Quantum Gravity from dimensional analysis of the Planck scale

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Abstract

This article is proposed to demonstrate in an informal and conceptual way that the equations of Quantum Mechanics are derived from the dimensional analysis of Planck units, this gives the possibility of extracting the equations of Quantum Gravity through the same process of reverse engineering, although it seems something simple this is extremely powerful for visualizing the possibility of a unification.

Introduction

Severe problems are reported about the unification of quantum mechanics with gravitation, among them is the emergence of singularities, making the General Theory of Relativity non-renormalizable, we will not deal here with how to solve this, instead we can analyze the situation in another way, without forcing gravitation to be a standard quantum theory. Many theories propose to solve this question, such as String Theory and Loop Quantum Gravity, the approach treated here is not as rigorous and formal as the previous ones, in advance, unprepared readers should already be warned that some concepts that will be presented here can seem quite wrong from a mathematical point of view, such as equating differential operators with scalar/vector quantities or treating them algebraically as fractions, but throughout the article we will discuss that these transformations are dimensionally valid.

Infinitesimal elements in mathematics

In mathematics small quantities called infinitesimals ("infinitely small") or infinitesimal changes are elementary quantity that serve to define the concept of non-standard calculus[1], then a change in the value of x is often denoted Δx, likewise a change in y is given as Δy. For example, consider the function $y = x^2$, given the change at points (x_0, y_0) so $(x_0 + \Delta x, y_0 + \Delta y)$, the average slope between those points is given as:

$$\text{Average slope} = \frac{\Delta y}{\Delta x}. \quad (1)$$

Developing the definition algebraically we have:

$$\begin{aligned}
&y_0 = (x_0)^2 \\
&y_0 + \Delta y = \left(x_0 + \Delta x\right)^2 \\
&\Delta y = \left(x_0 + \Delta x\right)^2 - x_0^2 \\
&\frac{\Delta y}{\Delta x} = \frac{\left(x_0 + \Delta x\right)^2 - x_0^2}{\Delta x} \\
&\frac{\Delta y}{\Delta x} = \frac{x_0^2 + 2x_0\Delta x + (\Delta x)^2 - x_0^2}{\Delta x} = 2x_0 + \Delta x
\end{aligned} \qquad (2)$$

Which is only valid when $\Delta x \neq 0$, otherwise the quotient $\Delta y/\Delta x$ is undefined, the term Δx is naturally disregarded because it is too small.

A rigorous and precise treatment was given by Abraham Robinson in 1960 through non-standard analysis[2], an infinitesimal number is a quantity that infinitely close to 0 but not equal to 0, for example, given a real function $f(x)$ that is defined for all x in the open interval $(0,1)$ and let x_0 be a number that belongs to that interval, then the derivative of $f(x)$ at x_0 is a.

$$f'(x) = \left(\frac{df}{dx}\right)_{x=x_0} = a \quad (3)$$

if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = a. \quad (4)$$

Then for ε all there is a positive number δ such that

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - a\right| < \varepsilon \quad (5)$$

for all x in $(0,1)$, then $0 < |x - x_0| < \delta$. So $dx = x - x_0$ are infinitely close to 0 but not equal to 0 and $df = f(x) - f(x_0)$ is infinitely close to a.
In Abraham's own words:

"G. W. Leibniz argued that the theory of infinitesimals implies the introduction of ideal numbers that can be infinitely small or infinitely large compared to the real numbers, but which should possess the same properties as the latter. "

If a number ε is said to be infinitely small then $-n < \varepsilon < n$, for all n positive and real. Infinitesimals do not exist in the standard real number system, but they do exist in hyperreal number system, The definitions given by K. D. Stroyan[3] of infinitesimals and hyperreal numbers are:

A number δ in an ordered field is called infinitesimal if it satisfies

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots > \frac{1}{m} > \dots > |\delta| \quad (6)$$

for any ordinary natural counting number m=1, 2, 3, ... We write $a \approx b$ *and say a is infinitely close to b if the number* $b - a \approx 0$ *is infinitesimal.*

The hyperreal numbers contain the real numbers, but also contain nonzero infinitesimal numbers, that is, numbers $\delta \approx 0$, *positive,* $\delta > 0$, *but smaller than all the real positive numbers.* The set of hyperreal numbers is denoted by R*, every real number is a member of R* including the real number 0. The symbols $\Delta x, \Delta t,...$ and the Greek letters ε and δ are often used to describe infinitesimals. We can have $\varepsilon, -\varepsilon$ and also $\frac{1}{\varepsilon}, -\frac{1}{\varepsilon}$ which is an infinite positive or negative number that is greater or less than any real number.

The transfer principle[4] ensures that the hyperreals get the declarations of the reals:

"Every real statement that holds for one or more particular real functions holds for the hyperrealnatural extensions of these functions."

This means that the axiom: *"for any number x, x + 0 = x"* still applies, the same goes for quantification; *"for any number x and y, xy=yx."*

One of the main advantages of using hyperreal numbers and the principle of transfer is that they allow for more intuitive and elegant proofs of certain mathematical theorems and results. They provide a natural framework for dealing with limits, derivatives, and integrals without the need for traditional epsilon-delta arguments. However, it is important to note that non-standard analysis and hyperreal numbers are not without their challenges and criticisms. Some mathematicians question the philosophical and foundational aspects of introducing infinitesimals and infinitely large numbers, and debates regarding the rigor of the theory have persisted.

The analogue of Infinitesimals in physical space: the Planck Scale

Zeno's paradox is an ancient philosophical puzzle that challenges our understanding of motion and continuity. It consists of a series of paradoxes, the most famous being Achilles and the Tortoise. In this paradox, Achilles, who is incredibly fast, races against a tortoise, which is much slower. Zeno argues that Achilles will never be able to overtake the tortoise because every time Achilles reaches the point where the tortoise was, the tortoise will have moved a bit further. This infinite series of movements should lead to an infinite number of steps for Achilles to complete, and thus he would never reach the tortoise. The geometric description of space on very small scales has been the subject of debate in philosophy and physics, we can see that part of the problems in the mathematics of modern physics is due to the appearance of singularities with terms that explode when a dependent variable becomes zero in the denominator, one way to try to get around this problem is to think of physical space as being discrete on very small scales, Planck units play a key role in this task, John L. Bell[5] quotes astrophysicist Martin Rees[6] in his book trying to compare the concept of infinitesimal and the Planck scale:

“In his book Just Six Numbers the astrophysicist Martin Rees comments on the microstructure of space and time, and the possibility of developing a theory of quantum gravity. In particular he says:

‘Some theorists are more willing to speculate than others. But even th boldest acknowledge the “Planck scales” as an ultimate barrier. We cannot measure distances smaller than the Planck length [about 10^{19} *times smaller than a proton]. We cannot distinguish two events (or even decide which came first) when the time interval between them is less than the Planck time (about* 10^{-43} *seconds).’*

On this account, Planck scales seem very similar in certain respects to Δ. In particular, the sentence above seems to be an exact embodiment of the idea that we cannot decide of two “events” in Δ which came first; in fact it makes the stronger assertion that actually neither comes “first”. Could Δ serve as a suitable model for “Planck scales”? While Δ is unquestionablysmall enough to play the role, it inhabits a domain in which everything is smooth and continuous, while Planck scales live in the quantum world which, if not outright discrete, is far from being universally continuous.”

The concept of a discrete spacetime at the Planck scale aligns with the development of non-standard analysis in mathematics. Non-standard analysis explores infinitesimals and infinitely small quantities in a framework that differs from traditional mathematical analysis. Just as the concept of discrete spacetime challenges the idea of continuous spacetime in physics, non-standard analysis challenges the conventional approach to calculus and mathematical continuity. The idea of a discrete space is completely contrary to Zeno's concept in which the notion of movement or change would be something totally absurd in a continuous space, the idea of going through infinite steps in a finite displacement would make movement something paradoxical.

Planck units

The Planck Length[7] is a fundamental concept in physics that arises from the combination of three constants: the speed of light (c), Newton's gravitational constant (G), and Planck's constant (ħ). These constants appear in general relativity and quantum field theory, and it is suspected that any theory reconciling these two areas will involve all three constants. Proposed in 1899, Max Planck suggested a system of fundamental natural units using the constants G, h and c, for quantities of length, mass, time and energy, Planck describes the universality of his units[8] as:

...it is possible to install units of length, mass, time and temperature, independent of bodies or special substances, necessarily retaining their meaning for all times and for all civilizations, including extraterrestrials and non-humans, which can be called "units natural measure".

Name	Dimension	Expression	Value(SI units)
Planck length	length (L)	$l_p = \sqrt{\frac{\hbar G}{c^3}}$	$1.616255(18) \times 10^{-35} m$
Planck mass	mass (M)	$m_p = \sqrt{\frac{\hbar c}{G}}$	$2.176434(24) \times 10^{-8} kg$
Planck time	time (T)	$t_p = \sqrt{\frac{\hbar G}{c^5}}$	$5.391247(60) \times 10^{-44} s$

Name	Dimension	Expression	Value(SI units)
Planck temperature	temperature (Θ)	$T_p = \sqrt{\frac{\hbar c^5}{G k_B^2}}$	$1.416784(16) \times 10^{32} K$

Table 01: Modern values of Planck units, current definitions differ by a factor of $\sqrt{2\pi}$ because modern definitions use $\hbar$ instead of h.

There are additional units referring to less frequently used electromagnetic quantities such as electrical charge:

$$q_p = \sqrt{4\pi\epsilon_0 \hbar c} \backsimeq 1.875546 \times 10^{-18} C \approx 11.7e \quad (7)$$

In SI units, the h, c and k_B values are exact and the ϵ_0 and G values have uncertainties of 1.5×10^{-10} and 2.2×10^{-5}, so the uncertainties in the Planck values come entirely from the uncertainty in the SI value of G.

Through the basic units mentioned above, the derived units can be obtained, which will be, like the others, widely used to establish the dimensional analysis of the equations studied in this article.

Derived unit of	Expression	Approximate SI equivalent
Area (L^2)	$l_p^2 = \frac{\hbar G}{c^3}$	$2.6121\times10^{-70} m^2$
Volume (L^3)	$l_p^3 = \left(\frac{\hbar G}{c^3}\right)^{\frac{3}{2}} = \sqrt{\frac{(\hbar G)^3}{c^9}}$	$4.2217\times10^{-105} m^3$
Momentum (LMT^{-1})	$m_p c = \frac{\hbar}{l_p} = \sqrt{\frac{\hbar c^3}{G}}$	$6.5249\ kg \cdot m/s$
Energy (L^2MT^{-2})	$E_p = m_p c^2 = \frac{\hbar}{t_p} = \sqrt{\frac{\hbar c^5}{G}}$	$1.9561\times10^{9}\ J$
Force (LMT^{-2})	$F_p = \frac{E_p}{l_p} = \frac{\hbar}{l_p \cdot t_p} = \frac{c^4}{G}$	$1.2103\times10^{44}\ N$
Density ($L^{-3}M$)	$\rho_p = \frac{m_p}{l_p^3} = \frac{\hbar \cdot t_p}{l_p^5} = \frac{c^5}{\hbar G^2}$	$5.1550\times10^{96}\ kg/m^3$

Derived unit of	Expression	Approximate SI equivalent
Acceleration (LT^{-2})	$a_p = \frac{c}{t_p} = \sqrt{\frac{c^7}{\hbar G}}$	$5.5608 \times 10^{51} m/s^2$

Table 02: Units derived from standard Planck units, there are several others but those cited so far will be most useful throughout this text.

These dimensions are so small for modern accelerators that a device the size of the Milky Way would be needed to measure them, one of the obstacles to proving modern theories of quantum gravity would be of an experimental nature.

Lee Smolin[9] in his book Three Roads to Quantum Gravity says:
"It is defined as the scale at which the effects of gravity and quantum phenomena will be equally important. For larger things, we can happily forget about quantum theory and relativity. But when we get down to the Planck scale we have no choice but to take it all into account."

Note that the mass is a huge amount (approximately the mass of a flea egg), this mass must be concentrated in a region of linear size l_p for the effects of quantum gravity to appear, an approximation of the Planck mass can be obtained equating the Comptom length with a Schwarzschild radius[10].

$$\frac{\hbar}{m_p c} \backsimeq R_S = \frac{2Gm_p}{c^2} \quad (8)$$

The ratio of the proton mass (m_{pr}) to the Planck mass makes it clear how gravity is a very weak force compared to the others, the 'fine structure constant of gravity' is defined as:

$$\alpha_g = \frac{Gm_{pr}^2}{\hbar c} = \left(\frac{m_{pr}}{m_p}\right)^2 \approx 5.91 \times 10^{-39} \quad (9)$$

This number is used to calculate astrophysical quantities like stellar masses and stellar lifetimes, for example, the Chandrasekhar mass is given by $M_C \approx a_g^{-3/2} m_{pr} \approx 1.8 M_\odot$ (A more precise value is $M_C \approx 1.44 M_\odot$), and stellar lifetimes are on the order of $a_g^{-3/2} t_p$. Another curious fact is that the geometric mean of the Planck length and the size of the observable universe is about 0.1mm, which is an everyday scale of measurement[11]. Several aspects consider that space is not infinitely divisible but that it is granular on the Planck scale, somewhat analogous to the discrete energy values of a quantum harmonic oscillator. Geometry at scale is thought of as fuzzy and ill-defined, John Wheeler in the 1950s called this quantum foam[12], where space-time fluctuations would be a type of "quantum background noise", the Planck length would be the minimum size used in theories of Quantum gravity, Andres Gomberoff and Donald Marolf give the following definition[13]:

"The Hilbert-Einstein action is $S_{EH} = (\hbar/16\pi L_p^2)\int d^4x|g|^{1/2}R$*. For a spacetime region with radius of curvature* L *and 4-volume* L^4 *the action is* $\backsim \hbar(L/Lp)^2$*. This suggests that quantum curvature fluctuations with radius less than the Planck length* $L \lesssim L_p$ *are unsuppressed.*

Another way to view the significance of the Planck length is as the minimum localization length Δx*, in the sense that if* $\Delta x < L_p$ *a black hole swallows the* Δx*. To see this, note that the uncertainty relation* $\Delta x \Delta p \geq \hbar/2$ *implies* $\Delta p \gtrsim \hbar/\Delta x$ *which implies* $\Delta E \gtrsim \hbar c/\Delta x$*. Associated with this uncertain energy is a Schwarzschild radius* $R_s(\Delta x) = 2G\Delta M/c^2 = 2G\Delta E/c^4$*, hence quantum mechanics and gravity imply* $R_s(\Delta x) \gtrsim L_p^2/\Delta x$*. The uncertain* $R_s(\Delta x)$ *is less than* Δx *only if* $\Delta x \gtrsim L_p$*."*

Using another type of definition, Daniel Coumbe[14] arrives at the same conclusion:

$$\langle 0 \,|\, ds^2 \,|\, 0 \rangle = \frac{l_p^2}{4\pi^2} \lim_{dx \to 0} \frac{1}{(x-y)^2} \eta_{\mu\nu} dx^\mu dx^\nu = \frac{l_p^2}{4\pi^2} \quad (10)$$

"Therefore, a non-zero minimum length $\sim l_p$ *arises due to the divergent twopoint function of the scalar fluctuation counteracting any attempt to shrink the spacetime distance to zero."*

Fundamental constants and approximations of QG

The constants c, G, and ħ play important roles in their respective theories. The speed of light (c) defines the geometry of spacetime in special relativity, and nothing is allowed to travel faster than light. Newton's gravitational constant (G) quantifies the effect of spacetime geometry on other fields in general relativity. Planck's constant (ħ) appears in the uncertainty principle of quantum theory and is linked to our inability to simultaneously know all properties of particles.

