_p = \eta_G$ , if we start the calculation as the upright state.

These situations described above are exactly the same as in Eq. (2.21) of Section 2.4 for the submerged circular cylinder.

As a final step, let's find the center of pressure in the upright state by setting the heel angle to  $\theta \rightarrow 0$ , in order to make this result clearer. Then, since the  $\eta\zeta$ -coordinates tilted and fixed on the submerged body coincide with the  $yz$ -coordinates fixed in space and the cross-sectional area  $A^{(\gamma)}$  is invariant regardless of  $\theta$ , the Eqs. (3.22) and (3.23) become as :

$$\left. \begin{aligned} (y_p, z_p) &= \left( \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} y dA, \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} z dA \right) = (y_G, z_G) \\ \therefore C_p &= B \end{aligned} \right\} \dots\dots\dots (3.24)$$

Therefore, this proves that the center of hydrostatic pressure  $C_p$  coincides with the well-known “Center of Buoyancy,  $B$ ” for the submerged body.

#### 4. Concluding Remarks

In this 3<sup>rd</sup> report, we proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy for a submerged circular cylinder and an arbitrarily shaped submerged body by using the authors’ method of inclining the body and coordinate axes laterally, as in the case of floating bodies in the 1<sup>st</sup> and 2<sup>nd</sup> reports<sup>(52),(55)</sup>.



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In calculating the forces and moments acting on the submerged body, the case of the submerged circular cylinder in Chapter 2 was performed by using the circumferential argument as the integral variable, and the case of the arbitrarily shaped submerged body in Chapter 3 by applying Gauss's integral theorem.

The forces and moments, which were calculated in the direction of the inclined coordinate axes, were connected via the coordinates of the center of hydrostatic pressure by using the hydraulic method, so that we fully elucidated the physical meaning of the center of buoyancy acting on the submerged bodies, as well as the floating bodies reported in author's 1<sup>st</sup> and 2<sup>nd</sup> papers<sup>(53),(56)</sup>.

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In closing this paper, I would like to express my heartfelt gratitude to *Dr. Yoshihiro KOBAYASHI*, former professor of *Sojo University* and current president of *Como-Techno Co., Ltd.* in Nagasaki, Japan. He always gave warm encouragement to the author's research and recommended that this study should be published in English. I am greatly inspired by the vigorous academic spirit with which he writes about the results of his research in books<sup>(58)~(60)</sup>.

Finally from the 1<sup>st</sup> author<sup>(40)</sup>, let me express the following thanks. I would like to communicate my deepest gratitude to my late teacher, *Pr. Masato KURIHARA*<sup>(61)~(63)</sup>, who cordially taught me the theory of “*Hydrostatics of Ships*” with detailed figures and formulas on the blackboard when I was a 1<sup>st</sup> year undergraduate student and learned my 1<sup>st</sup> specialized subject of naval architecture in *the College of Naval Architecture of Nagasaki* in Japan. Therefore, I am following the appearance of my teacher at that time from more than 40 years ago as an exemplary example, when I currently lecture *Hydrostatics of Floating Bodies* and *Theory of Ship Stability* to 2<sup>nd</sup> year students at my university<sup>(38),(39)</sup>.

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