

Proof that the Center of Buoyancy is Equal to the Center of Hydrostatic Pressure

(Part 3:) Submerged Circular Cylinder and Arbitrary Shaped Submerged Body

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Summary

We recently proved that “the center of buoyancy is equal to the center of hydrostatic pressure” for floating bodies. This subject was an unsolved problem in physics and naval architecture, even though the buoyancy taught by Archimedes' principle can be obtained clearly by the surface integral of hydrostatic pressure. Then we thought that the reason why the vertical position of the center of pressure could not be determined was that the horizontal force would be zero due to equilibrium in the upright state.

As a breakthrough, we dared to assume the left-right asymmetric pressure field by inclining the floating body with heel angle θ . In that state, the forces and moments due to hydrostatic pressure were calculated correctly with respect to the tilted coordinate system fixed to the body. By doing so, we succeeded in determining the center of pressure. Then, by setting the heel angle θ to zero in order to make it upright state, it could be proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy, *i.e.*, the centroid of the cross-sectional area under the water surface.

As mentioned above, we have already proved this problem for rectangular and arbitrarily shaped cross-sections, and published them here on *viXra.org* in English. Following that, in the 2nd report, a semi-submerged circular cylinder and a triangular prism were also published here. Thus, we have completed the proof for floating bodies, so in this 3rd report, we aim to prove for submerged bodies. We first prove for a submerged circular cylinder, and then apply Gauss's integral theorem to prove it clearly for an arbitrarily shaped submerged body.

*Keywords : Center of Buoyancy, Hydrostatic Pressure, Archimedes' Principle,
Surface Integral, Tilted Coordinate System,
Submerged Circular Cylinder, Arbitrary shaped Submerged Body*

1. Introduction

It is a well-known fact in physics and naval architecture that the position of “Center of Buoyancy” acting on a ship is equal to the center of the volume of the geometric shape under the water surface.

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The buoyancy taught by Archimedes' principle⁽¹⁾ is clearly obtained by the surface integral of the hydrostatic pressure, but the position of the center of buoyancy is described in every textbook (on physics⁽²⁾, fluid dynamics^{(3),(4)}, hydraulics⁽⁵⁾, naval architecture^{(6),(7),(8),(9),(10),(11)} and nautical mechanics⁽¹²⁾, *etc.*) as the center of gravity where the volume under the water surface is replaced by water. There is no explanation that it is the center of pressure due to hydrostatic pressure^{(13),(14)}.

Recently, Komatsu⁽¹⁵⁾ raised the issue of “the center of buoyancy \neq the center of pressure?” at 2007 in Japan, and it was actively discussed by Seto^{(16),(17)}, K. Suzuki⁽¹⁸⁾, Yoshimura and Yasukawa⁽¹⁹⁾, Komatsu⁽²⁰⁾, Yabushita and Watanabe⁽²¹⁾ and others in research committees and academic meetings of the Japan Society of Naval Architects and Ocean Engineers (hereinafter abbreviated as *JASNAOE*). At the same time, in Europe, the problem was studied in detail by M egel and Kliava^{(22),(23)} in terms of potential energy. However, no one was able to solve this issue.

On the other hand, it is also an indisputable fact that the well-known center of buoyancy (*i.e.* the volume center of the underwater portion) is correct from the viewpoint of ship's hydrostatic stability^{(24)–(29)} (that is to say, positioning of the metacenter by calculating the metacentric radius^{(30)–(33)} \overline{BM}).

In response to this unsolved problem, we considered that the reason why the vertical center of pressure could not be determined was because the horizontal forces equilibrated to zero in the upright state. To solve this problem, Hori^{(34),(35)} attempted in 2018 to integrate the hydrostatic pressure acting on the ship surface at the inclined state with heel angle θ . Then, the forces and moments acting on the ship were calculated with respect to a tilted coordinate system fixed to the ship. In this case, both orthogonal components of the force acting on the ship are not zero. Therefore, it was shown that the center of pressure at the inclined state can be determined. By setting the heel angle θ to zero, we proved that the center of hydrostatic pressure coincides with the centroid of cross-sectional area under the water surface in the upright state, *i.e.*, the well-known center of buoyancy. First, a columnar ship with the rectangular cross-section⁽³⁴⁾ was proved and its proof is lectured^{(36),(37)} to 2nd year students of the naval architectural engineering course^{(38),(39)} in the “*Hydrostatics of Floating Bodies*” of the university where one of the authors⁽⁴⁰⁾ works. And then an arbitrary cross-sectional shape⁽³⁵⁾ was proved and published in the Journal “*NAVIGATION*” of Japan Institute of Navigation (hereinafter abbreviated as *JIN*).

In other way, as many researchers are studying this issue with various approaches^{(41)–(45)}, the discussions have deepened in *JASNAOE*. To sublate these discussions, we have illustrated that “the center of buoyancy is equal to the center of pressure” for a semi-submerged circular cylinder^{1st half of (46)} and a submerged circular cylinder⁽⁴⁷⁾ which does not change its shape under the water even if it is inclined, and for a triangular prisms⁽⁴⁸⁾, using the same method⁽⁴⁹⁾.

In order to put an end to the above discussions, we proved that “the center of buoyancy = the center of pressure” for a submerged body with arbitrary shape^{1st half of (50)} using Gauss's integral theorem in 2021. Furthermore, it was published in the same journal “*NAVIGATION*” of *JIN* that it is easier to prove for a floating body with arbitrary shape^{2nd half of (50)} than author's previous paper⁽³⁵⁾ by using Gauss's theorem in the same way⁽⁵¹⁾.

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We subsequently summarized the proofs in English for the case of the rectangular cross-section⁽³⁴⁾, which is the easiest to understand, and for the floating body of arbitrary cross-sectional shape^{2nd half of (50)} by applying Gauss's integral theorem. And we published them on this *viXra.org*⁽⁵²⁾ and in the bulletin of our university, *Nagasaki Institute of Applied Science*⁽⁵³⁾. Furthermore, we showed an extension to the center of buoyancy for a 3-D floating body. More recently, the authors have summarized the above as a new developments for the fundamental theory of hydrostatics of floating body and published it here on *viXra.org*⁽⁵⁴⁾.