The Bekenstein-Hawking equation is a fundamental result in theoretical physics that establishes the relationship between the entropy (S) of a black hole and its surface area (A). This equation is a key milestone in the study of black hole thermodynamics, bridging the gap between general relativity and quantum mechanics. Discovered independently by Jacob Bekenstein and Stephen Hawking in the 1970s, the equation revolutionized our understanding of black holes. Prior to this breakthrough, black holes were considered objects devoid of any thermodynamic properties. However, Bekenstein and Hawking's work demonstrated that black holes possess entropy, a measure of the number of possible microstates associated with the system.

The Bekenstein-Hawking equation is given by[15]:

$$S_{BH} = \frac{k_B A}{4 l_p^2} = \frac{k_B c^3 A}{4 \hbar G} \quad (11)$$

Where S is the entropy of the black hole, k_B is the Boltzmann constant, A is its event horizon's surface area, and G is Newton's gravitational constant. The equation reveals a remarkable result: the entropy of a black hole is proportional to its surface area and is inversely related to the gravitational constant. As a consequence, larger black holes have more entropy and a greater number of microstates, while smaller black holes have lower entropy.

The Diósi-Penrose model[16][17] addresses the wave function collapse by incorporating the effects of gravity into quantum mechanics. According to their proposal, the process of wave function collapse is non-unitary and occurs when the superposition of different states reaches a certain critical mass, typically in macroscopic systems. This critical mass is believed to be associated with the gravitational self-energy of the superposed states.

In this model the evolution of a quantum system is described by a non-linear and stochastic equation that involves both the Schrödinger equation of quantum mechanics and the classical terms of gravity.

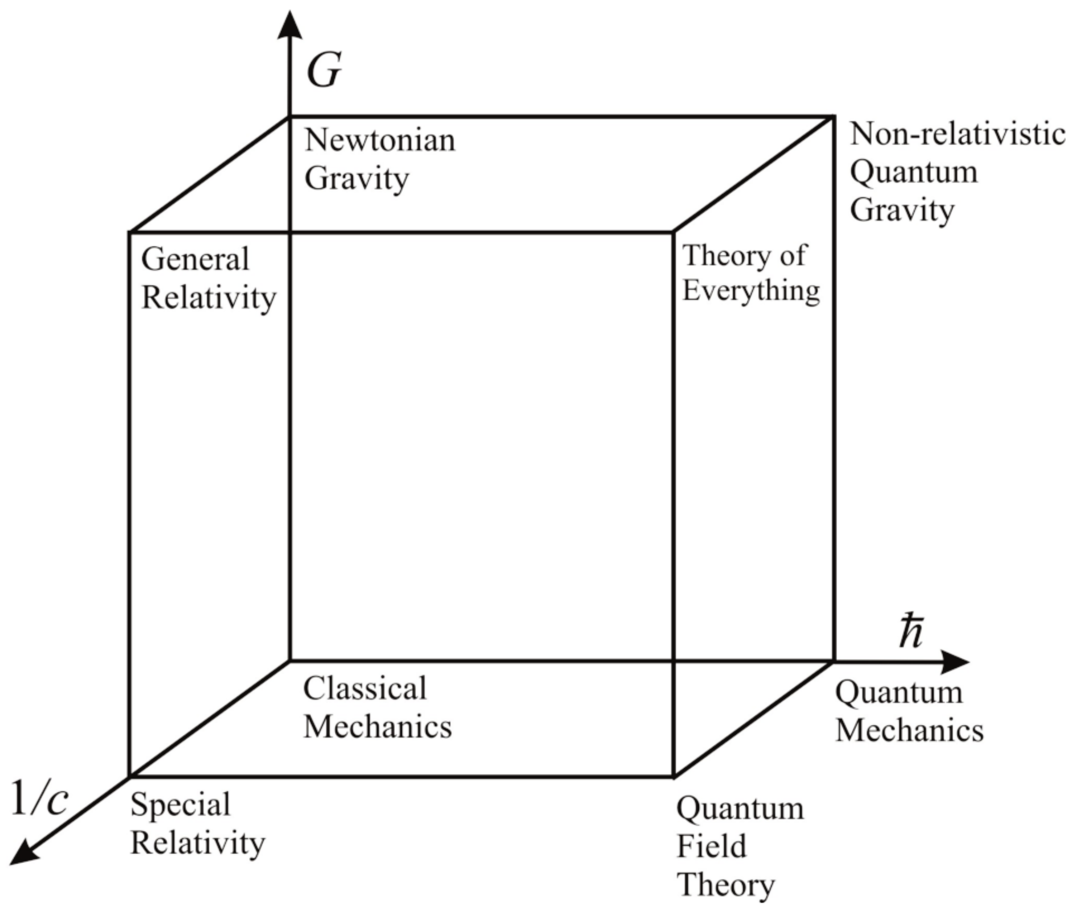
$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla\psi(x,t)-Gm^2\int\frac{|\psi(x',t)|^2}{|x-x'|}d^3x'\psi(x,t)+V\psi(x,t) \quad (12)$$

The Schrödinger-Newton equation, is a nonlinear modification of the Schrödinger equation with a Newtonian gravitational potential, where m is the mass of a quantum system, V is an external potential, G is Newton's gravitational constant, this equation describes the gravitational selfinteraction of a single quantum system, in which the mass density $m|\psi(x,t)|^2$ *is the source of the classical gravitational potential.*

This novel approach introduces a new type of dynamics that links the behavior of the quantum wave function to gravitational effects. Which suggests that gravity plays a fundamental role in the emergence of classical behavior from the underlying quantum realm. When a system becomes large enough for gravitational effects to become significant, the wave function of the system collapses in a way that resembles classical behavior. This mechanism provides a potential resolution to the measurement problem, as it naturally explains why we observe classical behavior in our macroscopic world, even though the underlying reality is governed by quantum mechanics. In his book Lectures on Gravitation, Feynman comments on the possibility of gravity-induced collapse[18]:

"I would like to suggest that it is possible that quantum mechanics

fails at large distances and for large objects. Now, mind you, I do not say that I think that quantum mechanics does fail at large distances, I only say that it is not inconsistent with what we do know. If this failure of quantum mechanics is connected with gravity, we might speculatively expect this to happen for masses such that $GM^2/\hbar c = 1$, of M near 10^{-15} grams, which corresponds to some 10^{18} particles."



The Bronstein cube of quantum gravity[19] is a theoretical construct represented in the $cG\hbar$ space, with axes labeled by Newton's gravitational constant G, the velocity of light c, and Planck's constant $\hbar$. Its dimensions are not fixed, and it encompasses various physical theories and models. Moving along the axes, one explores different theoretical frameworks. Along the G-axis, gravitational physics is included, leading to classical Newtonian gravity. Along the $1=c$ axis, relativistic effects due to the finite propagation speed of physical signals are considered, leading to General Relativity. Moving along the $\hbar$-direction, quantum mechanics is encountered, which becomes the modern framework for all physical sys-

tems when combined with relativistic features in the domain of quantum field theory.
However, modern physics cannot be confined to any single domain within the Bronstein cube, as it lacks a complete quantum theory of gravity. The ultimate goal is to unify both quantum and gravitational effects in a coherent description, leading to a “theory of everything” that describes quantum, relativistic, and gravitational phenomena. The complexities and subtleties involved in moving between different frameworks make this a challenging task. It’s important to note that the sketch provided is a simplified representation of theoretical physics, and it does not capture the complexity and intricacies of the actual phenomena described by these frameworks. Moreover, the dimensional quantities in the picture are not directly observable, and the nature of the systems considered significantly influences their description within each framework. Some entities may only exist in specific corners of the Bronstein cube.

Sciama’s inertia theory and the concept of gravitational potential

In Sciama’s theory[20], the inertia of a body arises from its interaction with the rest of the matter in the cosmos. According to this concept, the inertial properties of an object are not inherent to the object itself, as posited by Newton’s classical theory of mechanics, but rather influenced by the distribution and motion of all other matter in the universe (Mach’s principle). In other words, an object’s resistance to changes in motion is a result of its interaction with the gravitational fields generated by all the other masses in the cosmos. Therefore, the origin of inertia is tied to the overall mass distribution of the universe, implying a more holistic view of inertia compared to classical Newtonian mechanics.

He derived the following relationship in which the dependence of the constant G on distance becomes evident, and also the relationship of the gravitational potential with the square of the speed of light is evident.

$$G=\frac{c^2}{\sum_i \frac{m_i}{r_i}} \quad (13)$$

$$\phi = -G\sum_i \frac{m_i}{r_i} = -c^2 \quad (14)$$

The following excerpt is taken from Book *Mach's Principle and the Origin of Inertia* and also relates the square of the speed of light to Mach's principle of inertia[21]:

"Mach's principle underlines the essential umbilical link between local physics and the Universe as a whole. One significant implication of the truth of Mach's principle is that the Universe contains enormous amounts of matter, such that the physical effects of motion of a test body with respect to this Universal frame is equivalent to that of motion of the entire Universe with respect to the test body in the reverse sense. This reciprocity can be true quantitatively only if the evolving and expanding Universe contains a precise quantity of matter. In fact, only if it contains enough matter to satisfy the 'criticality' condition, $8\pi G\rho R^2/3c^2 \cong GM_U/c^2R_U \approx 1$, *$\rho$ is the average density of the Universe, and R_U is size scale of the presently observable Universe. M_U is the total mass-energy contained within a causal size of R_U."*

Berry[22] poses a Machian query regarding a body of mass m interacting with a larger mass M at a distance r, where the larger mass has an acceleration a relative to the smaller one. To satisfy Mach's principle, the force exerted on the small mass by the larger mass must include a part proportional to $m \cdot a$. Using dimensional analysis, it is determined that the correct force should be proportional to $m \cdot a$ and other terms such as M, r, G, and c, each raised to certain powers. According to Newton's third law, the powers of M and m must be equal, ensuring their symmetry in the force equation[23]:

$$F = -GM\frac{m}{c^2 r}a \quad (15)$$

The force described resembles the force that measures mutually accelerated charges. For the gravitational case, Sciama termed it the "law of inertial induction." The analogy indicates that if electromagnetic radiation is possible, then gravitational radiation should also exist. For the law to apply to the distant masses of the Universe in the Machian framework and for Newton's second law to hold, a specific Brans-Dicke relation needs to be valid:

$$\begin{aligned} \frac{GM}{c^2 r} &= 1 \\ F &= -GM\frac{m}{c^2 r}a = -ma \end{aligned} \tag{16}$$

In this scenario, every particle in the Universe experiences a combined force of inertial force and gravitational force, resulting in a zero-total force on each particle (inertial force + gravitational force = 0). To address the theoretical requirement of accounting for the inertia properties of matter, Sciama proposes that gravitation is analogous to electrodynamics. His working hypothesis suggests that inertial and gravitational forces cancel each other out at every point in space, resulting in a null total field.

Mathematical considerations on differential operators

By default, no apparent value is assigned to the differential operator $\frac{d}{dx}$ when it is applied to a function, the rules of calculus are generally followed to transform a function into its rate of change, informally, we can make clear the value attributed to $\frac{d}{dx}$ by defining a limit or using the differentiation/integration rules themselves, the differential operator acquires value when it acts on a function, we can associate this value obtained with another function, here are some simple examples of the power rule[24], for now we can ignore the constant of integration:

$$\begin{aligned} \frac{d}{dx}(x^n) &= n \cdot x^{n-1} \\ \frac{d}{dx}(x^n) &= \frac{n}{x} \cdot x^n \\ \int x^n \frac{dx}{d} &= x^n \cdot \frac{x}{n+1} \end{aligned} \tag{17}$$

It is evident that in this case the operator acquires the value n/x and in the case of integration $x/(n+1)$, here are some examples:

$$
\begin{aligned}
&\frac{d}{dx}x=1\\
&\frac{1}{x}x=1\\
&\frac{d}{dx}x^2=2x\\
&\frac{2}{x}x^2=2x\\
&\frac{d}{dx}x^3=3x^2\\
&\frac{3}{x}x^3=3x^2
\end{aligned}
\tag{18}
$$

We can also find the value assigned to the operator using the definition of limits:

$$\frac{d}{dx}=\left|\underset{h\to 0}{Lim}\frac{f(x+h)-f(x)}{h}\right|\frac{1}{f(x)} \tag{19}$$

$$
\begin{aligned}
&\frac{d}{dx}x=\underset{h\to 0}{Lim}\frac{(x+h)-x}{h}\\
&\frac{d}{dx}x=\underset{h\to 0}{Lim}\frac{h}{h}\\
&\frac{d}{dx}=\frac{1}{x}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx}x^2=\underset{h\to 0}{Lim}\frac{(x+h)^2-x^2}{h}\\
&\frac{d}{dx}x^2=\underset{h\to 0}{Lim}\frac{x^2+2xh+h^2-x^2}{h}\\
&\frac{d}{dx}x^2=\underset{h\to 0}{Lim}\frac{2xh+h^2}{h}\\
&\frac{d}{dx}=\underset{h\to 0}{Lim}\frac{(2x+h)}{x^2}\\
&\frac{d}{dx}=\frac{2}{x}
\end{aligned}
\tag{20}
$$

More complicated functions usually require the use of limits or the application of advanced rules like the chain rule, the differential operator d or ∂ acquires value depending on the function, for example:

$$
\begin{aligned}
&d(x^2)=xd(x)+xd(x)\\
&2(x^2)=x\cdot 1(x)+x\cdot 1(x)\\
&2x^2=x^2+x^2
\end{aligned}
\tag{21}
$$

