The Floor and Ceiling Functions

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ABSTRACT: In this note we give some properties of the Floor and Ceiling functions

Key words: Integer functions, computational science, pi formulas, mathematical reasoning.

Notation:

Integers numbers: $\mathbb{Z} = \{ ..., -3, -2, -1, 0, 1, 2, 3, ...\}$

Natural numbers: $\mathbb{N} = \{1, 2, 3, ...\}$

Real numbers: $\mathbb{R}$

I. Introduction: Floor and Ceiling Definitions

Floor Definition

For any $x \in \mathbb{R}$ we define

$\lfloor x \rfloor$ = the greatest integer less than or equal to x

Ceiling Definition

For any $x \in \mathbb{R}$ we define

$\lceil x \rceil$ = the least (smallest) integer greater than or equal to x

Definition written Symbolicaly

Floor

$$\lfloor x \rfloor = \max \{a \in \mathbb{Z} \ : \ a \leq x\} \tag{1}$$

Ceiling

$$\lceil x \rceil = \min \{a \in \mathbb{Z} \ : \ a \geq x\} \tag{2}$$

Remark: For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ exist and are unique

II. Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

$$\lfloor x \rfloor = x \text{ if and only if } x \in \mathbb{Z} \tag{3}$$

$$\lceil x \rceil = x \text{ if and only if } x \in \mathbb{Z} \tag{4}$$

$$x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1 \quad , \; x \in \mathbb{R} \tag{5}$$

$$\lfloor -x \rfloor = -\lceil x \rceil \ , \ x \in \mathbb{R} \tag{6}$$

$$\lceil -x \rceil = -\lfloor x \rfloor \ , \ x \in \mathbb{R} \tag{7}$$

$$\lceil x \rceil - \lfloor x \rfloor = 0 \ \text{ for } x \in \mathbb{Z} \tag{8}$$

$$\lceil x \rceil - \lfloor x \rfloor = 1 \ \text{ for } x \notin \mathbb{Z} \tag{9}$$

$$\lfloor x \rfloor = n \ \text{ if and only if } \ n \le x < n+1 \ \text{for } x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{10}$$

$$\lfloor x \rfloor = n \ \text{ if and only if } \ x-1 < n \le x \ \text{for } x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{11}$$

$$\lceil x \rceil = n \ \text{ if and only if } \ n-1 < x \le n \ \text{ for } x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{12}$$

$$\lceil x \rceil = n \ \text{ if and only if } \ x \le n < x+1 \ \text{ for } x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{13}$$

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n \ \text{ and } \ \lceil x+n \rceil = \lceil x \rceil + n \ \text{ for } x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{14}$$

$$x < n \ \text{ if and only if } \ \lfloor x \rfloor < n \ , \ x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{15}$$

$$n < x \ \text{ if and only if } \ n < \lceil x \rceil \ , \ x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{16}$$

$$x \le n \ \text{ if and only if } \ \lceil x \rceil \le n \ , \ x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{17}$$

$$n \le x \ \text{ if and only if } \ n \le \lfloor x \rfloor \ , \ x \in \mathbb{R}, \ n \in \mathbb{Z} \tag{18}$$

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \left\lfloor \sqrt{x} \right\rfloor \ , \ x \in \mathbb{R}, x \ge 0 \tag{19}$$

$$\lfloor x\,y \rfloor \ge \lfloor x \rfloor \lfloor y \rfloor \ , \ x, y \ge 0 \tag{20}$$

$$\left\lfloor \frac{y}{x} \right\rfloor \le \frac{\lfloor y \rfloor}{\lfloor x \rfloor} \ , \ x \ge 1, y > 0 \tag{21}$$

$$n \lfloor x \rfloor \le \lfloor n\,x \rfloor \ , n \in \mathbb{N} \ , \ x \in \mathbb{R} \tag{22}$$

III. Fractional Part of x

Definition

We define:

$$\{x\} = x - \lfloor x \rfloor \tag{23}$$

$\{x\}$ is called a fractional part of x

$\lfloor x \rfloor$ is called the integer part of x

By definition

$$0 \le \{x\} < 1 \tag{24}$$

and we write

$$x = \lfloor x \rfloor + \{x\} \tag{25}$$

$$\text{if } \ x = n + \theta \ , \ n \in \mathbb{Z} \text{ and } \ 0 \le \theta < 1 \ \text{ then } \ n = \lfloor x \rfloor \ \text{ and } \ \theta = \{x\} \tag{26}$$

some properties

$$\text{for any } \ x, y \in \mathbb{R} \ \ \lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \ \text{ when } \ 0 \le \{x\} + \{y\} < 1 \tag{27}$$

$$\text{for any } \ x, y \in \mathbb{R} \ \ \lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1 \ \text{ when } \ 1 \le \{x\} + \{y\} < 2 \tag{28}$$

$$n \lfloor x \rfloor = \lfloor n\,x \rfloor \ \Leftrightarrow \ n\{x\} < 1 \ , \ n \in \mathbb{N} \ , \ x \in \mathbb{R} \tag{29}$$

Notation:

Floor function: $\text{floor}(x) = \lfloor x \rfloor$

Ceiling function: $\text{Ceiling}(x) = \text{Ceil}(x) = \lceil x \rceil$

IV. Floors and Ceilings Applications

Recall that

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n + 1} = 4 \int_0^1 \frac{1}{1 + x^2}\, dx \tag{30}$$

Entry 1.