As mentioned above, we have already proved this problem for rectangular and arbitrarily shaped cross-sections, and published them here on *viXra.org*⁽⁵²⁾ in English. Following that, in the 2nd report⁽⁵⁵⁾, a semi-submerged circular cylinder and a triangular prism were also published here. Thus, we have completed the proof for floating bodies, so in this 3rd report, we aim to prove for submerged bodies. We first prove for a submerged circular cylinder, and then apply Gauss's integral theorem to prove it clearly for an arbitrarily shaped submerged body.

We would like to report all of you smart readers about the two proofs for submerged bodies.

**2. Positioning of the Center of Hydrostatic Pressure C_p
Acting on the Submerged Circular Cylinder**

In Chapter 2, we apply to a submerged circular cylinder the same method developed for floating bodies (e.g., rectangular^{(34),(36),(37),(49),(52)~(54)}, triangular⁽⁴⁸⁾, 2nd half of (55), semi-submerged circular cylinder⁽⁴⁶⁾, 1st half of (55) and an arbitrary shaped^{(35),(49)~(54)} cross-section) by the authors already, in which the center of hydrostatic pressure is positioned by inclining the floating body laterally. It should be noted, however, in this submerged cylinder, as in the case of the semi-submerged cylinder, the cross-sectional shape does not change when it is laterally inclined.

Fig. 2.1 shows that the cross-section of a circular cylinder with radius R is submerged at water depth f to the top, and is inclined laterally by heel angle θ to the starboard side. The origin o is placed at the center of circle, which depth is $f + R$. The coordinate system fixed in space with the z -axis pointing vertically downward is $o-yz$, and that fixed to the cylinder and tilted clockwise by θ is $o-\eta\zeta$. The following analysis is performed for the latter $o-\eta\zeta$ coordinate system, using the argument ϕ measured counterclockwise from the ζ -axis as a variable.

The water depth $Z(\phi)$, denoted by capital letter, from the still water surface is expressed as :

$$Z(\phi) = f + R + z(\phi) \dots\dots\dots(2.1)$$

Here, the small letter $z(\phi)$ on the right side is the water depth z measured from the origin o downward, and is obtained on the cylinder surface $(\eta, \zeta) = (R \sin \phi, R \cos \phi)$ as follows:

$$\begin{aligned} z(\phi) &= (\zeta + \eta \tan \theta) \cos \theta \\ &= (R \cos \phi + R \sin \phi \cdot \tan \theta) \cos \theta \\ &= R (\cos \phi \cos \theta + \sin \phi \sin \theta) \\ &= R \cos(\phi - \theta) \dots\dots\dots(2.2) \end{aligned}$$

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Here, $p'(\phi)$ on the right-hand side of the above equation is the relative pressure to hydrostatic pressure $p_0 + \gamma f$ at the top of the cylinder and is defined as follows :

$$\begin{aligned}
 p'(\phi) &\equiv p(\phi) - (p_0 + \gamma f) \\
 &= \gamma (R + z(\phi)) \dots\dots\dots(2.4)
 \end{aligned}$$

The pressure shown by vector in Fig. 2.1 is this $p'(\phi)$. It acts in the $-\mathbf{n}$ direction perpendicular to the cylinder surface and is zero at the top of the cylinder.

And in the figure, the outward unit normal vector \mathbf{n} , standing on the cylinder surface, can be written using the argument ϕ , as follows :

$$\begin{aligned}
 \mathbf{n} &= n_\eta \mathbf{j} + n_\zeta \mathbf{k} \\
 &= \sin \phi \mathbf{j} + \cos \phi \mathbf{k} \dots\dots\dots(2.5)
 \end{aligned}$$

Here, n_η and n_ζ mean the directional cosines with respect to the cylinder-fixed coordinates η and ζ respectively.

2.1 Forces $F_{-\eta}$ and $F_{-\zeta}$ due to pressure in the $-\eta$ and $-\zeta$ directions acting on the surface of submerged cylinder

The pressure $p(\phi)$ acting on the cylinder surface can be expressed by Eqs. (2.2), (2.3) and (2.4) in the previous section as follows :

$$\begin{aligned}
 p(\phi) &= p_0 + \gamma f + p'(\phi) \\
 &= p_0 + \gamma (f + R) + \gamma z(\phi) \\
 &= p_c + \gamma R \cos(\phi - \theta) \dots\dots\dots(2.6)
 \end{aligned}$$

In the above equation, p_c is defined as follows, and means the hydrostatic pressure on the y -axis, placed on water depth $f + R$, passing through the center o of circle.

$$p_c \equiv p_0 + \gamma (f + R) \dots\dots\dots(2.7)$$

The forces $F_{-\eta}$ in $-\eta$ direction and $F_{-\zeta}$ in the $-\zeta$ direction due to hydrostatic pressure acting on the cylinder surface are obtained by integrating the $-\eta$ and $-\zeta$ components of $p(\phi)$ in Eq. (2.6), as shown in Fig. 2.1, over the circumference of cylinder as follows :

$$\left. \begin{aligned}
 F_{-\eta} &= \oint_{r=R} p(\phi) n_\eta d\ell \\
 F_{-\zeta} &= \oint_{r=R} p(\phi) n_\zeta d\ell
 \end{aligned} \right\} \dots\dots\dots(2.8)$$

Here, on the cylinder surface ($r = \sqrt{\eta^2 + \zeta^2} = R$), the line element is $d\ell = R d\phi$, and the directional cosines in the η and ζ directions of the normal vector \mathbf{n} are $n_\eta = \sin \phi$ and $n_\zeta = \cos \phi$ according to Eq. (2.5), so that both $F_{-\eta}$ and $F_{-\zeta}$ can be written by integration with respect to the argument ϕ .