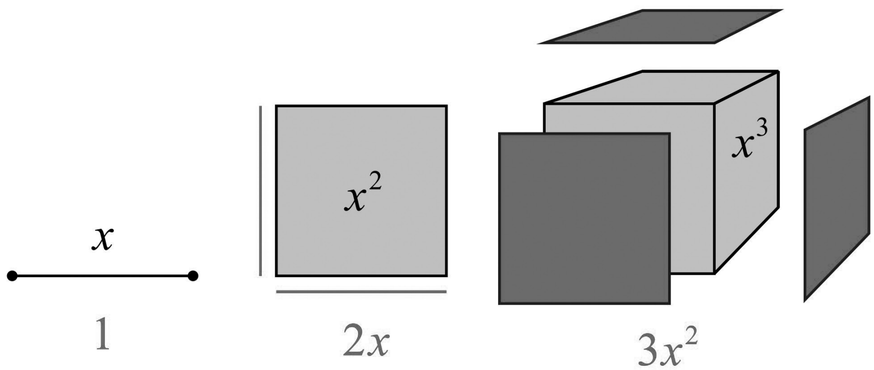
$$
\begin{aligned}
&\frac{d}{dx}e^{2x}=\frac{d}{d(2x)}e^{2x}\cdot\frac{d(2x)}{dx}\\
&\frac{d}{dx}e^{2x}=\underset{h\to 0}{Lim}\frac{e^{2x+h}-e^{2x}}{h}\cdot 2\frac{d(x)}{dx}\\
&\frac{d}{dx}e^{2x}=e^{2x}\cdot\underset{h\to 0}{Lim}\frac{e^{h}-1}{h}\cdot 2\\
&\frac{d}{dx}=2
\end{aligned}
\tag{22}
$$

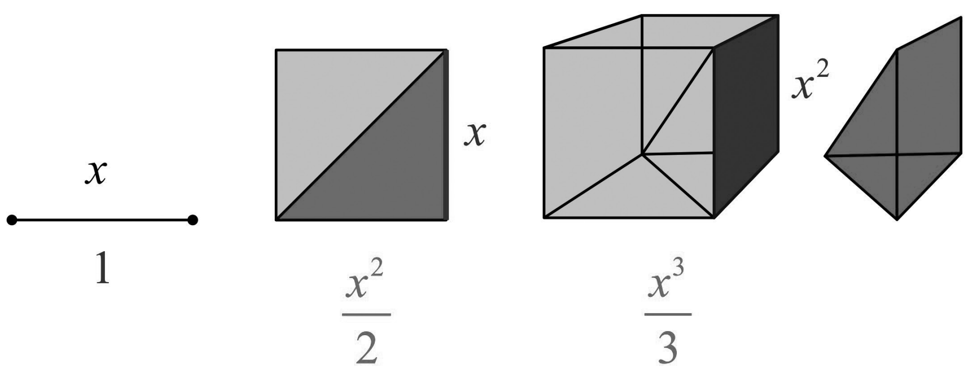
Because it is more laborious, we will not use the definition of integrals through limits here, we can use the same resources to find $\frac{dx}{d}$ through the inverse transformation of the derivative, here are some simple examples:

$$
\begin{aligned}
&\frac{d}{dx}x=1\\
&x+C=1\frac{dx}{d}\\
&\frac{d}{dx}x^2=2x\\
&x^2=2x\frac{dx}{d}\\
&\frac{x^2}{2}+C=x\frac{dx}{d}\\
&dy=2x^2\\
&dx=x\\
&\frac{d}{dx}x^4=4x^3\\
&x^4=4x^3\frac{dx}{d}\\
&\frac{x^4}{4}+C=x^3\frac{dx}{d}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}\\
&2\sqrt{x}+C=\frac{1}{\sqrt{x}}\frac{dx}{d}\\
&\frac{d}{dx}x\sqrt{x}=\frac{3\sqrt{x}}{2}\\
&\frac{2}{3}x\sqrt{x}+C=\sqrt{x}\frac{dx}{d}\\
&\frac{d}{dx}x(\ln x-1)=\ln x\\
&x(\ln x-1)+C=\ln x\frac{dx}{d}\\
&\frac{d}{dx}2e^{\frac{x}{2}}=e^{\frac{x}{2}}\\
&2e^{\frac{x}{2}}+C=e^{\frac{x}{2}}\frac{dx}{d}
\end{aligned}
\tag{24}
$$

We can show the geometric meaning of $\frac{d}{dx}$ acting on a function of the type $C \cdot x^n$ as being the multiplicative inverse of the length of the sides of a hypercube, where n is the number of sides referring to each dimension of the function, and also the integration as being the volume of the hyperpyramids that form the hypercube.





Eigenvalues and eigenvectors[25][26] are crucial concepts in linear algebra. When we have a square matrix A, an eigenvector v and its corresponding eigenvalue λ satisfy the following equation:

$$A \cdot v = \lambda \cdot v \quad (25)$$

In this context, λ is the eigenvalue, and v is the eigenvector. The eigenvector represents a direction that remains unchanged (up to scaling) under the linear transformation represented by the matrix A, and the eigenvalue scales the eigenvector. Linear transformations can have diverse representations, mapping vectors across various vector spaces, leading to a variety of eigenvectors. For instance, if the linear transformation takes the form of a differential operator like $\frac{d}{dx}$, the corresponding eigenvectors are called eigenfunctions, which are scaled by that specific differential operator, such as:

$$\frac{d}{dx}e^{\lambda x}=\lambda e^{\lambda x} \quad (26)$$

The concepts of eigenvalue and eigenvectors for a linear transformation T hold true even in infinite-dimensional Hilbert or Banach spaces. A common type of linear transformations on infinite-dimensional spaces are the differential operators acting on function spaces. Consider D as a linear differential operator on the space C∞ of infinitely differentiable real functions with a real argument t. The eigenvalue equation for D is a differential equation.

$$Df(t)=\lambda f(t) \quad (27)$$

Functions that satisfy the given differential equation are eigenvectors of D and are referred to as eigenfunctions.

Rewriting Special Relativity

In mathematics very small changes can be associated with limits tending to zero $\underset{\Delta x\to 0}{Lim}\Delta x=\underset{x_0\to x}{Lim}(x_0-x)=dx$ or infinitesimals ε, but when it comes to physical quantities this generates problems such as the appearance of singularities in the equations, in order to remove this we can associate these infinitesimal changes with Planck units to maintain the dimensional analysis of the equations and remove non-physical and undefined results such as $\frac{1}{0},\frac{1}{\infty},\frac{0}{0},\frac{\infty}{\infty},\infty$.

One of the pillars of Special Relativity is the invariance of the speed of light, regardless of the referential it remains the same, Einstein in his original article[27] says:

"The following considerations are based on the principle of relativity and the principle of the constancy of the velocity of light. We define these two principles as follows:

1. If two coordinate systems are in uniform parallel translational motion relative to each other, the laws according to which the states of a physical system change do not depend on which of the two systems these changes are related to.

2. Every light ray moves in the "rest" coordinate system with a fixed velocity V, independently of whether this ray of light is emitted by a body at rest or in motion. Hence,

$$velocity = \frac{light\ path}{time\ interval}$$

where "time interval" should be understood in the sense of the definition given in section 1."

The existence of an upper limit on the speed of light is a direct consequence of a lower limit on the length of space and time, the original theory only takes into account the constancy of the speed of light, so we can associate the Planck length with the path taken by light and the Planck time with the time interval, therefore:

$$\begin{aligned}
c &= \frac{\sqrt{\frac{\hbar G}{c^3}}}{\sqrt{\frac{\hbar G}{c^5}}} \\
c &= \sqrt{\frac{\hbar G}{c^3}}\sqrt{\frac{c^5}{\hbar G}} \\
c &= \sqrt{\frac{c^5}{c^3}} \qquad\qquad (28) \\
c &= \sqrt{c^2} \\
c &= c
\end{aligned}$$

Using this principle we can derive the other transformations so that the dimensional analysis remains consistent and avoid zero and infinity appearing in the results. The distance between two events in a flat four-dimensional spacetime (Minkowski space)[28][29][30][31] is defined as:

$$(x^4-x_0^4)^2-(x^1-x_0^1)^2-(x^2-x_0^2)^2-(x^3-x_0^3)^2=0 \quad (29)$$

Substituting 0 for l_p^2, we then have:

$$\begin{aligned}
&(x^4-x_0^4)^2-(x^1-x_0^1)^2-(x^2-x_0^2)^2-(x^3-x_0^3)^2+\frac{\hbar G}{c^3}=\frac{\hbar G}{c^3}\\
&c^2(t_1-t_2)^2-(x_1-x_2)^2-(y_1-y_2)^2-(z_1-z_2)^2+\frac{\hbar G}{c^3}=\frac{\hbar G}{c^3}\\
&c^2(\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2+\frac{\hbar G}{c^3}=\frac{\hbar G}{c^3}\\
&c^2dt^2-dx^2-dy^2-dz^2+\frac{\hbar G}{c^3}=ds^2\\
&0+\frac{\hbar G}{c^3}=ds^2\\
&\sqrt{\frac{\hbar G}{c^3}}=ds
\end{aligned} \quad (30)$$

We can associate this $ds=cdt$, so:

$$\begin{aligned}
&ds^2=\frac{\hbar G}{c^3}\\
&c^2=\frac{\hbar G}{c}\frac{d^2}{ds^2}\\
&c^2=\frac{\hbar G}{c^3}\frac{d^2}{dt^2}\\
&c^2=\frac{ds^2}{d^2}\frac{d^2}{dt^2}
\end{aligned} \quad (31)$$

or equivalently:

$$\begin{aligned}
&ds=\sqrt{\frac{\hbar G}{c^3}}\\
&ds^2=\frac{\hbar G}{c^3}\\
&\frac{ds^2}{dt^2}=\frac{\hbar G}{c^3}\frac{c^5}{\hbar G}\\
&\frac{ds^2}{dt^2}=c^2
\end{aligned} \quad (32)$$

The space-time interval is one of the fundamental expressions of this article, through it we can extract all the elements of relativity, such as the Lorentz Factor that is widely used in relativistic transformations[32][33][34], disregarding the Planck length at the moment, we can obtain it as follows:

$$\begin{aligned}
&ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2\\
&\frac{ds^2}{c^2dt^2} = \frac{c^2dt^2}{c^2dt^2} - \frac{dx^2}{c^2dt^2} - \frac{dy^2}{c^2dt^2} - \frac{dz^2}{c^2dt^2}\\
&\frac{ds^2}{c^2dt^2} = 1 - \frac{1}{c^2}\left(\frac{dx^2}{dt^2} - \frac{dy^2}{dt^2} - \frac{dz^2}{dt^2}\right)\\
&\frac{ds^2}{c^2dt^2} = 1 - \frac{v^2}{c^2}\\
&\frac{c^2dt^2}{ds^2} = \frac{1}{1 - \frac{v^2}{c^2}}\\
&\gamma = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}
\end{aligned} \tag{33}$$

Lorentz transformations are a set of equations in special relativity that describe how coordinates and time measurements change when observed from different inertial reference frames. These transformations play a crucial role in connecting the physics of observers moving at constant velocities relative to one another.
Consider two inertial reference frames, S and S', where S' is moving with a constant velocity v relative to S along the x-axis. Let (x, y, z, t) be the coordinates of an event in frame S, and (x', y', z', t') be the coordinates of the same event in frame S'.

The Lorentz transformations for a boost along the x-axis are as follows:

1. Length Contraction:
The length L' of an object measured in frame S' is related to its length L measured in frame S by:

$$
\begin{aligned}
L' &= \frac{L}{\gamma} \\
L' &= L\sqrt{1-\frac{v^2}{c^2}}
\end{aligned}
\quad (34)
$$

where γ is the Lorentz factor given by (33), and c is the speed of light in vacuum.

2. Time Dilation:
The time interval Δt' between two events measured in frame S' is related to the time interval Δt measured in frame S by:

$$
\begin{aligned}
\Delta t' &= \gamma \Delta t \\
\Delta t' &= \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}
\end{aligned}
\quad (35)
$$

3. Relativity of Simultaneity:
Two events that are simultaneous in frame S may not be simultaneous in frame S'.

4. Lorentz Transformations (Position and Time):
The coordinates (x', y', z', t') in frame S' are related to the coordinates (x, y, z, t) in frame S by:

$$
\begin{aligned}
x' &= \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}
\end{aligned}
\quad (36)
$$

5. Inverse Lorentz Transformations:
The coordinates (x, y, z, t) in frame S are related to the coordinates (x', y', z', t') in frame S' by:

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$y = y' \qquad (37)$$
$$z = z'$$
$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Infinitesimals are quantities that are infinitely small but not zero. In the context of Lorentz transformations, infinitesimals are used to describe infinitesimal changes in spacetime coordinates. We can also adjust the Lorentz transformations so that they acquire an "infinitesimal" value, so we have the modified form:

$$\begin{aligned} x' &= \frac{x\left(1 - \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}\right)}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^3}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t\left(1 - \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}\right)}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^3}}} \end{aligned} \qquad (38)$$

$$\begin{aligned} x &= \frac{x'\left(1 + \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}\right)}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^3}}} \\ y &= y' \\ z &= z' \\ t &= \frac{t'\left(1 + \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}\right)}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^3}}} \end{aligned} \qquad (39)$$

Eliminating l_p^2 or considering it as the usual zero we return to the traditional form of the equations, for example:

$$\begin{aligned} x' &= \frac{x\left(1 - \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}\right)}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^3}}} \\ x' &= \frac{x\left(1 - \sqrt{\frac{v^2}{c^2} - 0}\right)}{\sqrt{1 - \frac{v^2}{c^2} + 0}} \end{aligned} \qquad (40)$$

$$\begin{aligned} t' &= \frac{t\left(1 - \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}\right)}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^3}}} \\ t' &= \frac{t\left(1 - \sqrt{\frac{v^2}{c^2} - 0}\right)}{\sqrt{1 - \frac{v^2}{c^2} + 0}} \end{aligned} \qquad (41)$$

$$x' = \frac{x\left(1-\sqrt{\frac{v^2}{c^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t' = \frac{t\left(1-\sqrt{\frac{v^2}{c^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x' = \frac{x-x\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t' = \frac{t-t\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t' = \frac{t-\frac{x}{c}\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Note that the most characteristic change is the replacement of $\beta = v/c$ by $\beta = \sqrt{v^2/c^2 - l_p^2}$, expressing in matrix form:

$$\beta = \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}} \quad (42) \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}}} \quad (43)$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\sqrt{\frac{v_x^2}{c^2}-\frac{\hbar G}{c^3}} & 0 & 0 \\ -\gamma\sqrt{\frac{v_x^2}{c^2}-\frac{\hbar G}{c^3}} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (44)$$

For simplicity we accept the fact that there are positions in the matrix that are zero for v^i=0, using the non-standard definition these positions have a very small value, follows the form for boosts in arbitrary directions, some factors have been abbreviated to keep it simple:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\sqrt{\frac{v_x^2}{c^2}-\frac{\hbar G}{c^3}} & -\gamma\sqrt{\frac{v_y^2}{c^2}-\frac{\hbar G}{c^3}} & -\gamma\sqrt{\frac{v_z^2}{c^2}-\frac{\hbar G}{c^3}} \\ -\gamma\sqrt{\frac{v_x^2}{c^2}-\frac{\hbar G}{c^3}} & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} \\ -\gamma\sqrt{\frac{v_y^2}{c^2}-\frac{\hbar G}{c^3}} & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\sqrt{\frac{v_z^2}{c^2}-\frac{\hbar G}{c^3}} & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (45)$$