Define s by

$$s = 1 - \frac{1 - i\sqrt{3}}{2^{2/3}\left(1 + i\sqrt{3}\right)^{1/3}} - \frac{\left(1 + i\sqrt{3}\right)^{4/3}}{2 \cdot 2^{1/3}} \;,\; i = \sqrt{-1} \tag{31}$$

$$s = 1 - 2\sin\left(\frac{\pi}{18}\right) \tag{32}$$

$$s^3 = \left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \ldots\right)^{3/2}\right)^{3/2}\right)^{3/2} \tag{33}$$

Then

$$\pi = 6\,s + 6 \sum_{n=2}^{\infty} s^{n+1} \sum_{k=\text{Ceiling}(n/3)}^{\text{Floor}(n/2)} \frac{(-1)^k\, 2^{-2\,n+4\,k}}{(6\,k - 2\,n + 1)\binom{6\,k - 2\,n}{3\,k - n}} \binom{k}{n - 2\,k} \binom{2\,k}{k} \tag{34}$$

$$\pi = 6\,s - 2\,s^3 - 3\,s^4 + \frac{6}{5}\,s^5 + 3\,s^6 + \frac{39}{28}\,s^7 - \ldots \tag{35}$$

Entry 2.

Define u by

$$u = -1 + \frac{1}{2}\left(1 - i\sqrt{3}\right)\left(\frac{1 + i\sqrt{3}}{2}\right)^{1/3} + \left(\frac{1 + i\sqrt{3}}{2}\right)^{2/3} \;,\; i = \sqrt{-1} \tag{36}$$

$$u = -1 + 2\sin\left(\frac{5\,\pi}{18}\right) \tag{37}$$

$$u = \left(3 + \left(3 + (3 + \ldots)^{-1/2}\right)^{-1/2}\right)^{-1/2} \tag{38}$$

Then

$$\pi = 6\,u + 6 \sum_{n=2}^{\infty} (-1)^n\, u^{n+1} \sum_{k=\text{Ceiling}(n/3)}^{\text{Floor}(n/2)} \frac{(-1)^k\, 2^{-2\,n+4\,k}}{(6\,k - 2\,n + 1)\binom{6\,k - 2\,n}{3\,k - n}} \binom{k}{n - 2\,k} \binom{2\,k}{k} \tag{39}$$

$$\pi = 6u - 2u^3 + 3u^4 + \frac{6}{5}u^5 - 3u^6 + \frac{39}{28}u^7 + \ldots \tag{40}$$

Entry 3. for $k = 1, 2, 3, \ldots$, we have

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\left\lceil\frac{n+1}{k}\right\rceil(-1)^n\left(\frac{1}{3}\right)^n\left(\frac{1}{2n+1} - \frac{(-1)^k 3^{-k}}{2n+2k+1}\right) \tag{41}$$

Entry 4.

$$\pi = 2\sum_{n=0}^{\infty}\left\lceil\frac{n+1}{2}\right\rceil\left(\frac{i}{\sqrt{3}}\right)^n\left(\frac{\sqrt{3}}{n+1} - \frac{i}{n+2}\right), \; i = \sqrt{-1} \tag{42}$$

Entry 5.

$$\pi = \frac{2}{3\sqrt{3}}\sum_{n=0}^{\infty}\left\lceil\frac{n+1}{2}\right\rceil 3^{-n}\left(\frac{9}{2n+1} - \frac{6}{2n+3} + \frac{1}{2n+5}\right) \tag{43}$$

Entry 6. for $i = \sqrt{-1}$, we have

$$\pi = 12\,i\sum_{n=1}^{\infty}\left\lfloor\frac{n}{3}\right\rfloor\left(\sqrt[6]{2-\sqrt{2}}\right)^n\left(\frac{\sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}}{2}\right)^n\left(\frac{1}{n} - \frac{\sqrt[6]{2-\sqrt{2}}\left(\sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}\right)}{2n+2}\right) \tag{44}$$

Entry 7. for $k = 1, 2, 3, \ldots$, we have

$$\frac{\sqrt[k]{3}\,\pi}{3\sqrt{3}\left(\sqrt[k]{3} - 1\right)} = \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor n/k\rfloor}}{2\lfloor n/k\rfloor + 1}\left(\frac{1}{3}\right)^{n/k} \tag{45}$$

V. References

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