Therefore, $F_{-\eta}$ acting in the $-\eta$ direction is expressed as :

$$\begin{aligned}
 F_{-\eta} &= \int_{-\pi}^{\pi} (p_c + \gamma R \cos(\phi - \theta)) \sin \phi \cdot R d\phi \\
 &= p_c R \int_{-\pi}^{\pi} \sin \phi d\phi + \gamma R^2 \int_{-\pi}^{\pi} \cos(\phi - \theta) \sin \phi d\phi \\
 &= p_c R \int_{-\pi}^{\pi} \sin \phi d\phi + \frac{1}{2} \gamma R^2 \cos \theta \int_{-\pi}^{\pi} \sin 2\phi d\phi \\
 &\quad + \frac{1}{2} \gamma R^2 \sin \theta \left(\int_{-\pi}^{\pi} d\phi - \int_{-\pi}^{\pi} \cos 2\phi d\phi \right) \dots\dots\dots(2.9)
 \end{aligned}$$

Similarly, $F_{-\zeta}$ acting in the $-\zeta$ direction is expressed as :

$$\begin{aligned}
 F_{-\zeta} &= \int_{-\pi}^{\pi} (p_c + \gamma R \cos(\phi - \theta)) \cos \phi \cdot R d\phi \\
 &= p_c R \int_{-\pi}^{\pi} \cos \phi d\phi + \gamma R^2 \int_{-\pi}^{\pi} \cos(\phi - \theta) \cos \phi d\phi \\
 &= p_c R \int_{-\pi}^{\pi} \cos \phi d\phi + \frac{1}{2} \gamma R^2 \sin \theta \int_{-\pi}^{\pi} \sin 2\phi d\phi \\
 &\quad + \frac{1}{2} \gamma R^2 \cos \theta \left(\int_{-\pi}^{\pi} d\phi + \int_{-\pi}^{\pi} \cos 2\phi d\phi \right) \dots\dots\dots(2.10)
 \end{aligned}$$

In both Eqs. (2.9) and (2.10) above, after expansion and integration by terms, the integrals of the 1st, 2nd, and 4th terms of the 3rd line are zero, and the integral value 2π results from the 3rd term only. Thus, both $F_{-\eta}$ and $F_{-\zeta}$ are computed respectively, as follows :

$$\left. \begin{aligned}
 F_{-\eta} &= \gamma \cdot \pi R^2 \cdot \sin \theta \\
 F_{-\zeta} &= \gamma \cdot \pi R^2 \cdot \cos \theta
 \end{aligned} \right\} \dots\dots\dots(2.11)$$

Therefore, in both forces above, since the integral with respect to p_c in the 1st term is zero, it is indicated that the forces acting on the submerged body is not dependent of atmospheric pressure p_0 or submerged depth f . In addition, according to the results of Eq.(2.12) in the next section, $F_{-\eta}$ and $F_{-\zeta}$ are obtained as $-\eta$ and $-\zeta$ directional components of the buoyancy F_{-z} acting vertically upward, respectively.

2.2 Forces F_{-y} and F_{-z} converted in the $-y$ and $-z$ directions

In this section, let us find the horizontal and vertical components of the force acting on the submerged circular cylinder. By using $F_{-\eta}$ and $F_{-\zeta}$ obtained in Eq. (2.11) in the previous section, the horizontal component F_{-y} in the $-y$ direction and the vertical component F_{-z} in the $-z$ direction are converted as follows :

$$\left. \begin{aligned}
 F_{-y} &= F_{-\eta} \cos \theta - F_{-\zeta} \sin \theta \\
 &= \gamma \cdot \pi R^2 (\sin \theta \cdot \cos \theta - \cos \theta \cdot \sin \theta) \\
 &= 0 \\
 F_{-z} &= F_{-\zeta} \cos \theta + F_{-\eta} \sin \theta \\
 &= \gamma \cdot \pi R^2 (\cos^2 \theta + \sin^2 \theta) \\
 &= \gamma \cdot \pi R^2 (= \text{Buoyant Force})
 \end{aligned} \right\} \dots\dots\dots(2.12)$$

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The above results show that the horizontal component F_{-y} does not act as the combined force due to pressure integration. The vertical component F_{-z} is the product of the specific gravity γ of water and the area πR^2 of submerged circle, and is indeed the buoyant force itself generated vertically upward, as Archimedes' principle⁽¹⁾ teaches.

2.3 Moments M_η and M_ζ due to pressure in the η and ζ directions acting on the surface of submerged cylinder

The clockwise moment M_η about the origin o due to the pressure in the $-\eta$ direction acting on the cylinder surface and the counterclockwise moment M_ζ due to the pressure in the $-\zeta$ direction can be obtained by circumferentially integrating Eq. (2.8) multiplied by ζ or η as the lever of moment respectively, as follows :

$$\left. \begin{aligned} M_\eta &= \oint_{r=R} p(\phi) \zeta \cdot n_\eta d\ell \\ M_\zeta &= \oint_{r=R} p(\phi) \eta \cdot n_\zeta d\ell \end{aligned} \right\} \dots\dots\dots(2.13)$$

Here, if the above moments are expressed by contour integrals with respect to the argument ϕ , as in Eqs. (2.9) and (2.10) for $F_{-\eta}$ and $F_{-\zeta}$ in the Section 2.1, M_η becomes as :

$$\begin{aligned} M_\eta &= \int_{-\pi}^{\pi} (p_c + \gamma R \cos(\phi - \theta)) R \cos \phi \cdot \sin \phi \cdot R d\phi \\ &= p_c R^2 \int_{-\pi}^{\pi} \sin \phi \cos \phi d\phi + \gamma R^3 \int_{-\pi}^{\pi} \cos(\phi - \theta) \sin \phi \cos \phi d\phi \quad \dots\dots\dots(2.14) \end{aligned}$$

And, M_ζ becomes as :

$$\begin{aligned} M_\zeta &= \int_{-\pi}^{\pi} (p_c + \gamma R \cos(\phi - \theta)) R \sin \phi \cdot \cos \phi \cdot R d\phi \\ &= p_c R^2 \int_{-\pi}^{\pi} \sin \phi \cos \phi d\phi + \gamma R^3 \int_{-\pi}^{\pi} \cos(\phi - \theta) \sin \phi \cos \phi d\phi \quad \dots\dots\dots(2.15) \end{aligned}$$