We can also express this with hyperbolic functions[35][36], so we have:

$$\cosh^2\psi - \sinh^2\psi = 1$$
$$\sinh\psi = \frac{\tanh\psi}{\sqrt{1-\tanh^2\psi}} \quad (46)$$
$$\cosh\psi = \frac{1}{\sqrt{1-\tanh^2\psi}}$$

$$\tanh\psi = \sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}} \quad (47) \qquad \sinh\psi = \frac{\sqrt{\frac{v^2}{c^2} - \frac{\hbar G}{c^3}}}{\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}}} \quad (48)$$

$$\cosh\psi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}}} \quad (49)$$

The general solution of: $(ct)^2 - x^2 = (ct')^2 - x'^2$ is:

$$x = x'\cosh\psi + ct'\sinh\psi \quad (50) \qquad ct = x'\sinh\psi + ct'\cosh\psi \quad (51)$$

Length contraction can be derived from time dilation, which states that the rate of a single “moving” clock (indicating its proper time T_0 is slower compared to two synchronized “resting” clocks (indicating T). Time dilation has been experimentally confirmed multiple times and is represented by the following relationship:

$$T = T_0 \cdot \gamma \quad (52)$$

The text discusses a scenario where a rod of proper length L_0 is at rest in frame S, and a clock in frame S' is moving towards the rod with a relative velocity v. Using the principle of relativity, the relative velocity is the same in any reference frame. The respective travel times for the clock between the endpoints of the rod are given by $T = L_0/v$ in frame S and $T_0' = L'/v$ in frame S'. It follows that $L_0 = Tv$ and $L' = T'_0 v$. By incorporating the formula for time dilation, the ratio between these lengths is obtained.

$$\frac{L'}{L_0}=\frac{T'_0 v}{Tv}=\frac{1}{\gamma} \quad (53)$$

Therefore, the length measured in S' is given by :

$$L'=L_0\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}} \quad (54)$$

where L_0 represents the proper length of the rod (length at rest), and v is the relative velocity between reference frames S and S'. We then make the following equivalence:

$$\begin{aligned} T'_0 &= T\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}} \\ \frac{L'}{v} &= \frac{L_0}{v}\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}} \qquad (55) \\ \frac{L'}{v} &= L_0\sqrt{\frac{1}{v^2}-\frac{1}{c^2}+\frac{1}{v^2}\frac{\hbar G}{c^3}} \end{aligned}$$

And finally we can define the modified versions of length contraction and time dilation:

$$L=L_0\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}} \quad (56) \qquad T=\frac{T_0}{\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar G}{c^3}}} \quad (57)$$

Relativistic Dynamics

In his seminal paper *"Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?"* (Does the Inertia of a Body Depend Upon Its Energy Content?)[37], published in 1905, Einstein discussed the equivalence of mass and energy, which later led to the famous equation $E=mc^2$. However, as Einstein delved deeper into the theory of relativity, his perspective on relativistic mass evolved.

He introduced the concept of "rest mass" and emphasized that this mass remains constant regardless of the object's velocity. He wrote:

"If a body gives off the energy L in the form of radiation, its mass diminishes by L/c^2*."*

In 1908, chemist G.N. Lewis was probably the first to express relativistic mass in the form we know it today, before Planck and Einstein. Lewis and R.C. Tolman worked independently between 1908 and 1912 to derive equations that described the variation of mass with velocity, based on dynamics[38][39][40]. From the 1920s onwards, relativistic mass became common in texts on relativity written by Pauli, Eddington and Born. However, although rest mass is widely used in various areas of physics, relativistic mass is mainly applied to dynamics in special relativity. For this reason, the rest mass of a body is often simply called its "mass."

In 1948, Einstein expressed his thoughts on relativistic mass in a letter to Lincoln Barnett—an American journalist[41][42]:

"It is not good to introduce the concept of the mass $M = m/\sqrt{1 - v^2/c^2}$ *of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the 'rest mass'* m*. Instead of introducing* M *it is better to mention the expression for the momentum and energy of a body in motion."*

Einstein, too, moved away from the notion of relativistic mass when he wrote a letter addressed to Dale B. Swanson[43], the letter says:

"Dear Sir, One should speak of mass only in the sense of something characteristic for the body and independent of its motion [rest mass]. Now it is true that the energy of a finite mass becomes infinite [in motion], provided the mass is finite. In the case of the photon the mass has to be assumed zero or infinitely small in such a way that in spite of its having the light velocity its energy is finite."

This statement highlights the connection between energy and mass without explicitly referring to relativistic mass. Overall, Einstein and other prominent physicists expressed varying opinions on the concept of relativistic mass. While the term was initially used and explored, Einstein's later writings and the general consensus among physicists favored the notion of rest mass and its connection to energy, as exemplified by the famous equation $E=mc^2$. Einstein's established a fundamental link between energy and mass. This equation signifies that energy and mass are interchangeable, and that energy can be generated from matter and vice versa. In his words,

"It followed from the special theory of relativity that mass and energy are both but different manifestations of the same thing - a somewhat unfamiliar conception for the average mind." (Albert Einstein, "Does the Inertia of a Body Depend Upon Its Energy Content?" 1905)

As a result, the concept of relativistic mass is now considered less favored in modern physics, with the focus being on the concept of energy and rest mass as fundamental quantities. Many modern authors, like Taylor and Wheeler, choose not to use the concept of relativistic mass[44]:

"The concept of "relativistic mass" is subject to misunderstanding. That's why we don't use it. First, it applies the name mass – belonging to the magnitude of a 4-vector – to a very different concept, the time component of a 4-vector. Second, it makes increase of energy of an object with velocity or momentum appear to be connected with some change in internal structure of the object. In reality, the increase of energy with velocity originates not in the object but in the geometric properties of spacetime itself."

Relativistic mass is the total energy in a body or system divided by the speed of light squared c^2:

$$\begin{aligned} E &= m_{rel}c^2 \\ m_{rel} &= \frac{E}{c^2}. \end{aligned} \quad (58)$$

As an object with rest mass approaches the speed of light (c) in the theory of special relativity, its relativistic mass (m_{rel}) increases significantly. This effect is described by the following equation:

$$m_{rel}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad (59)$$

where:

- m_{rel} is the relativistic mass of the object.
- m_0 is the rest mass of the object (mass at rest).

As the object's velocity (v) approaches the speed of light (c), the denominator of the equation approaches zero. When the velocity of the object reaches c, the denominator becomes zero, making the relativistic mass infinite:

$$m_{rel}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-1}}=\frac{m_0}{0}=\infty \quad (60)$$

This result implies that an object with rest mass would require an infinite amount of energy to accelerate it to the speed of light. According to the theory of special relativity, no massive object can ever reach or exceed the speed of light, as doing so would require an unattainable amount of energy. This does not seem to be satisfactory from a mathematical point of view, and it gets worse when it comes to a physical quantity, a dimensional quantity with dimensionless results. We can avoid this by adding the Planck mass squared to the Lorentz factor so that the quantity remains dimensionally correct, so we have:

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar c}{G}}} \quad (61)$$

So taking to the limit that $v \to c$, it results:

$$m = \frac{m_0}{\sqrt{0 + \frac{\hbar c}{G}}} = m_0 \sqrt{\frac{G}{\hbar c}} = \frac{m_0}{m_p} \quad (62)$$

We got rid of infinity and kept the result dimensionally correct, this modification has an undetectable impact in current experiments, so even at relativistic speeds the measurements remain practically the same as in standard theory because the amount added is very small.
The 4-momentum[45][46] is defined as $p = (p^0, p^1, p^2, p^3) = \left(\frac{E}{c}, p_x, p_y, p_z\right)$, so we have:

$$\begin{aligned} p^u p_u &= - m_0^2 c^2 \\ -\frac{E^2}{c^2} + |p|^2 &= - m_0^2 c^2 \end{aligned} \quad (63)$$

In terms of relativistic mass m and invariant mass m_0 we do:

$$\begin{aligned} -\frac{E^2}{c^2} + |p|^2 - m^2 \frac{\hbar c^3}{G} &= - m_0^2 c^2 \\ m_0^2 c^2 &= \frac{E^2}{c^2} - |p|^2 + m^2 \frac{\hbar c^3}{G} \end{aligned} \quad (64)$$

We then conclude that the modified momentum becomes:

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}}} \quad (65)$$

In a similar way we can derive one of the main equations of special relativity, the energy-momentum relationship, just as the momentum of the Planck mass remains in the Lorentz factor, hence:

$$\begin{aligned} E^2 &= p^2 c^2 + m_0^2 c^4 - m^2 c^4 \frac{\hbar c}{G} \\ E &= \sqrt{p^2 c^2 + m_0^2 c^4 - m^2 \frac{\hbar c^5}{G}} \end{aligned} \quad (66)$$

From this we arrive at the form:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}}} \quad (67)$$

In the realm of Special Relativity, the classical concept of kinetic energy undergoes a profound transformation. As particles approach relativistic speeds, the traditional equation for kinetic energy, $\frac{1}{2}mv^2$, becomes inadequate. Instead, the relativistic formulation of kinetic energy considers the interplay between mass, velocity, and the speed of light, fundamentally altering our understanding of motion and energy.
According to Albert Einstein, in "Relativity: The Special and General Theory"[47]: *"A body moving with the velocity v, which absorbs an amount of energy E_0 in the form of radiation without suffering an alteration in velocity in the process, has, as a consequence, its energy increased by an amount*

$$\frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}}.$$

In consideration of the expression given above for the kinetic energy of the body, the required energy of the body comes out to be

$$\frac{\left(m+\frac{E_0}{c^2}\right)c^2}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Thus the body has the same energy as a body of mass $\left(m+\frac{E_0}{c^2}\right)$ moving with the velocity v. Hence we can say: If a body takes up an amount of energy E_0, then its inertial mass increases by an amount E_0/c^2; the inertial mass of a body is not a constant, but varies according to the change in the energy of the body."

Renowned physicist Richard P. Feynman, in "Six Easy Pieces"[48], elaborates: *"We start with the body at rest, when its energy is m_0c^2. Then we apply a force to the body, which starts it moving and gives it kinetic energy; therefore, since the energy has increased, the mass has increased—this is implicit in the original assumption. So long as the force continues, the* *energy and the mass both continue to increase."*

We have that the kinetic energy is the total relativistic energy minus the remaining energy:

$$E_k = E - m_0 c^2 = \sqrt{p^2 c^2 + m_0^2 c^4 - m^2 \frac{\hbar c^5}{G}} - m_0 c^2 \quad (68)$$

equivalently:

$$\begin{aligned} E_k &= m_0 \gamma c^2 - E_0 = m_0 \gamma c^2 - m_0 c^2 \\ E_k &= (\gamma - 1) m_0 c^2 \end{aligned} \quad (69)$$

The modified equation then becomes:

$$E_k = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}}} - m_0 c^2 \quad (70)$$

Something to note is that for very low speeds we can consider:

$$m_0 \cong \frac{m_0}{\sqrt{1 + \frac{\hbar c}{G}}} \quad (71)$$

since the Planck term in γ is very small, and when v tends to c the γ factor does not go to infinity but results in:

$$m_0 \sqrt{\frac{G}{\hbar c}} = \frac{m_0}{m_p} \quad (72)$$

which would be the mass maximum that the particle would obtain at that speed. So we arrive at the following approximation:

$$\begin{aligned} T &= mc^2 - m_0 c^2 & -\frac{1}{2} mc^2 &= -m_0 c^2 \\ \frac{1}{2} mc^2 &= mc^2 - m_0 c^2 & -m &= -2m_0 \\ \frac{1}{2} mc^2 - mc^2 &= -m_0 c^2 & m &= 2m_0 \approx \frac{2m_0}{\sqrt{1 + \frac{\hbar c}{G}}} \end{aligned} \quad (73)$$

Tachyons in Relativistic Physics

Tachyons are hypothetical particles that have captured the imagination of physicists for their intriguing property of superluminal motion. They were first introduced by Gerald Feinberg[49] in 1967 as a theoretical construct within the framework of special relativity. Tachyons are characterized by having an imaginary rest mass, m_0, such that $m^2 < 0$. This property implies that the speed of a tachyon, denoted as v_t, exceeds the speed of light in vacuum c.

For tachyons, where $v_t > c$, the Lorentz factor becomes imaginary:

$$\gamma_t = \frac{1}{\sqrt{1 - \frac{v_t^2}{c^2}}} = i\sqrt{\frac{v_t^2}{c^2} - 1} \quad (74)$$

The imaginary nature of γ_t has profound implications. Tachyons do not follow the usual relativistic kinematics, such as time dilation and length contraction, which are governed by real Lorentz factors. Instead, tachyons exhibit a reverse behavior known as "Lorentz contraction" and "time advance," which means they appear to gain energy as they accelerate and lose energy as they decelerate. This apparent violation of causality has led to significant theoretical challenges.

One of the most critical issues associated with tachyons is their stability and causality. While they are solutions to certain field equations, it is unclear whether they can exist in a physical sense. Various quantum field theories have explored the concept of tachyons, including string theory and some approaches to dark matter.

The relativistic energy–momentum relation of a bradyon (massive particle inside the light cone), can never exceed c:

$$E^2 = p^2c^2 + m_0^2c^4 \quad (75)$$

For a tachyon the speed is always greater than c, its energy-momentum equation is:

$$E^2 = p^2c^2 - m_0^2c^4 \quad (76)$$

Luxon or a massless particle, traveling at the speed of light, the energy-momentum equation is:

$$E^2 = p^2c^2 = \left(\frac{h}{\lambda}\right)^2 c^2 \qquad E = \frac{hc}{\lambda} = hf = hv \quad (77)$$

Deriving quantum mechanics from Planck units

The Heisenberg Uncertainty Principle[50] is a fundamental concept in quantum mechanics, introduced by German physicist Werner Heisenberg in 1927. This principle asserts that it is impossible to simultaneously know both the position and momentum of a quantum particle with arbitrary precision. The more precisely we determine the position of a particle, the less accurately we can know its momentum, and vice versa. Heisenberg's Uncertainty Principle is mathematically expressed as follows: $\Delta x \cdot \Delta p \geq \hbar/2$,
where Δx represents the uncertainty in position, Δp represents the uncertainty in momentum, and $\hbar$ (h-bar) is the reduced Planck constant, approximately equal to 1.054571×10^{-34} Joule-seconds.

$$\hbar = \frac{h}{2\pi} \quad (78)$$

In other words, the product of the uncertainties in position and momentum must always be greater than or equal to half of the reduced Planck constant. This principle implies that it is fundamentally impossible to simultaneously measure both the position and momentum of a particle with arbitrary precision. Converting the infinitesimal quantities into Planck units we can obtain the same result, considering:

$$m_{p^*} = \frac{i}{2}\sqrt{\frac{\hbar c}{G}} \qquad (79)$$
$$m_{p^*} = i\sqrt{\frac{hc}{8\pi G}}$$

(later we will justify this definition of mass, we can disregard i for the moment) we then have:

$$\begin{aligned}
&\Delta x \cdot \Delta p\\
&\Delta x \cdot m_p c\\
&\sqrt{\frac{\hbar G}{c^3}}\frac{1}{2}\sqrt{\frac{\hbar c}{G}}c\\
&\sqrt{\frac{\hbar G}{c^3}}\frac{1}{2}\sqrt{\frac{\hbar c^3}{G}} \qquad (80)\\
&\frac{1}{2}\sqrt{\hbar^2}\\
&\frac{\hbar}{2}
\end{aligned}$$

We have the energy-time uncertainty relation, and likewise we have:

$$\begin{aligned}
&\Delta E \cdot \Delta t\\
&m_p c^2 \cdot \Delta t\\
&\frac{1}{2}\sqrt{\frac{\hbar c}{G}}c^2\sqrt{\frac{\hbar G}{c^5}}\\
&\frac{1}{2}\sqrt{\frac{\hbar c^5}{G}}\sqrt{\frac{\hbar G}{c^5}} \qquad (81)\\
&\frac{1}{2}\sqrt{\hbar^2}\\
&\frac{\hbar}{2}
\end{aligned}$$

In the early 20th century, the quest to understand the dual nature of matter and light marked a turning point in the field of physics. Louis de Broglie's groundbreaking proposal, which suggested that matter particles possess wave-like properties[51][52], laid the foundation for quantum mechanics. In this paper, we shall delve into the mathematical formalism

behind de Broglie relations and their interplay with quantum operators.
The de Broglie relations express the wavelength (λ) associated with a particle's momentum (p) and its energy (E) in a quantum system. For a particle with mass (m) and speed (v), the relations are given by:

$$\begin{aligned} \lambda &= \frac{h}{mv} \\ \lambda &= \frac{h}{p} \\ p &= \frac{h}{\lambda} \\ p &= \hbar \boldsymbol{k} \end{aligned} \quad (82)$$

These relations assert that any particle, regardless of its mass, exhibits wave-like properties. Moreover, the wavelength becomes significant in the quantum realm, particularly when dealing with subatomic particles.
In the early 1900s, the field of physics underwent a revolutionary transformation with the advent of quantum mechanics. At the heart of this paradigm shift lies the Planck-Einstein relation[53][54][55], which relates the energy of a photon to its frequency.
The Planck-Einstein relation is expressed as follows:

$$\begin{aligned} E &= h\nu \\ E &= h\frac{c}{\lambda} \\ E &= \hbar\omega \end{aligned} \quad (83)$$

where E represents the energy of a photon, ν denotes its frequency. This relation signifies a fundamental connection between the wave nature of light and its discrete energy packets, known as photons.
The derivation of the Planck-Einstein relation begins with Max Planck's groundbreaking work on blackbody radiation. Planck introduced the concept of quantized energy levels to explain the observed spectral distribution of electromagnetic radiation. He proposed that energy is quantized in discrete units, or quanta, and derived the expression for blackbody radiation[56][57]:

$$\rho(\upsilon,T)\,d\upsilon=(8\pi\upsilon^2/c^3)\cdot[h\upsilon/(e^{h\upsilon/kT}-1)]d\upsilon \quad (84)$$

where ρ(ν, T) represents the energy density of radiation at frequency ν and temperature T, c is the speed of light, and k is Boltzmann's constant. Albert Einstein further developed this concept by applying Planck's quantization of energy to explain the photoelectric effect[58][59][60]. By considering light as composed of discrete packets of energy (photons), he deduced the relation between the energy of a photon and its frequency.
This groundbreaking relation revolutionized our understanding of light and its interactions with matter, paving the way for the modern quantum theory.

Quantum mechanics relies on the framework of mathematical operators to represent physical observables and their corresponding quantum states[61][62][63].
In quantum mechanics, an operator is a mathematical entity that operates on a quantum state vector to produce another quantum state vector. Operators are often represented as linear, Hermitian, or unitary matrices, depending on the physical observable they represent. For example, the position operator ($\hat{x}$) and momentum operator ($\hat{p}$) are defined as:

$$\hat{x}|\psi\rangle=x|\psi\rangle \quad (85)$$

$$\hat{p}|\psi\rangle=-i\hbar\nabla|\psi\rangle \quad (86)$$

where $|\psi\rangle$ represents a quantum state vector, x is the position, ∇ is the del operator.
Hermitian operators play a central role in quantum mechanics due to their real eigenvalues and orthogonal eigenvectors. An operator $\hat{A}$ is Hermitian if it satisfies the condition:

$$\hat{A}^\dagger=\hat{A} \quad (87)$$

where $\hat{A}^\dagger$ denotes the Hermitian adjoint (conjugate transpose) of $\hat{A}$. The

observables associated with Hermitian operators have real eigenvalues, which represent measurable quantities in quantum experiments.
Unitary operators are also essential in quantum mechanics, as they preserve the norm of the quantum state vectors, ensuring that the total probability of finding the system in any state remains constant. A unitary operator $\hat{U}$ satisfies the condition:

$$\hat{U}^{\dagger}\hat{U} = \hat{U}\hat{U}^{\dagger} = \hat{I} \quad (88)$$

where $\hat{U}^{\dagger}$ is the adjoint of $\hat{U}$, and $\hat{I}$ is the identity operator.

The commutation relation between two operators $\hat{A}$ and $\hat{B}$ is defined as:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad (89)$$

The commutation relation between certain operators is fundamental in quantum mechanics and leads to uncertainty principles, such as the Heisenberg Uncertainty Principle, which states that certain pairs of observables cannot be simultaneously known with arbitrary precision.

Quantum operators find applications in a wide range of quantum mechanical phenomena. For instance, the Hamiltonian operator ($\hat{H}$) represents the total energy of a quantum system and plays a crucial role in time evolution and quantum dynamics. Additionally, the angular momentum operators ($\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$) describe the quantized angular momentum of particles, leading to the quantization of angular momentum in quantum systems.

In 1925, Werner Heisenberg formulated matrix mechanics, which was one of the first consistent mathematical descriptions of quantum phenomena. Heisenberg introduced a new set of mathematical entities called "matrices" to represent observable quantities. These matrices later evolved into what we now recognize as quantum operators, with each operator corresponding to a measurable physical quantity.
The first appearance of the expression for the momentum operator $\hat{p} = -i\hbar\nabla$

can be attributed to the groundbreaking work of Erwin Schrödinger. In 1926, Schrödinger published his paper titled “Quantization as an Eigenvalue Problem”[64], in which he presented his wave mechanics formulation of quantum mechanics.

$\hat{x}, \hat{y}, \hat{z}$	Position	x, y, z
$\hat{p} = -i\hbar\nabla$ $\hat{p}_x, \hat{p}_y, \hat{p}_z$	Momentum	$-i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}$
$\hat{T} = \frac{\hat{p}\cdot\hat{p}}{2m} = -\frac{\hbar^2}{2m}\nabla^2$ $\hat{T}_x, \hat{T}_y, \hat{T}_z$	Kinetic energy	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}, -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}, -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}$
$\hat{E} = i\hbar\frac{\partial}{\partial t}$	Total energy	Time-dependent
$\hat{H} = \hat{T} + \hat{V}$ $= \frac{\hat{p}^2}{2m} + V(x)$	Hamiltonian	Time-independent
$\hat{L} = r \times -i\hbar\nabla$ $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$ $\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$ $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$	Angular momentum operator	$-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$ $-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$ $-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

Using the concept of Planck units as infinitesimals we can express these operators in a way that the dimensional analysis is maintained, we can define the momentum operator from the Planck scale for example:

$$\begin{aligned} &2m_p c \\ &2\frac{i}{2}\sqrt{\frac{\hbar c}{G}}c \\ &i\sqrt{\frac{\hbar c^3}{G}} \\ &i\sqrt{\frac{\hbar^2 c^3}{\hbar G}} \end{aligned} \tag{90}$$

$$i\hbar\sqrt{\frac{c^3}{\hbar G}}$$
$$i\hbar\frac{\partial}{\partial x}$$

It's the same way with the total energy:

$$\begin{aligned}
&2m_p c^2\\
&2\frac{i}{2}\sqrt{\frac{\hbar c}{G}}c^2\\
&i\sqrt{\frac{\hbar c^5}{G}}\\
&i\sqrt{\frac{\hbar^2 c^5}{\hbar G}}\\
&i\hbar\sqrt{\frac{c^5}{\hbar G}}\\
&i\hbar\frac{\partial}{\partial t}
\end{aligned} \quad (91)$$

So the modified quantum operators become:

$$2mc = i\hbar\frac{\partial}{\partial x} \quad (92) \qquad\qquad 2mc^2 = i\hbar\frac{\partial}{\partial t} \quad (93)$$

Schrödinger introduced what is now known as Schrödinger's equation. He derived the time-dependent form of the equation, describing how the wavefunction of a quantum system evolves over time under the influence of a Hamiltonian operator[65][66].

$$\begin{aligned}
&\hat{H}\psi = i\hbar\frac{\partial}{\partial t}\psi\\
&i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\cdot\psi
\end{aligned} \quad (94)$$

For a free particle in one dimension:

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi \quad (95)$$

Converting the differential operators to Planck units the equality remains dimensionally true, ignoring the ψ on both sides we then have:

$$\begin{aligned} i\hbar\sqrt{\frac{c^5}{\hbar G}} &= -\hbar^2\frac{1}{2}\frac{2}{i}\sqrt{\frac{G}{\hbar c}\frac{c^3}{\hbar G}} \\ i\hbar\sqrt{\frac{c^5}{\hbar G}} &= i\hbar\sqrt{\frac{G}{\hbar c}\frac{c^3}{G}} \\ i\hbar\sqrt{\frac{c^5}{\hbar G}} &= i\hbar\sqrt{\frac{Gc^6}{\hbar c G^2}} \\ i\hbar\sqrt{\frac{c^5}{\hbar G}} &= i\hbar\sqrt{\frac{c^5}{\hbar G}} \end{aligned} \tag{96}$$

And we also arrive at the following relation using the fact that $ds = cdt$, therefore:

$$\begin{aligned} 2m\frac{\partial}{\partial t}\Psi &= \frac{-\hbar^2}{i\hbar}\frac{\partial^2}{\partial x^2}\Psi \\ 2m\frac{\partial}{\partial t}\Psi &= i\hbar\frac{\partial^2}{\partial x^2}\Psi \\ 2m\frac{\partial x}{\partial t}\Psi &= i\hbar\frac{\partial}{\partial x}\Psi \\ 2mc\Psi &= i\hbar\frac{\partial}{\partial x}\Psi \\ 2mc^2\Psi &= i\hbar\frac{\partial}{\partial t}\Psi \end{aligned} \tag{97}$$

In quantum mechanics, the wave function, typically denoted as Ψ (Psi), is a complex-valued function that depends on the coordinates of a particle in space and time. It characterizes the behavior of particles, such as electrons and photons, as both particles and waves simultaneously, as described by the famous wave-particle duality.

Born's Rule states that the probability density of finding a particle in a particular state is proportional to the squared magnitude of the particle's wave function. Mathematically, for a particle described by a wave function Ψ, the probability density P(x, t) of finding the particle at a position x and time t is given by:

$$\begin{aligned} P(x,t) &= \psi(x,t) * \psi(x,t) \\ P(x,t) &= |\psi(x,t)|^2 \geq 0 \end{aligned} \tag{98}$$

where $|\psi(x,t)|^2$ represents the modulus squared of the wave function and Ψ^* is complex conjugate of wave function.

Born's Rule, named after the German physicist Max Born[67], is a fundamental postulate in quantum mechanics that provides a probabilistic interpretation of the wave function. This rule bridges the gap between the wave-like behavior of particles and their discrete, particle-like interactions, leading to a profound understanding of quantum phenomena.

The dimensional analysis of the wave function involves examining the units of the various quantities involved in the function. Since the wave function represents the probability amplitude, its dimensions must be such that it leads to a dimensionless probability when squared.

The wave function Ψ's dimensionality can be expressed as $L^{-3/2}$, where L represents length (in meters). This is because the wave function's squared magnitude ($|\psi|^2$) must have units of probability, which is dimensionless.
For instance, the normalization condition requires the integral of $|\psi|^2$ over all space to equal unity.

$$P(x,t)dx = \int |\psi(x,t)|^2 dx \quad (99)$$

$$\int_{-\infty}^{+\infty} \int |\psi(x,t)|^2 dx = 1 \quad (96)$$

This condition ensures that the total probability of finding a particle in all possible states is 100%. By imposing this normalization condition, we can interpret the wave function's magnitude squared as a probability density function.

$$\begin{aligned} |\psi(x)|^2 &= \frac{1}{L^n} \\ |\psi(x)|^2 &= \frac{\partial^n}{\partial x^n} \\ \psi(x) &= \sqrt{\frac{\partial^n}{\partial x^n}} \\ \psi(x) &= \frac{\partial^{\frac{1}{n}}}{\partial x^{\frac{1}{n}}} ? \end{aligned} \quad (100)$$

$$\begin{aligned} |\psi(x)|^2 &= \frac{1}{L^n} \\ |\psi(x)|^2 &= \frac{\partial^n}{\partial x^n} \\ \psi(x) &= \sqrt{\left(\frac{C^3}{\hbar G}\right)^n} ? \end{aligned} \quad (101)$$

Could the fundamental wavefunctions be expressed in Planck units or could their deeper meaning be explained through the concept of fractional derivatives?

Relativistic Quantum Mechanics

The Klein-Gordon equation emerged as a result of the efforts to reconcile quantum mechanics with special relativity. In 1926, Erwin Schrödinger formulated the non-relativistic wave equation, which successfully described the behavior of non-relativistic particles, but it failed to account for relativistic effects. In 1926, Oskar Klein proposed a relativistic wave equation for spinless particles, but his attempt was not entirely satisfactory. Later, in 1928, Walter Gordon, building upon Klein's work, successfully formulated the equation for scalar particles with spin-zero. This equation is now known as the Klein-Gordon equation. The Klein-Gordon equation can be mathematically expressed as follows[68][69][70]:

$$(\partial^{\mu}\partial_{\mu}+m^{2})\phi(x)=0 \quad (102)$$

where ∂_u represents the four-gradient:

$$\frac{\partial}{\partial x^{u}}=\left(\frac{1}{c}\frac{\partial}{\partial t},\vec{\nabla}\right)=(\partial_0,\partial_i)=\left(\frac{\partial_t}{c},\partial_x,\partial_y,\partial_x\right), \quad (103)$$

m is the mass of the particle, and $\phi(x)$ is the scalar field representing the particle's quantum state as a function of spacetime coordinates $x=(ct,r)$. However, this equation presented a challenging feature known as negative probabilities. It allowed for solutions that could not be interpreted as probabilities in the traditional sense, raising questions about its physical significance.