The above results show that both equations for M_η and M_ζ are equivalent, as follows :

$$M_\eta = M_\zeta \quad \dots\dots\dots(2.16)$$

Further expanding Eqs. (2.14) and (2.15) for both moments and proceeding with the calculation, we obtain as follows :

$$\begin{aligned} M_\eta &= M_\zeta \\ &= \frac{1}{2} p_c R^2 \int_{-\pi}^{\pi} \sin 2\phi d\phi + \gamma R^3 \cos \theta \int_{-\pi}^{\pi} \sin \phi \cos^2 \phi d\phi \\ &\quad + \gamma R^3 \sin \theta \int_{-\pi}^{\pi} \sin^2 \phi \cos \phi d\phi \\ &= 2\gamma R^3 \sin \theta \int_0^{\pi} \sin^2 \phi \cos \phi d\phi \quad \dots\dots\dots(2.17) \end{aligned}$$

Here, in the above equation, the 1st and 2nd terms are zero because the integrand is a odd function with respect to ϕ . And the 3rd term is an even function, so it is written with the integral interval folded in half.

Furthermore, by replacing the integrating variable from ϕ to φ by $\varphi = \phi - \frac{\pi}{2}$, the above equation is computed as follows :

$$\begin{aligned}
 M_\eta &= M_\zeta \\
 &= -2\gamma R^3 \sin\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi d\varphi \\
 &= 0 \quad \dots\dots\dots(2.18)
 \end{aligned}$$

This is because the integrand in the above equation is an odd function with respect to φ , so the integral value is zero. This result shows that the η and ζ components of the hydrostatic pressure do not cause moments around the center of circle in the case of the submerged cylinder, unlike the semi-submerged cylinder ^{(46), 1st half of (55)}.

2.4 Positioning of the center of hydrostatic pressure C_p for the submerged circular cylinder

To locate the center of pressure C_p in $o-\eta\zeta$ coordinate system fixed to circular cylinder, the hydraulic method used in the authors' previous papers ^{(34),(35),(46)~(55)} is applied. This method was used by Ohgushi ⁽⁹⁾ for an example problem of the rolling gate.

Since the forces $F_{-\eta}$ and $F_{-\zeta}$ due to the hydrostatic pressure obtained in Section 2.1 act on the center of pressure $C_p (\eta_p, \zeta_p)$, the moments M_η and M_ζ due to the corresponding pressure obtained in Section 2.3 can be expressed respectively, as follows :

$$\left. \begin{aligned}
 M_\eta &= F_{-\eta} \zeta_p \\
 M_\zeta &= F_{-\zeta} \eta_p
 \end{aligned} \right\} \dots\dots\dots(2.19)$$

Therefore, the unknown coordinate (η_p, ζ_p) of the center of pressure C_p can be determined by Eq. (2.19). Hence, the η -coordinate, η_p , can be determined by the combined force $F_{-\zeta}$ and the moment M_ζ due to the hydrostatic pressure in the $-\zeta$ direction, and the ζ -coordinate, ζ_p , by $F_{-\eta}$ and M_η in the $-\eta$ direction, by using Eqs. (2.11) and (2.18) respectively, as follows :

$$\left. \begin{aligned}
 \eta_p &= \frac{M_\zeta}{F_{-\zeta}} = \frac{0}{\gamma \cdot \pi R^2 \cdot \cos\theta} = 0 \\
 \zeta_p &= \frac{M_\eta}{F_{-\eta}} = \frac{0}{\gamma \cdot \pi R^2 \cdot \sin\theta} = 0
 \end{aligned} \right\} \dots\dots\dots(2.20)$$

Let us consider the above equation. For the latter ζ_p of vertical component, if we assume the upright state $\theta=0$ from the beginning, $\sin\theta$ in the denominator $F_{-\eta}$ will be zero, so the fraction becomes indeterminate forms and ζ_p cannot be determined. We were able to locate the vertical component ζ_p

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of the center of pressure, because the submerged cylinder was laterally inclined along with its $\eta\zeta$ - coordinate axes, even though the shape did not change when inclined.

On the other hand, for the former η_p , even if the heel angle is $\theta=0$ from the beginning, the denominator $F_{-\zeta}$ can take a finite value because of $\cos\theta=1$, and horizontal component η_p can be determined.

As a result, the center of pressure C_p of the submerged cylinder is obtained in the $o-\eta\zeta$ coordinate system fixed and inclined to the cylinder as follows. It is found that the C_p is located at the origin o , which is the center of the submerged circle.

$$C_p(\eta_p, \zeta_p) = (0, 0) \dots\dots\dots(2.21)$$

Furthermore, its C_p is located in the following position in the space-fixed $o-yz$ coordinate system and can be determined without any need for coordinate transformation.

$$C_p(y_p, z_p) = (0, 0) \dots\dots\dots(2.22)$$

The both Eqs. (2.21) and (2.22) above is correctly indicates the figure centroid of the submerged circle. Hence, we were able to prove that the center of hydrostatic pressure is equal to the well-known center of buoyancy even for submerged bodies, as in the case of floating bodies^{(34),(35),(46)~(55)} published by authors already.

3. Positioning of the Center of Hydrostatic Pressure Acting on an Inclined Submerged Body with Arbitrary Shape

In Chapter 3, we apply the same method as used in the previous Chapter 2, in which the submerged circular cylinder is inclined laterally, to the submerged body with the arbitrary shape. It is then proved that the center of hydrostatic pressure is equal to well-known center of buoyancy by using Gauss's integral theorem, which has already been applied to floating bodies with arbitrary shape by the authors^{(35),(49)~(54)}.

Fig. 3.1 shows the cross-section of an arbitrary shaped body submerged at water depth f to the top, inclined laterally about its top by θ to the starboard side. The origin o is placed at the top of submerged body. The coordinate system fixed in space with the z -axis pointing vertically downward is $o-yz$, and that fixed to the body and tilted clockwise by θ is $o-\eta\zeta$. The following analysis is performed for the latter inclined $o-\eta\zeta$ coordinate system.