In 1928, Paul Dirac addressed the problems of negative probabilities and introduced the Dirac equation, providing a comprehensive description of relativistic electrons with spin-1/2. Dirac's primary motivation was to combine special relativity and quantum mechanics in a consistent manner while preserving the principle of the probabilistic interpretation of quantum theory. The Dirac equation can be expressed mathematically as follows:

$$(i\gamma^{\mu}\partial_{\mu}-m)\psi(x)=0 \quad (104)$$

where i is the imaginary unit, γ^{u} are four 4x4 matrices known as the Dirac gamma matrices, m is the mass of the electron, and $\psi(x)$ is the four-component spinor representing the electron's quantum state as a function of spacetime coordinates $x=(ct,r)$.

The gamma matrices γ^{u} satisfy the anticommutation relation:

$$\{\gamma^{\mu},\gamma^{\upsilon}\}=\gamma^{\mu}\gamma^{\upsilon}+\gamma^{\upsilon}\gamma^{\mu}=2g^{uv}I_4 \quad (105)$$

where $\{.,.\}$ denotes the anticommutator, g^{uv} is the Minkowski metric tensor, and I_4 is the 4x4 identity matrix.

Dirac's equation predicted the existence of positive and negative energy solutions, leading to the discovery of the concept of antimatter, with the positron (antielectron) being the antiparticle of the electron.

The Klein-Gordon equation with all constants and expanded is:

$$\begin{aligned}
&\Box\psi=-\frac{m^2c^2}{\hbar^2}\psi\\
&\partial^{\mu}\partial_{\mu}\psi=-\frac{m^2c^2}{\hbar^2}\psi\\
&\eta^{\mu\upsilon}\partial_{\upsilon}\partial_{\mu}\psi=-\frac{m^2c^2}{\hbar^2}\psi\\
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\psi=-\frac{m^2c^2}{\hbar^2}\psi\\
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)\psi=-\frac{m^2c^2}{\hbar^2}\psi
\end{aligned} \quad (106)$$

Where $\Box$ is the d'Alembert operator, $\eta^{\mu\nu}$ is the Minkowski metric, ∇^2 is the 3-dimensional Laplacian operator.
The Dirac equation with all constants and expanded is:

$$\begin{aligned}
&(i\hbar\gamma^u\partial_u - mc)\psi = 0\\
&i\hbar\left(\gamma^0\frac{\partial}{\partial x^0}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}\right)\psi = mc\psi\\
&\left(\gamma^0\frac{1}{c}\frac{\partial}{\partial t}+\gamma^1\frac{\partial}{\partial x}+\gamma^2\frac{\partial}{\partial y}+\gamma^3\frac{\partial}{\partial z}\right)\psi = \frac{mc}{i\hbar}\psi
\end{aligned} \tag{107}$$

The Klein-Gordon equation can be obtained by simply substituting the standard quantum operators into the relativistic energy-momentum equation, doing this so we have:

$$\begin{aligned}
&E^2 = p^2c^2 + m_0^2c^4\\
&\frac{E^2}{c^2} = p^2 + m_0^2c^2\\
&\frac{E^2}{c^2} - p^2 = m_0^2c^2\\
&\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2 - \left(-i\hbar\frac{\partial}{\partial x}\right)^2 = m_0^2c^2\\
&-\frac{1}{c^2}\hbar^2\frac{\partial^2}{\partial t^2} + \hbar^2\frac{\partial^2}{\partial x^2} = m_0^2c^2\\
&\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -\frac{m_0^2c^2}{\hbar^2}
\end{aligned} \tag{108}$$

Using dimensional analysis of Planck units we can prove the equality of this equation, considering m as the Planck mass (we can disregard the 1/2 for the moment and keep the imaginary unit) and adding the inverse square of the Planck length on the left side we have:

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{c^3}{\hbar G} = -\frac{m_p^2c^2}{\hbar^2}\\
&0 + \frac{c^3}{\hbar G} = -\left(i\sqrt{\frac{\hbar c}{G}}\right)^2\frac{c^2}{\hbar^2}\\
&\frac{c^3}{\hbar G} = \frac{c^3}{\hbar G}
\end{aligned} \tag{109}$$

Now let's get the non-standard Klein-Gordon equation with an additional "infinitesimal" term using the modified energy-momentum relationship, so:

$$
\begin{aligned}
&E^2 = p^2c^2 + m_0^2c^4 - m^2c^4\frac{\hbar c}{G} \\
&\frac{E^2}{c^2} = p^2 + m_0^2c^2 - m^2c^2\frac{\hbar c}{G} \\
&-m_0^2c^2 + m^2c^2\frac{\hbar c}{G} = p^2 - \frac{E^2}{c^2} \\
&-m_0^2c^2 + m^2c^2\frac{\hbar c}{G} = \left(-i\hbar\frac{\partial}{\partial x}\right)^2 - \frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2 \qquad (110) \\
&-m_0^2c^2 + m^2c^2\frac{\hbar c}{G} = \frac{1}{c^2}\hbar^2\frac{\partial^2}{\partial t^2} - \hbar^2\frac{\partial^2}{\partial x^2} \\
&-\frac{m_0^2c^2}{\hbar^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\frac{\hbar c}{G}
\end{aligned}
$$

Expanding the terms of the total energy expression and highlighting the invariant mass we can highlight the additional term:

$$
\begin{aligned}
&\frac{E^2}{c^2} - p^2 = m_0^2c^2 \\
&m^2c^2 - m^2v^2 = m_0^2c^2 \\
&m^2c^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2c^2 \qquad (111) \\
&\frac{m_0^2c^2}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^2}\left(1 - \frac{v^2}{c^2}\right) = m_0^2c^2 \\
&\frac{m_0^2c^2}{\left(\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}}\right)^2}\left(1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}\right) = m_0^2c^2 \\
&m^2c^2\left(1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}\right) = m_0^2c^2 \\
&m^2c^2 - m^2v^2 + m^2c^2\frac{\hbar c}{G} = m_0^2c^2
\end{aligned}
$$

Something to note is the removal of the asymptotic behavior when v tends to c, so we have minimum and maximum values of in these limits, for example:

$$\lim_{v\to c}\frac{m_0^2c^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}=\frac{m_0^2c^2}{1-\frac{c^2}{c^2}}=\frac{m_0^2c^2}{0}=\infty$$

$$\lim_{v\to c}\ m_0^2c^2\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2=m_0^2c^2\left(1-\frac{c^2}{c^2}\right)=m_0^2c^2(0)=0$$

$$\lim_{v\to c}\frac{m_0^2c^2}{\left(\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar c}{G}}\right)^2}=\frac{m_0^2c^2}{1-\frac{c^2}{c^2}+\frac{\hbar c}{G}}=\frac{m_0^2c^2}{0+\frac{\hbar c}{G}}=m_0^2c^2\frac{G}{\hbar c} \tag{112}$$

$$\lim_{v\to c}\ m_0^2c^2\left(\sqrt{1-\frac{v^2}{c^2}+\frac{\hbar c}{G}}\right)^2=m_0^2c^2\left(1-\frac{c^2}{c^2}+\frac{\hbar c}{G}\right)=m_0^2c^2\frac{\hbar c}{G}$$

The Klein-Gordon equation can be extracted from the free particle Dirac equation as follows:

$$\left(\gamma^u\frac{\partial}{\partial x^u}+\frac{mc}{i\hbar}\right)\psi=0$$

$$\gamma^v\frac{\partial}{\partial x^v}\left(\gamma^u\frac{\partial}{\partial x^u}+\frac{mc}{i\hbar}\right)\psi=0$$

$$\gamma^v\frac{\partial}{\partial x^v}\gamma^u\frac{\partial}{\partial x^u}\psi+\gamma^v\frac{\partial}{\partial x^v}\frac{mc}{i\hbar}\psi=0 \quad (113)$$

$$\gamma^v\gamma^u\frac{\partial}{\partial x^v}\frac{\partial}{\partial x^u}\psi+\frac{mc}{i\hbar}\gamma^v\frac{\partial}{\partial x^v}\psi=0$$

$$\gamma^u\gamma^v\frac{\partial}{\partial x^u}\frac{\partial}{\partial x^v}\psi-\left(\frac{mc}{i\hbar}\right)^2\psi=0$$

$$(\gamma^u\gamma^v+\gamma^v\gamma^u)\frac{\partial}{\partial x^u}\frac{\partial}{\partial x^v}\psi-2\left(\frac{mc}{i\hbar}\right)^2\psi=0$$

$$2\delta^{vu}\frac{\partial}{\partial x^u}\frac{\partial}{\partial x^v}\psi-2\left(\frac{mc}{i\hbar}\right)^2\psi=0$$

$$2\frac{\partial}{\partial x^u}\frac{\partial}{\partial x^u}\psi-2\left(\frac{mc}{i\hbar}\right)^2\psi=0$$

$$\frac{\partial}{\partial x^u}\frac{\partial}{\partial x^u}\psi-\left(\frac{mc}{i\hbar}\right)^2\psi=0$$

The formalism of the Dirac equation embodies the mathematical structure of the square root of the Klein-Gordon equation, emphasizing the idea that solutions of the Dirac equation correspond to the positive and negative energy solutions of the Klein-Gordon equation.
Dirac himself rather supports this heuristic[71] in his Recollections of an exciting era, History of twentieth century physics, Academic Press 1977, pp. 109-146:

"I was playing around with the three components $\sigma_1, \sigma_2, \sigma_3$ which I had used to describe the spin of the electron, and I noticed that if you formed the expression $\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$ and squared it, p_1, p_2 and p_3 being the three components of momentum, you got just $p_1^2 + p_2^2 + p_3^2$, the square of the momentum (1). This was a pretty mathematical result. I was quite excited over it. It seemed that it must be of some importance. (...)

I suddenly realized that there was no need to stick to the quantities σ , which can be represented by matrices with just two rows and columns. Why not go over to four rows and columns? Mathematically there was no objection to this at all. Replacing the σ-matrices by four-row and column matrices, one could easily take the square root of the sum of four squares, or even five squares if one wanted to."

Using the quantum operators and taking as the square root of the total energy we have:

$$i\hbar \frac{\partial}{\partial t}\psi = c\left[\sqrt{-\frac{\hbar^2}{2m_0}\nabla^2 + m_0^2 c^2}\right]\psi \quad (114)$$

Equating the square root of the energy with the following expression[72]:

$$\sqrt{p^2c^2 + m_0^2 c^4} = \left(-i\hbar\alpha_x \frac{\partial}{\partial x} - i\hbar\alpha_y \frac{\partial}{\partial y} - i\hbar\alpha_z \frac{\partial}{\partial z} + \beta m_0 c\right) \quad (115)$$

squared both sides:

$$p^2c^2+m_0^2c^4=(\alpha_x p_x+\alpha_y p_y+\alpha_z p_z+\beta m_0 c)(\alpha_x p_x+\alpha_y p_y+\alpha_z p_z+\beta m_0 c) \quad (116)$$

to be consistent with the KG equation requires:

$$\begin{aligned} &\alpha_x^2=\alpha_y^2=\alpha_z^2=\beta^2=1 \\ &\alpha_j\beta+\beta\alpha_j=0 \\ &\alpha_j\alpha_k+\alpha_k\alpha_j=0 \quad (j\neq k) \end{aligned} \quad (117)$$

After some algebra we then have:

$$\begin{aligned} -\hbar^2\frac{\partial^2}{\partial t^2}&=-\hbar^2\alpha_x^2\frac{\partial^2}{\partial x^2}-\hbar^2\alpha_y^2\frac{\partial^2}{\partial y^2}-\hbar^2\alpha_z^2\frac{\partial^2}{\partial z^2}+\beta^2 m_0^2c^2 \\ -\hbar^2\frac{\partial^2}{\partial t^2}&=-\hbar^2\frac{\partial^2}{\partial x^2}-\hbar^2\frac{\partial^2}{\partial y^2}-\hbar^2\frac{\partial^2}{\partial z^2}+m_0^2c^2 \end{aligned} \quad (118)$$

Using the Pauli spin matrices we get:

$$\beta=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_j=\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (119)$$

with $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (120)

Now we can express the Dirac gamma matrices as:

$$\gamma^0=\beta; \quad \gamma^1=\beta\alpha_x; \quad \gamma^2=\beta\alpha_y; \quad \gamma^3=\beta\alpha_z \quad (121)$$

Multiplying the equation X by B then gives us:

$$-\beta\hbar\frac{\partial}{\partial t}\psi=\left(-i\hbar\beta\alpha_x\frac{\partial}{\partial x}-i\hbar\beta\alpha_y\frac{\partial}{\partial y}-i\hbar\beta\alpha_z\frac{\partial}{\partial z}+\beta^2 m_0 c\right)\psi \quad (122)$$

We get the standard Dirac equation:

$$-\gamma^0\hbar\frac{\partial}{\partial t}\psi=\left(-i\hbar\gamma^1\frac{\partial}{\partial x}-i\hbar\gamma^2\frac{\partial}{\partial y}-i\hbar\gamma^3\frac{\partial}{\partial z}+m_0 c\right)\psi \quad (123)$$

Some important properties of gamma matrices are:

$$
\begin{aligned}
&(\gamma^0)^2 = 1 \\
&(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\
&\gamma^0\gamma^j + \gamma^j\gamma^0 = 0 \\
&\gamma^k\gamma^j + \gamma^j\gamma^k = 0 \ \ (j \neq k)
\end{aligned} \quad (124)
$$

γ^0 is Hermitian, γ^i are anti-Hermitian:

$$\gamma^{0\dagger} = \gamma^0; \quad \gamma^{1\dagger} = -\gamma^1; \quad \gamma^{2\dagger} = -\gamma^2; \quad \gamma^{3\dagger} = -\gamma^3; \ (125)$$

In expanded form, the matrices are:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(126)

Due to the presence of these matrices the wave function is a Dirac Spinor with 4 components:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \ (127)$$

Expressing as the square root of the energy we have the modified form of the equation as:

$$
\begin{aligned}
&\sqrt{m^2c^2 - m^2v^2 + m^2c^2\frac{\hbar c}{G}} = m_0c \\
&mc\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}} = m_0c \\
&\frac{m_0c}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}}}\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}} = m_0c \\
&m_0c = m_0c
\end{aligned} \quad (128)
$$

The final equation with the additional term then remains (remembering that this term is very small so it is disregarded in the standard equation):

$$\left(i\hbar\gamma^{u}\partial_{u}-m_0c+mc\sqrt{\frac{\hbar c}{G}}\right)\psi=0$$
$$i\hbar\left(\gamma^0\frac{1}{c}\frac{\partial}{\partial t}+\gamma^1\frac{\partial}{\partial x}+\gamma^2\frac{\partial}{\partial y}+\gamma^3\frac{\partial}{\partial z}\right)\psi+mc\sqrt{\frac{\hbar c}{G}}\psi=m_0c\psi \quad (129)$$

Quantization

In the framework of quantum mechanics, commutators play a fundamental role in characterizing the non-commutative behavior of operators that represent physical observables. Additionally, the process of quantization bridges classical mechanics and quantum mechanics by establishing a systematic method for transforming classical variables into quantum operators.