The water depth Z , denoted by capital letter, on the surface (η, ζ) of submerged body from the still water line is expressed as :

$$\begin{aligned} Z &= f + z(\eta, \zeta) \\ &= f + (\zeta + \eta \tan \theta) \cos \theta \\ &= f + (\zeta \cos \theta + \eta \sin \theta) \dots\dots\dots(3.1) \end{aligned}$$

Here, in the above equation, the small letter z on the right - hand represents the water depth measured from the top o .

Hydrostatic pressure p at the body surface (η, ζ) can be written as follows, using p_0 for the atmospheric pressure, γ for the specific weight of water, and the 1st line of Eq. (3.1) for the water depth Z .

$$\begin{aligned}
 p &= p_0 + \gamma Z \\
 &= p_0 + \gamma f + \gamma z(\eta, \zeta) \\
 &\equiv p_0 + \gamma f + p'(\eta, \zeta) \dots\dots\dots(3.2)
 \end{aligned}$$

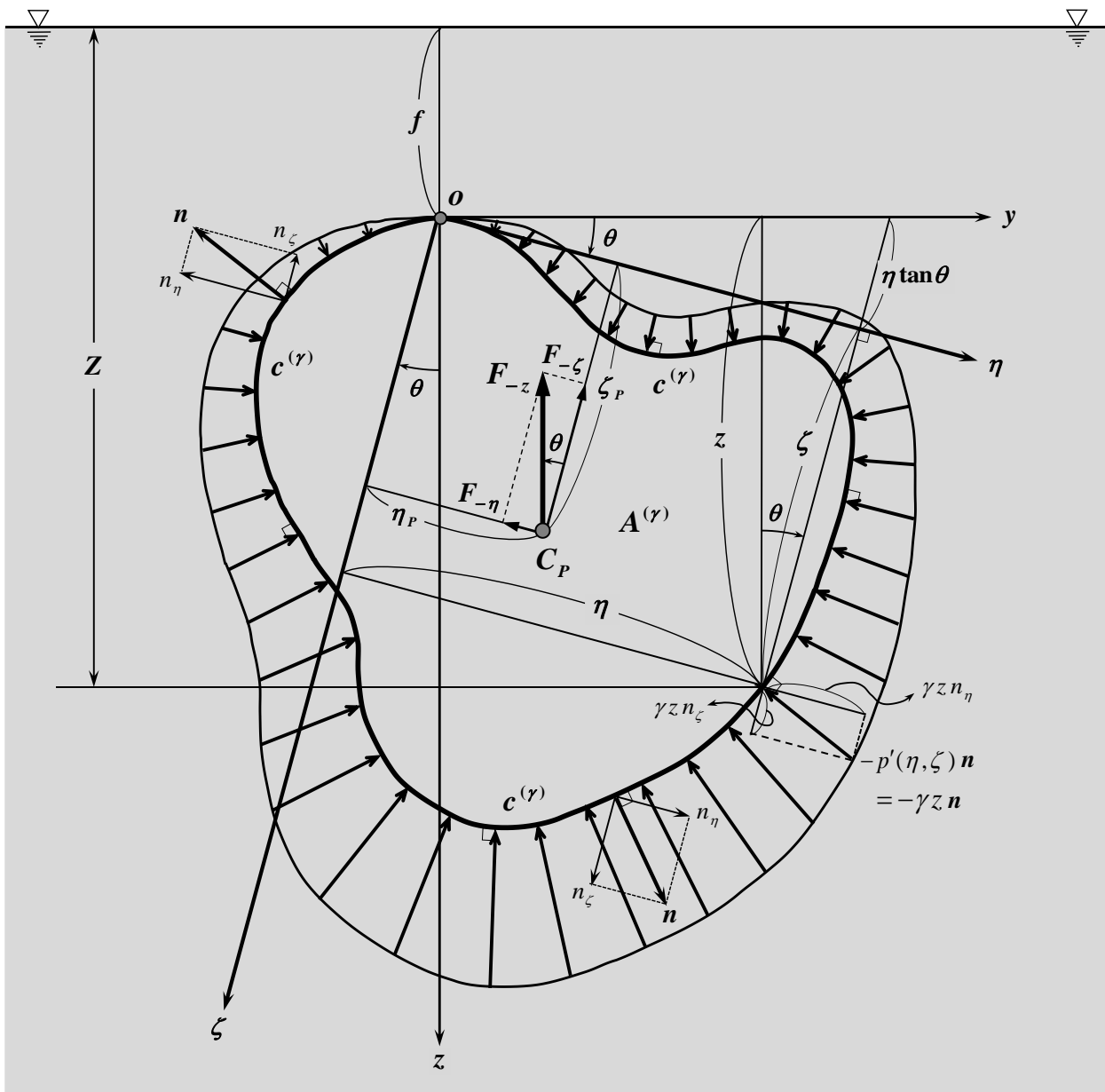


Fig. 3.1 Hydrostatic pressure $p' = p - (p_0 + \gamma f)$ and the center of pressure C_p acting on the cross-section of an inclined submerged body with arbitrary shape.

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Where, p' on the right-hand side of the above 3rd line is the relative pressure to hydrostatic pressure $p_0 + \gamma f$ at the top of submerged body and is defined by using Eq. (3.1), as follows :

$$\begin{aligned} p'(y, z) &\equiv p - (p_0 + \gamma f) \\ &= \gamma z (\eta, \zeta) \\ &= \gamma (\zeta \cos \theta + \eta \sin \theta) \dots\dots\dots(3.3) \end{aligned}$$

Here, the pressure shown by vector in Fig. 3.1 is the above relative pressure p' . It acts in the $-\mathbf{n}$ direction perpendicular to the body surface and is zero at the top of body.

Then, \mathbf{n} is the outward unit normal vector standing on the body surface, and as in the case of circular cylinder in the previous chapter, is written as follows :

$$\mathbf{n} = n_\eta \mathbf{j} + n_\zeta \mathbf{k} \dots\dots\dots(3.4)$$

Here, n_η and n_ζ mean the directional cosines with respect to the body-fixed coordinates η and ζ respectively.

3.1 Forces $F_{-\eta}$ and $F_{-\zeta}$ due to pressure in the $-\eta$ and $-\zeta$ directions acting on the surface of submerged body

The forces $F_{-\eta}$ in $-\eta$ direction and $F_{-\zeta}$ in the $-\zeta$ direction due to hydrostatic pressure acting on the cylinder surface can be obtained by integrating the $-\eta$ and $-\zeta$ components of p in Eq. (3.2), as shown in Fig. 3.1, over the entire circumference $c^{(\gamma)}$ of the body surface with the line element as $d\ell$, as follows :

$$\left. \begin{aligned} F_{-\eta} &= \oint_{c^{(\gamma)}} (p_0 + \gamma f + p'(\eta, \zeta)) n_\eta d\ell \\ F_{-\zeta} &= \oint_{c^{(\gamma)}} (p_0 + \gamma f + p'(\eta, \zeta)) n_\zeta d\ell \end{aligned} \right\} \dots\dots\dots(3.5)$$

In proceeding with the calculation, the 3rd line of Eq. (3.3) is used for the relative pressure p' and the termwise integration is performed respectively. Then, $F_{-\eta}$ is calculated as :

$$\begin{aligned} F_{-\eta} &= \oint_{c^{(\gamma)}} \{ p_0 + \gamma f + \gamma (\zeta \cos \theta + \eta \sin \theta) \} n_\eta d\ell \\ &= \oint_{c^{(\gamma)}} (p_0 + \gamma f) n_\eta d\ell + \gamma \cos \theta \oint_{c^{(\gamma)}} \zeta n_\eta d\ell + \gamma \sin \theta \oint_{c^{(\gamma)}} \eta n_\eta d\ell \dots\dots\dots(3.6) \end{aligned}$$

Similarly, $F_{-\zeta}$ is calculated as :

$$\begin{aligned} F_{-\zeta} &= \oint_{c^{(\gamma)}} \{ p_0 + \gamma f + \gamma (\eta \sin \theta + \zeta \cos \theta) \} n_\zeta d\ell \\ &= \oint_{c^{(\gamma)}} (p_0 + \gamma f) n_\zeta d\ell + \gamma \sin \theta \oint_{c^{(\gamma)}} \eta n_\zeta d\ell + \gamma \cos \theta \oint_{c^{(\gamma)}} \zeta n_\zeta d\ell \dots\dots\dots(3.7) \end{aligned}$$

Let us now apply the following two-dimensional ($\eta\zeta$ -plane) Gauss' integral theorem to the above contour integral in order to convert the line integral into an areal integral.

$$\left. \begin{aligned} \oint_c u(\eta, \zeta) n_\eta d\ell &= \iint_A \frac{\partial u}{\partial \eta} dA \\ \oint_c v(\eta, \zeta) n_\zeta d\ell &= \iint_A \frac{\partial v}{\partial \zeta} dA \end{aligned} \right\} \dots\dots\dots(3.8)$$

In the above theorem, n_η and n_ζ are the directional cosines of the outward unit normal vector \mathbf{n} in η and ζ directions, as shown in Eq.(3.4) and Fig.3.1.

Then, $F_{-\eta}$ in Eq. (3.6) can be converted to an areal integral and written as :

$$\begin{aligned} F_{-\eta} &= \iint_{A^{(\gamma)}} \frac{\partial(p_0 + \gamma f)}{\partial \eta} dA + \gamma \cos \theta \iint_{A^{(\gamma)}} \frac{\partial \zeta}{\partial \eta} dA + \gamma \sin \theta \iint_{A^{(\gamma)}} \frac{\partial \eta}{\partial \eta} dA \\ &= \gamma \sin \theta \iint_{A^{(\gamma)}} dA = \gamma A^{(\gamma)} \sin \theta \quad \dots\dots\dots(3.9) \end{aligned}$$

Similarly, $F_{-\zeta}$ in Eq. (3.7) can be written as :

$$\begin{aligned} F_{-\zeta} &= \iint_{A^{(\gamma)}} \frac{\partial(p_0 + \gamma f)}{\partial \zeta} dA + \gamma \sin \theta \iint_{A^{(\gamma)}} \frac{\partial \eta}{\partial \zeta} dA + \gamma \cos \theta \iint_{A^{(\gamma)}} \frac{\partial \zeta}{\partial \zeta} dA \\ &= \gamma \cos \theta \iint_{A^{(\gamma)}} dA = \gamma A^{(\gamma)} \cos \theta \quad \dots\dots\dots(3.10) \end{aligned}$$

From the results above, it can be seen that both forces are determined by the cross-sectional area $A^{(\gamma)}$ of the submerged body and the lateral inclination angle θ , and do not depend on the atmospheric pressure p_0 and the submerged depth f . The reason is why the integrands of 1st and 2nd terms in both Eqs. (3.9) and (3.10) become zero and vanished.

In addition, according to the results of Eq.(3.11) in the next section, $F_{-\eta}$ and $F_{-\zeta}$ are obtained as $-\eta$ and $-\zeta$ directional components of the buoyancy F_{-z} acting vertically upward, respectively.

3.2 Forces F_{-y} and F_{-z} converted in the $-y$ and $-z$ directions

In this section, let us find the horizontal and vertical components of the force acting on the submerged body. By using $F_{-\eta}$ and $F_{-\zeta}$ obtained in Eqs. (3.9) and (3.10) in the previous section, the horizontal component F_{-y} in the $-y$ direction and the vertical component F_{-z} in the $-z$ direction are converted as follows :

$$\left. \begin{aligned} F_{-y} &= F_{-\eta} \cos \theta - F_{-\zeta} \sin \theta \\ &= \gamma A^{(\gamma)} (\sin \theta \cdot \cos \theta - \cos \theta \cdot \sin \theta) \\ &= 0 \\ F_{-z} &= F_{-\zeta} \cos \theta + F_{-\eta} \sin \theta \\ &= \gamma A^{(\gamma)} (\cos^2 \theta + \sin^2 \theta) \\ &= \gamma A^{(\gamma)} (= \text{Buoyant Force}) \end{aligned} \right\} \dots\dots\dots(3.11)$$

The above results show that the horizontal component F_{-y} does not act as the combined force due to pressure integration, even when the pressure field is left-right asymmetric. The vertical component

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F_{-z} is the product of the specific gravity γ of water and the cross-sectional area $A^{(\gamma)}$ of submerged body, and is the buoyant force itself generated vertically upward, as Archimedes' principle⁽¹⁾ teaches. This situation is similar to Eq. (2.12) for the submerged circular cylinder in Section 2.2.