1.1 Poisson Brackets

In classical mechanics, the Poisson bracket of two functions F and G of canonical coordinates (q_i, p_i) is defined as[73]:

$$\{F,G\}=\sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \quad (130)$$

where the Poisson bracket encodes the dynamical behavior of classical observables.

Properties:
1. ***Antisymmetry:*** Poisson brackets are antisymmetric, implying $\{F,G\}=-\{G,F\}$. (131)
2. ***Jacobi Identity:*** Poisson brackets satisfy the Jacobi identity, ensuring consistency in their algebraic manipulation.

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0 \quad (132)$$

3. ***Bilinearity:***

$$\{AF+BG,H\}=A\{F,H\}+B\{G,H\}, \quad \{H,AF+BG\}=A\{H,F\}+B\{H,G\}, \\ A,B\in\mathbb{R} \tag{133}$$

3. ***Leibniz's rule:***

$$\{FG,H\}=\{F,H\}G+F\{G,H\} \tag{134}$$

Significance:
1. ***Hamilton's Equations:*** Poisson brackets are foundational in Hamilton's equations of motion, determining the time evolution of canonical coordinates and momenta.
2. ***Phase Space Dynamics:*** Poisson brackets characterize the evolution of a dynamical system's phase space coordinates.

1.2 Quantum Commutators: Defining Non-Commutive Behavior

In quantum mechanics, the commutator between two operators $\hat{A}$ and $\hat{B}$ is defined[74][75] as:

$$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A} \tag{135}$$

where the commutator measures the non-commutativity of the operators.

Properties:
1. ***Antisymmetry:*** Commutators are antisymmetric, implying
$[A,B]=-[B,A]$. (136)
2. ***Linearity:*** Commutators adhere to the distributive and associative properties of linearity.

Significance in Quantum Mechanics:
1. ***Uncertainty Principle:*** Commutators are foundational in Heisen-

berg's uncertainty principle, which restricts the simultaneous measurement of certain pairs of observables with high precision.
2. ***Quantum Dynamics:*** Commutators dictate how observables evolve in time within the framework of quantum mechanics.

1.3 Quantization: Transition from Classical to Quantum

Classical functions of q_i and p_i are mapped to corresponding quantum operators on a Hilbert space. Quantum observables are represented by Hermitian operators to ensure real eigenvalues and valid probability interpretations.

Mathematical Procedure:
1. ***Canonical Variables:*** Consider classical variables q_i (positions) and p_i (momenta), which are promoted to quantum operators $\hat{q}_i$ and $\hat{p}_i$.
2. ***Canonical Commutation Relations:*** Quantum operators are subject to canonical commutation relations, such as $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, where δ_{ij} is the Kronecker delta.
3. ***Correspondence Principle:*** In the classical limit, commutators converge to Poisson brackets, reconciling the classical and quantum descriptions.

1.4 Quantum Operators and Observables:

1. ***Non-Commutativity:*** Quantum operators do not generally commute, capturing the intrinsic uncertainty inherent in quantum mechanics.
2. ***Hermitian Operators:*** Quantum observables correspond to Hermitian operators, ensuring real eigenvalues and consistent probability interpretations.

For example; in quantum mechanics, the commutation relation between position $\hat{x}$ and momentum $\hat{p}$ is defined as:

$$
\begin{aligned}
[\hat{p},\hat{x}]\psi &= (\hat{p}\hat{x}-\hat{x}\hat{p})\psi \\
&= -i\hbar\frac{\partial}{\partial x}(x\psi)+i\hbar x\frac{\partial}{\partial x}(\psi) \\
&= -i\hbar\psi\frac{\partial}{\partial x}(x)-i\hbar x\frac{\partial}{\partial x}(\psi)+ix\hbar\frac{\partial}{\partial x}(\psi) \\
&= -i\hbar\psi
\end{aligned}
\tag{137}
$$

The same result can be obtained in a simpler way using the dimensional analysis of Planck units, then we have:

$$
\begin{aligned}
&-2m_pc\cdot\frac{dx}{d} \\
&-2\frac{i}{2}\sqrt{\frac{\hbar c}{G}}c\cdot\sqrt{\frac{\hbar G}{c^3}} \\
&-i\sqrt{\frac{\hbar c^3}{G}}\cdot\sqrt{\frac{\hbar G}{c^3}} \\
&-i\sqrt{\hbar^2} \\
&-i\hbar
\end{aligned}
\tag{138}
$$

Or more directly:

$$
\begin{aligned}
&-2m_pc\cdot\frac{dx}{d} \\
&-i\hbar\frac{d}{dx}\cdot\frac{dx}{d} \\
&-i\hbar
\end{aligned}
\tag{139}
$$

A brief summary of gravitation theory

General Relativity was proposed by Albert Einstein[76] in 1915, this theory revolutionized our understanding of gravity and the structure of the universe. Its development was marked by a combination of historical context and intricate mathematical formalism that reshaped the foundations of physics. The roots of General Relativity trace back to Einstein's earlier work on the Special Theory of Relativity in 1905. Special Relativity introduced the concept of spacetime as a unified entity, where the speed of light is a fundamental constant. However, it left the issue of gravity unresolved. Einstein realized that a deeper theory was needed to account for gravity's effects and proposed General Relativity as the answer.

The culmination of years of thinking, Einstein's theory was spurred by various historical and scientific influences, including the equivalence principle—a concept dating back to Galileo—that gravitational and inertial mass are equivalent. Additionally, the works of mathematicians like Riemann, Gauss, and Minkowski laid the mathematical foundations necessary for developing the intricate geometry of curved spacetime.
General Relativity's predictions were confirmed through a series of experimental tests and observations. The 1919 solar eclipse expedition led by Arthur Eddington validated the theory's prediction of light bending in a gravitational field[77][78], affirming the curvature of spacetime. Further, the precise measurement of the precession of Mercury's orbit confirmed General Relativity's superiority over Newtonian gravity at explaining such phenomena. Einstein's General Relativity is formulated using the language of differential geometry, particularly tensor calculus. At its core is the Einstein field equations, which describe how the curvature of spacetime is related to the distribution of matter and energy.

1.1 Manifolds

A manifold is a fundamental concept in the realm of mathematics and physics[79][80][81][82], playing a crucial role in the formulation and understanding of Einstein's theory of General Relativity. In the context of General Relativity, manifolds provide a rigorous mathematical framework for describing the curvature of spacetime caused by gravitational interactions.

Mathematically, a manifold is a topological space that locally resembles Euclidean space but may possess a more intricate global structure. Specifically, a manifold of dimension n is a Hausdorff topological space M such that each point p in M has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. This local "Euclidean-ness" enables us to define coordinates, or charts, which are mappings from open subsets of the manifold to Euclidean spaces.

In General Relativity, spacetime itself is modeled as a four-dimensional manifold, often denoted as M. Each point p in this manifold corresponds to a specific event in spacetime. The coordinates assigned to each point via the charts allow us to label events with numerical values, facilitating the formulation of equations and calculations. The transition between different charts' coordinates is governed by smooth functions known as transition maps, ensuring the compatibility of different coordinate systems.

The concept of a manifold enables the mathematical description of curvature intrinsic to General Relativity. The curvature of spacetime is encoded in the metric tensor field $g_{\mu\upsilon}$, which assigns a metric to each point on the manifold. This metric encodes information about how distances and angles are measured in the curved spacetime.

1.2 Defining Tensors

Tensors are multi-dimensional arrays of values that can represent various quantities and transformations[83][84][85][86][87]. Formally, an n -th order tensor in m-dimensional space is an object that can be fully described by n indices and m^n components. Tensors of different orders exhibit specific transformation behaviors under coordinate changes.
A tensor of rank n can be succinctly defined as a mathematical object that exhibits n indices, each ranging over its respective coordinate axis. It captures the geometrical and physical properties of a space and can represent quantities like scalars (rank-0 tensors), vectors (rank-1 tensors), matrices (rank-2 tensors), and more complex entities. The transformation behavior of tensors under changes of coordinate systems is a critical aspect that defines their utility and generality. A tensor type (r,s) can be written as:

$$T = T^{a1\ldots ar}{}_{b1\ldots bs} \frac{\partial}{\partial x^{a1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{ar}} \otimes dx^{b1} \otimes \ldots dx^{bs} \quad (140)$$

where $\frac{\partial}{\partial x^{ai}}$ is a basis for the i-th tangent space and dx^{bj} a basis for the j-th cotangent space.

1.2.1 Tensor Algebra

Tensor algebra involves operations on tensors, such as addition and multiplication. The sum of two tensors of the same order results in a new tensor with components obtained by adding corresponding components. The operation is performed on corresponding components:

$$C_{i1i2\ldots i_p}^{j1j2\ldots j_q} = A_{i1i2\ldots i_p}^{j1j2\ldots j_q} \pm B_{i1i2\ldots i_p}^{j1j2\ldots j_q}. \ (141)$$

The multiplication of a tensor by a scalar scales all its components:

$$A_{ik}^{j} = aB_{ik}^{j}, \ (142)$$

where a is a non-zero scalar.

1.2.2 Tensor Product

The tensor product, denoted by $\bigotimes$, combines tensors to create higher-order tensors. For two tensors A and B of orders n and m, respectively, their tensor product $A \bigotimes B$ is a tensor of order $(n+m)$. The tensor product combines two tensors to create a new tensor with indices determined by the original tensors:

$$C_{i1i2\ldots i_p k1k2\ldots k_r}^{j1j2\ldots j_q l1l2\ldots l_s} = A_{i1i2\ldots i_p}^{j1j2\ldots j_q} \cdot B_{k1k2\ldots k_r}^{l1l2\ldots l_s}. \ (143)$$

1.2.3 Tensor Contraction

Contraction involves summing over repeated indices in a tensor expression. This operation reduces the tensor's order and is often used to form inner products or trace operations. A contracted tensor retains the indices not involved in the contraction.

$$C^{j_1\ldots j_q}_{i_2\ldots i_p} = A^{j_1 j_2\ldots j_q}_{i_1 i_2\ldots i_p}\cdot B^{i_1},$$
$$A^j_i \xrightarrow{contraction} A^i_i, \qquad (144)$$
$$A^{jk}_{il} \xrightarrow{contraction\ on\ jl} A^{mk}_{im}.$$

1.2.4 Inner Product of Tensors

In tensor calculus, the concept of an inner product enables the comparison of tensors and plays a pivotal role in various mathematical and physical applications. The inner product, also known as the contraction or the scalar product, allows the pairing of indices in a meaningful way to yield scalar quantities or lower-rank tensors.

Consider two tensors, A of type (p,q) and B of type (r,s). The inner product of these tensors can be defined as follows:

$$A\cdot B = A^{j_1 j_2\ldots j_q}_{i_1 i_2\ldots i_p}\cdot B^{k_1 k_2\ldots k_s}_{j_1 j_2\ldots j_q}. \quad (145)$$

This operation involves contracting the common indices $j_1, j_2, \ldots, j_q$ between the two tensors. The result is a tensor of type $(p+r-2q, k_1 k_2\ldots k_s)$.

The inner product can also be expressed in index-free notation using Einstein's summation convention:

$$A\cdot B = A^{j_1 j_2\ldots j_q}_{i_1 i_2\ldots i_p} B^{k_1 k_2\ldots k_s}_{j_1 j_2\ldots j_q} = \sum_{j_1 j_2\ldots j_q} A^{j_1 j_2\ldots j_q}_{i_1 i_2\ldots i_p} B^{k_1 k_2\ldots k_s}_{j_1 j_2\ldots j_q}. \quad (146)$$

This operation carries several important properties:
Commutativity: The inner product is commutative, i.e., $A\cdot B = B\cdot A$.
Distributivity: The inner product distributes over tensor addition, i.e., $A\cdot(B+C) = A\cdot B + A\cdot C$.
Linearity: The inner product is linear with respect to scalar multiplication, i.e., $(cA)\cdot B = c(A\cdot B)$, where c is a scalar.
Associativity: The inner product is not associative, unlike the tensor product.

The inner product is fundamental in a wide range of disciplines. In physics, it is used to calculate physical quantities such as work, energy, and momentum. In geometry, it plays a role in defining lengths, angles, and projections. Additionally, the inner product's contraction property is often utilized in simplifying tensor equations and expressing physical laws concisely.

1.2.5 Tensor Transformation

Tensors transform under changes of basis or coordinate systems according to specific rules. For a tensor T with components $T^{i1i2\ldots in}$ in one coordinate system and $T^{`j1j2\ldots jn}$ in another, the transformation law is given by:

$$T^{`j1j2\ldots jn} = \frac{\partial x^{`j1}}{\partial x^{i1}} \frac{\partial x^{`j2}}{\partial x^{i2}} \ldots \frac{\partial x^{`jn}}{\partial x^{in}} T^{i1i2\ldots in} \quad (147)$$

where $x^{`j}$ and x^{i} represent the new and old coordinate systems, respectively.

1.2.6 Covariant Transformations

When a coordinate system undergoes a change, the components of a tensor may transform accordingly. Covariant transformations pertain to changes in the basis of the domain space of a tensor. If we denote the original tensor's components as T with indices i, j, ..., and the transformed tensor's components as T' with indices k, l, ..., the covariant transformation rule is given by:

$$\begin{aligned} T'_{k,l,\ldots} &= \frac{\partial x^{i}}{\partial x'^{k}} \frac{\partial x^{j}}{\partial x'^{l}} \ldots T_{i,j,\ldots} \\ \bar{A}_i &= \frac{\partial x^{j}}{\partial \bar{x}^{i}} A_j \end{aligned} \quad (148)$$

where $\frac{\partial x^{i}}{\partial x'^{k}}$ represents the Jacobian matrix of the coordinate transformation.

1.2.7 Contravariant Transformations

Contravariant transformations, on the other hand, concern the changes in the basis of the codomain (or dual) space of a tensor. The transformed components T' are related to the original components T through:

$$T'^{k,l,\dots} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j}\dots T^{i,j,\dots}$$
$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \quad (149)$$

The key distinction between covariant and contravariant transformations lies in the transformation rules for the components of the tensor under changes of basis.

1.2.8 Mixed Tensor Transformations

In the realm of tensor calculus, mixed tensors play a significant role in expressing various physical phenomena, especially in the context of multiple coordinate systems. A mixed tensor of type (p,q) is characterized by its contravariant rank p and covariant rank q. The transformation laws for such tensors under a change of coordinates can be succinctly described using index notation.