3.3 Moments M_η and M_ζ due to pressure in the η and ζ directions acting on the surface of submerged body

In this section, we shall calculate the total counterclockwise moment M_o around the origin o due to hydrostatic pressure acting on the surface of the submerged body. It can be calculated by superimposing the clockwise moment M_η due to the pressure component in the $-\eta$ direction and the counterclockwise moment M_ζ due to in the $-\zeta$ direction, as follows :

$$M_o = -M_\eta + M_\zeta \quad \dots\dots\dots(3.12)$$

Here, M_η and M_ζ can be obtained by multiplying the integrand in Eq. (3.5) by ζ or η as the lever of moment respectively, in the following form :

$$\left. \begin{aligned} M_\eta &= \oint_{c^{(\gamma)}} (p_0 + \gamma f + p'(\eta, \zeta)) \zeta \cdot n_\eta \, d\ell \\ M_\zeta &= \oint_{c^{(\gamma)}} (p_0 + \gamma f + p'(\eta, \zeta)) \eta \cdot n_\zeta \, d\ell \end{aligned} \right\} \dots\dots\dots(3.13)$$

Now, as in the case of forces $F_{-\eta}$ and $F_{-\zeta}$ in Eqs. (3.6) and (3.7) by using Eq. (3.3) for the relative pressure p' , M_η can be expressed as the superposition of the contour integrals around $c^{(\gamma)}$, as follows :

$$\begin{aligned} M_\eta &= \oint_{c^{(\gamma)}} \{ p_0 + \gamma f + \gamma(\zeta \cos\theta + \eta \sin\theta) \} \zeta n_\eta \, d\ell \\ &= (p_0 + \gamma f) \oint_{c^{(\gamma)}} \zeta n_\eta \, d\ell + \gamma \cos\theta \oint_{c^{(\gamma)}} \zeta^2 n_\eta \, d\ell + \gamma \sin\theta \oint_{c^{(\gamma)}} \eta \zeta n_\eta \, d\ell \quad \dots\dots\dots(3.14) \end{aligned}$$

Similarly, M_ζ can be expressed as :

$$\begin{aligned} M_\zeta &= \oint_{c^{(\gamma)}} \{ p_0 + \gamma f + \gamma(\eta \sin\theta + \zeta \cos\theta) \} \eta n_\zeta \, d\ell \\ &= (p_0 + \gamma f) \oint_{c^{(\gamma)}} \eta n_\zeta \, d\ell + \gamma \sin\theta \oint_{c^{(\gamma)}} \eta^2 n_\zeta \, d\ell + \gamma \cos\theta \oint_{c^{(\gamma)}} \eta \zeta n_\zeta \, d\ell \quad \dots\dots\dots(3.15) \end{aligned}$$

Therefore, we can apply Gauss's integral theorem in Eq. (3.8) to the above contour integrals, as in the case of forces $F_{-\eta}$ and $F_{-\zeta}$ in Section 3.1, and convert them into area integrals.

Then, the clockwise moment M_η in Eq. (3.14) can be converted to an areal integral and written as follows, and is consequently obtained in proportion to the areal moment about the η - axis.

$$\begin{aligned} M_\eta &= (p_0 + \gamma f) \iint_{A^{(\gamma)}} \frac{\partial \zeta}{\partial \eta} \, dA + \gamma \cos\theta \iint_{A^{(\gamma)}} \frac{\partial \zeta^2}{\partial \eta} \, dA + \gamma \sin\theta \iint_{A^{(\gamma)}} \frac{\partial(\eta \zeta)}{\partial \eta} \, dA \\ &= \gamma \sin\theta \iint_{A^{(\gamma)}} \zeta \, dA \quad \dots\dots\dots(3.16) \end{aligned}$$

On the other hand, the counterclockwise moment M_ζ in Eq. (3.15) can be written as follows, and is consequently obtained in proportion to the areal moment about the ζ - axis.

$$\begin{aligned}
 M_\zeta &= (p_0 + \gamma f) \iint_{A^{(\gamma)}} \frac{\partial \eta}{\partial \zeta} dA + \gamma \sin \theta \iint_{A^{(\gamma)}} \frac{\partial \eta^2}{\partial \zeta} dA + \gamma \cos \theta \iint_{A^{(\gamma)}} \frac{\partial(\eta \zeta)}{\partial \zeta} dA \\
 &= \gamma \cos \theta \iint_{A^{(\gamma)}} \eta dA \quad \dots\dots\dots(3.17)
 \end{aligned}$$

From the both results above, it can be found that both moments do not depend on the atmospheric pressure p_0 and the submerged depth f , as in the case of forces $F_{-\eta}$ and $F_{-\zeta}$ in Eqs. (3.9) and (3.10). The reason is why the integrands of 1st and 2nd terms in both Eqs. (3.16) and (3.17) become zero and vanished.

3.4 Positioning of the center of hydrostatic pressure C_p for the floating body with an arbitrary form

For the positioning of the center of pressure of the submerged body, we will use the hydraulic method by Ohgushi⁽⁹⁾, as in the case of the submerged circular cylinder in the previous Chapter 2.

Since the forces $F_{-\eta}$ and $F_{-\zeta}$ due to the hydrostatic pressure obtained in Section 3.1 act on the center of pressure $C_p(\eta_p, \zeta_p)$, the clockwise moment M_η and the counterclockwise moment M_ζ due to the corresponding pressure obtained in Section 3.3 can be expressed respectively, as follows :

$$\left. \begin{aligned}
 M_\eta &= F_{-\eta} \zeta_p \\
 M_\zeta &= F_{-\zeta} \eta_p
 \end{aligned} \right\} \dots\dots\dots(3.18)$$

Then, the total counterclockwise moment M_o around the origin o in Eq. (3.12) can be calculated as :

$$M_o = -F_{-\eta} \zeta_p + F_{-\zeta} \eta_p \quad \dots\dots\dots(3.19)$$

On the other hand, the moment M_{C_p} around the point C_p , at which $F_{-\eta}$ and $F_{-\zeta}$ act, is computed as follows, and becomes zero.

$$M_{C_p} = -F_{-\eta} \times 0 + F_{-\zeta} \times 0 = 0 \quad \dots\dots\dots(3.20)$$

This correctly indicates that C_p is the center of hydrostatic pressure due to the forces $F_{-\eta}$ and $F_{-\zeta}$.

Therefore, the unknown coordinate (η_p, ζ_p) of this center of pressure C_p can be determined by Eq. (3.18). First, the η -coordinate, η_p , can be determined by using Eq. (3.10) for $F_{-\zeta}$ and Eq. (3.17) for M_ζ , as follows :

$$\begin{aligned}
 \eta_p &= \frac{M_\zeta}{F_{-\zeta}} = \frac{\gamma \cos \theta \iint_{A^{(\gamma)}} \eta dA}{\gamma A^{(\gamma)} \cos \theta} \\
 &= \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} \eta dA \quad (= \eta_G) \quad \dots\dots\dots(3.21)
 \end{aligned}$$

Next, the ζ -coordinate, ζ_p , can be determined by using Eq. (3.9) for $F_{-\eta}$ and Eq. (3.16) for M_η , as follows :

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$$\begin{aligned} \zeta_p &= \frac{M_\eta}{F_{-\eta}} = \frac{\gamma \sin \theta \iint_{A^{(\gamma)}} \zeta \, dA}{\gamma A^{(\gamma)} \sin \theta} \\ &= \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} \zeta \, dA \quad (= \zeta_G) \quad \dots\dots\dots(3.22) \end{aligned}$$

As a result, since the η_p and ζ_p are obtained with the form in which the areal moment about the ζ - and η - axis is divided by the cross-sectional area $A^{(\gamma)}$ respectively, it can be seen that they are the η_G and ζ_G of the figure centroid position for submerged body geometrically. And, in the both equations, the specific weight γ of water and the heel angle θ have been cancelled out in the denominator and numerator respectively, so that the both results are independent of θ . Furthermore, the geometric shape of a submerged body does not change when it is laterally inclined, unlike the case of a floating body. Therefore, this shows that the center of pressure (η_p, ζ_p) of the submerged body in the inclined state always coincides with the centroid (η_G, ζ_G) of the cross-sectional area $A^{(\gamma)}$, i.e., the well-known center of buoyancy, regardless of whether it is laterally inclined or not.

Considering the above, ζ_p of vertical component can be obtained by offsetting the zero factor $\sin \theta$ at the heel angle $\theta \rightarrow 0$ with the denominator and numerator, as shown in Eq. (3.22). Here, if we start the calculation as the upright state $\theta=0$, both the denominator $F_{-\eta}$ and the numerator M_η are in equilibrium and become zero, so the fraction becomes indeterminate forms and ζ_p cannot be determined. This is the reason why we were able to determine the position of the center of pressure in the ζ direction as $\zeta_p = \zeta_G$ by inclining the submerged body laterally.

On the other hand, in the calculation of η_p in Eq. (3.21), even if the heel angle is $\theta=0$ from the beginning, the denominator $F_{-\zeta}$ takes a finite value as the cosine component of the buoyancy. Therefore, the horizontal component η_p can be determined as $\eta_p = \eta_G$, if we start the calculation as the upright state.

These situations described above are exactly the same as in Eq. (2.20) of Section 2.4 for the submerged circular cylinder.

As a final step, let's find the center of pressure in the upright state by setting the heel angle to $\theta \rightarrow 0$, in order to make this result clearer. Then, since the $\eta\zeta$ - coordinates tilted and fixed on the submerged body coincide with the yz - coordinates fixed in space and $A^{(\gamma)}$ is invariant, the Eqs. (3.21) and (3.22) become as :

$$\left. \begin{aligned} (y_p, z_p) &= \left(\frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} y \, dA, \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} z \, dA \right) = (y_G, z_G) \\ \therefore C_p &= B \end{aligned} \right\} \dots\dots\dots(3.23)$$

Therefore, this proves that the center of pressure C_p due to hydrostatic pressure coincides with the well-known "Center of Buoyancy, B ".

4. Concluding Remarks

In this 3rd report, we proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy for a submerged circular cylinder and an arbitrarily shaped submerged body by using the authors' method of inclining the body and coordinate axes laterally, as in the case of floating bodies in the 1st and 2nd reports^{(52),(55)}.

In calculating the forces and moments acting on the submerged body, the case of the submerged circular cylinder in the 1st half was performed by using the circumferential argument as the integral variable, and the case of the arbitrarily shaped submerged body in the 2nd half by applying Gauss's integral theorem.

The forces and moments, which were calculated in the direction of the inclined coordinate axes, were connected via the coordinates of the center of hydrostatic pressure by using the hydraulic method, so that we fully elucidated the physical meaning of the center of buoyancy acting on the submerged bodies, as well as the floating bodies reported in authors' 1st and 2nd papers^{(52),(55)}.

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Finally from the 1st author⁽⁴⁰⁾, let me express the following thanks. I would like to communicate my deepest gratitude to my late teacher, *Pr. Masato KURIHARA*^{(59)~(61)}, who cordially taught me the theory of “*Hydrostatics of Ships*” with detailed figures and formulas on the blackboard when I was a 1st year undergraduate student and learned my 1st specialized subject of naval architecture in *the College of Naval Architecture of Nagasaki* in Japan. Therefore, I am following the appearance of my teacher at that time from more than 40 years ago as an exemplary example, when I currently lecture *Hydrostatics of Floating Bodies* and *Theory of Ship Stability* to 2nd year students at my university^{(38),(39)}.

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