Let $T^{j_1 j_2 \dots j_q}_{i_1 i_2 \dots i_p}$ be a mixed tensor of type (p,q) defined in the initial coordinate system x^i, and let x'^i represent the transformed coordinates. The transformation of this mixed tensor can be formulated as follows:

$$T'^{j_1 j_2 \dots j_q}_{i_1 i_2 \dots i_p} = \frac{\partial x'^{j_1}}{\partial x^{k_1}}\frac{\partial x'^{j_2}}{\partial x^{k_2}}\cdots\frac{\partial x'^{j_q}}{\partial x^{k_q}}\frac{\partial x^{l_1}}{\partial x'^{i_1}}\frac{\partial x^{l_2}}{\partial x'^{i_2}}\cdots\frac{\partial x^{l_p}}{\partial x'^{i_p}} T^{k_1 k_2 \dots k_q}_{l_1 l_2 \dots l_p}. \quad (150)$$

$$\bar{A}^{ln}_{m} = \frac{\partial \bar{x}^l}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial \bar{x}^n}{\partial x^k} A^{ik}_{j}$$

Here, the Kronecker delta symbol δ_{mn} is often employed to simplify the transformation laws:

$$\frac{\partial x'^j}{\partial x^i} = \frac{\partial x^j}{\partial x'^i} = \delta^j_i. \quad (151)$$

It's important to note that the transformation laws for mixed tensors directly follow from the chain rule of multivariable calculus. These transformations are crucial in diverse areas of physics and engineering, ranging from fluid dynamics to general relativity, where physical quantities are often described in terms of mixed tensors to account for multiple coordinate systems and frames of reference.

1.2.9 Symmetry and Antisymmetry

Tensors can possess symmetry or antisymmetry properties with respect to their indices. A tensor is symmetric if permuting any pair of indices doesn't change its value. Conversely, a tensor is antisymmetric if swapping any pair of indices results in a sign change.

Let's consider a tensor T of type (p,q). The tensor T is symmetric with respect to its contravariant indices $i_1, i_2, \ldots, i_p$ and covariant indices $j_1, j_2, \ldots, j_q$ if it satisfies the symmetry condition:

$$T_{i_1 i_2 \ldots i_p}^{j_1 j_2 \ldots j_q} = T_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(p)}}^{j_{\pi(1)} j_{\pi(2)} \ldots j_{\pi(q)}}, \ (152)$$

where σ represents a permutation of the contravariant indices and π represents a permutation of the covariant indices. In index notation, this can be represented as:

$$T_{(i_1 i_2 \ldots i_p)}^{(j_1 j_2 \ldots j_q)} = T_{(i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(p)})}^{(j_{\pi(1)} j_{\pi(2)} \ldots j_{\pi(q)})}. \ (153)$$

On the other hand, a tensor T is antisymmetric if it satisfies the antisymmetry condition:

$$T_{i_1 i_2 \ldots i_p}^{j_1 j_2 \ldots j_q} = -T_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(p)}}^{j_{\pi(1)} j_{\pi(2)} \ldots j_{\pi(q)}}, \ (154)$$

or equivalently:

$$T_{[i_1 i_2 \ldots i_p]}^{[j_1 j_2 \ldots j_q]} = -T_{[i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(p)}]}^{[j_{\pi(1)} j_{\pi(2)} \ldots j_{\pi(q)}]}. \ (155)$$

These symmetry and antisymmetry conditions have profound implications for the transformation properties of tensors. Symmetric tensors remain symmetric after a change of coordinates, while antisymmetric tensors switch sign under coordinate transformations.

The significance of these concepts extends to various fields such as physics, where symmetric tensors often represent physical quantities with certain conservation properties, and antisymmetric tensors are prevalent in describing quantities like angular momentum and electromagnetic fields.

1.3 Vectors: Geometric Entities

In a vector space V, a vector $\mathbf{v}$ is an element that possesses both magnitude and direction. Vectors can be represented as a linear combination of basis vectors, each scaled by a corresponding component.

Mathematical Representation: A vector $\mathbf{v}$ can be expressed as $\mathbf{v} = v^i \mathbf{e}_i$, where v^i are the components of the vector in a chosen coordinate basis and $\mathbf{e}_i$ are the basis vectors.

Transformation: Under a change of coordinates, the components of a vector transform according to the transformation rules, ensuring that the vector's geometric properties remain invariant.

1.3.1 Covectors: Dual Space Perspectives

The dual space V^* to a vector space V is the set of all linear functions (covectors) that map vectors from V to the field of scalars $\mathbb{R}$ or $\mathbb{C}$. Covectors associate scalar values to vectors.

Mathematical Representation: A covector $\boldsymbol{\omega}$ can be expressed as $\boldsymbol{\omega} = \omega_i dx^i$, where ω_i are the components of the covector in the dual basis and dx^i are the corresponding basis covectors.

Transformation: Covectors transform contravariantly to vectors, ensuring that the pairing of a covector with a vector yields an invariant scalar value.

The covariant and contravariant basis vectors are defined by:

$$e_i = \frac{\partial r}{\partial u^i}, \quad e^i = \nabla u^i, \ (156)$$

where r is the position vector in Cartesian coordinates $(x^1, x^2, \ldots)$, and u^i are generalized curvilinear coordinates.

1.4 Kronecker delta tensor

The Kronecker delta, often denoted as δ_{ij}, is a fundamental mathematical construct within the realm of linear algebra and tensor calculus. It serves as a discrete symbol that encapsulates the relationship between two indices, facilitating concise representations of various mathematical expressions and operations.

Formally, the Kronecker delta tensor is defined as follows:

$$\begin{aligned}
&\delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases} \\
&\delta_{ij} = \delta_{ji} \\
&\delta^i_j = \delta^j_i = \delta^{ij} = \delta_{ij} \\
&\frac{\partial x^i}{\partial x^j} = \frac{\partial x^j}{\partial x^i} = \delta_{ij} = \delta^{ij}
\end{aligned} \qquad (157)$$

where i and j are indices taking values in a specified index set. This tensor is typically used in contexts where a compact representation of identity or orthogonality conditions is required.
In a vector space equipped with a basis, the Kronecker delta tensor plays a pivotal role in defining the components of other tensors. For instance, the components of a vector v in terms of its basis vectors e_i can be ex-

pressed using the Kronecker delta as follows: $v = v^i e_i$
where v^i are the components of the vector v. The Kronecker delta naturally emerges when raising and lowering indices through contraction with the metric tensor or when expressing tensors in a coordinate-free manner. Furthermore, the Kronecker delta tensor can be employed to establish the orthogonality relations between vectors and tensors. In Euclidean spaces, the dot product of two vectors v and w can be compactly expressed as:

$$v \cdot w = v^i w^j \delta_{ij} \quad (158)$$

In tensor calculus, the Kronecker delta is invaluable for raising and lowering indices, manipulating tensor components, and defining operations such as the inner product and matrix trace.
For example, in 3D Cartesian coordinates, the distance from the origin to a point (x, y, z) can be calculated as follows:

$$s^2 = x^2 + y^2 + z^2 \quad (159)$$

The line element is:

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (160)$$

writing more compactly (where both i and j are summed over from 1 to 3):

$$ds^2 = \delta_{ij} dx^i dx^j \quad (161)$$

We can write δ_{ij} as a matrix form:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (162)$$

expanding then we have (indices here do not represent exponents):

$$\begin{gathered} ds^2 = \delta_{ij} dx^i dx^j = \delta_{11} dx^1 dx^1 + \delta_{12} dx^1 dx^2 + \delta_{13} dx^1 dx^3 + \delta_{21} dx^2 dx^1 + \dots \\ + \delta_{33} dx^3 dx^3 \end{gathered} \quad (163)$$

Making the necessary cancellations, we recover the usual Pythagorean theorem: $ds^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2$.

1.5 Defining the Metric Tensor

The metric tensor $g_{\mu\nu}$ is a symmetric, rank-2 tensor that encodes the local geometry of a manifold. For a given point P on the manifold, the metric tensor provides a rule for constructing the inner product of tangent vectors at that point. Mathematically, the metric tensor satisfies the properties:

1. ***Symmetry:*** $g_{\mu\nu} = g_{\nu\mu}$
2. ***Positivity:*** $g_{\mu\nu} v^\mu v^\nu \geq 0$ for any tangent vector v^μ
3. ***Non-degeneracy:*** $g_{\mu\nu} v^\mu v^\nu = 0$ if and only if $v^\mu = 0$

The metric is essentially characterized by the dot product of basis vectors:

$$g_{\mu\nu} = \vec{e_\mu} \cdot \vec{e_\nu} \quad (164)$$

Generally the metric tensor in expanded form can be represented by a matrix:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \quad (165)$$

1.5.1 Computing Distances and Angles

The metric tensor allows the calculation of infinitesimal distances and angles on the manifold. The line element ds in a curved space is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (166)$$

where dx^μ are the differentials of the coordinates. This formula generalizes the Pythagorean theorem for curved spaces, enabling precise distance measurements.

1.5.2 Metric Tensor and Coordinate Transformations

The metric tensor's components depend on the choice of coordinates, reflecting the manifold's local geometry. Under a change of coordinates, the metric tensor transforms as a covariant tensor of rank 2. Specifically, for a transformation from coordinates x^μ to x'^μ, the metric tensor components change as:

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \quad (167)$$

1.5.3 Diagonal and Non-diagonal Metric Tensors

In certain coordinate systems, the metric tensor takes a diagonal form, simplifying distance calculations. However, in more general cases, the metric tensor components may involve cross-terms, indicating interactions between different coordinate directions.

1.5.4 Applications in General Relativity and Beyond

In general relativity, the metric tensor plays a central role in Einstein's field equations, linking the curvature of spacetime to the distribution of matter and energy. It defines the spacetime interval, which governs the causal structure of the universe.

1.5.5 Duality and the Metric Tensor

The metric tensor g_{ij} is a fundamental object in tensor calculus that establishes a connection between vectors and covectors. It defines an inner product between vectors and allows for raising and lowering indices:

$$v_i = g_{ij} v^j \text{ and } \omega^i = g^{ij} \omega_j \quad (168)$$

Here, g^{ij} are the components of the inverse metric tensor.

1.6 Definition of Christoffel Symbols

Given a smooth manifold with a coordinate system $\{x^\mu\}$, the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ of the second kind are defined as the coefficients that describe how the basis vectors ∂_μ change as one moves along the coordinate directions. Mathematically, they can be expressed as:

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x^\mu \partial x^\nu} \quad (169)$$

where ∂_μ is the partial derivative with respect to the coordinate x^μ, expressing in terms of basis vectors:

$$\Gamma^\lambda_{\mu\upsilon} = \frac{\partial \vec{e_\mu}}{\partial x^\upsilon} \cdot \vec{e^\lambda} \quad (170)$$

1.6.1 Geometric Interpretation and Properties

1. ***Affine Connection:*** Christoffel symbols define an affine connection on the manifold, which represents the way tangent spaces are connected as one moves along the manifold's surface.
2. ***Symmetry:*** The Christoffel symbols are symmetric in their lower indices:

$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} \quad (171)$$

3. ***Metric Compatibility:*** In a metric-compatible connection, the Christoffel symbols can be expressed in terms of the metric tensor $g_{\mu\nu}$:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \quad (172)$$

4. ***Curvature and Geodesics:*** Christoffel symbols are used to compute the curvature of a manifold and define geodesics—the paths that extremize the length between two points.

1.7 Definition of the Covariant Derivative

Given a vector field V^μ on a curved manifold with a connection represented by the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, the covariant derivative ∇_ν of V^μ along the coordinate direction ν is defined as:

$$\begin{aligned}
&\nabla_c T^{a\ldots}_{b\ldots} = \partial_C T^{a\ldots}_{b\ldots} + \Gamma^a_{dc} T^{d\ldots}_{b\ldots} + \ldots - \Gamma^d_{bc} T^{a\ldots}_{d\ldots} - \ldots \\
&\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\alpha\nu} V^\alpha \qquad (173) \\
&\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\alpha_{\mu\nu} V_\alpha
\end{aligned}$$

Here, ∂_ν represents the partial derivative with respect to the coordinate x^ν, and $\Gamma^\mu_{\alpha\nu}$ are the connection coefficients that encapsulate the curvature of the manifold.

1.7.1 Properties of the Covariant Derivative

1. ***Linearity:*** The covariant derivative is linear, obeying the Leibniz rule for tensor products and addition of tensors.
2. ***Metric Compatibility:*** A metric-compatible connection, also known as a metric-compatible covariant derivative, preserves the metric tensor's properties during differentiation. Mathematically, it satisfies:

$$\nabla_\rho g_{\mu\nu} = 0 \quad (174)$$

3. ***Covariant Derivative of Tensors:*** The covariant derivative of a tensor of any rank involves the connection coefficients and derivatives of its components.
4. ***Parallel Transport:*** The covariant derivative plays a key role in the concept of parallel transport, describing how a vector or tensor field can be transported along a curve while maintaining its intrinsic properties. Consider a curve γ on a manifold and a vector V at a point P on this curve. Parallel transport of V along γ to a nearby point Q involves maintaining the direction of V as it is transported while accounting for

the manifold's curvature. Mathematically, parallel transport can be expressed as:

$$\nabla_{\dot{\gamma}} V = 0 \quad (175)$$

Here, $\nabla_{\dot{\gamma}}$ represents the covariant derivative along the tangent vector $\dot{\gamma}$ to the curve γ , and 0 indicates that the transported vector remains unchanged.
In the realm of differential geometry and general relativity, the Riemann curvature tensor plays a pivotal role in describing the intrinsic curvature of a manifold. This tensor encapsulates the geometric effects of spacetime curvature, providing a deeper understanding of how gravity arises as a consequence of spacetime deformation.

1.8 Definition of the Riemann Curvature Tensor

The Riemann curvature tensor, often denoted as $R^{\alpha}_{\beta\mu\nu}$, is a multidimensional array of components that characterizes the curvature of a manifold. It quantifies the non-commutativity of covariant derivatives and is defined as:

$$R^{\alpha}_{\beta\mu\nu} = \partial_{\mu} \Gamma^{\alpha}_{\nu\beta} - \partial_{\nu} \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\sigma} \Gamma^{\sigma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\sigma} \Gamma^{\sigma}_{\mu\beta} \quad (176)$$

where $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols of the second kind, representing the connection coefficients of the manifold.

1.8.1 Geometric Interpretation

The Riemann curvature tensor encodes how the intrinsic geometry of a manifold deviates from Euclidean space. It characterizes the tidal gravitational forces experienced by extended objects in a curved spacetime. The components $R^{\alpha}_{\beta\mu\nu}$ denote the change in the α-component of a vector transported along an infinitesimal loop in the $\mu\nu$-plane, compared to transporting it directly along the β-direction.
The failure of parallel transport to maintain the vector's direction around closed curves reveals the curvature of the manifold. The Riemann cur-

vature tensor R^a_{bcd} quantifies this curvature by measuring the difference between the result of parallel transporting a vector around two different paths.
The Riemann curvature tensor is derived from the commutation of covariant derivatives. Starting with the covariant derivative of the covariant derivative of a vector V^a, we have:

$$\nabla_b \nabla_c V^a - \nabla_c \nabla_b V^a = R^a_{\ bcd} V^d \quad (177)$$

The left-hand side represents the non-commutativity of covariant derivatives, which captures the effect of curvature on parallel transport. Solving for R^a_{bcd}, we obtain the Riemann curvature tensor.

1.8.2 Properties of the Riemann Curvature Tensor

1. ***Symmetry:*** The Riemann curvature tensor exhibits symmetries in its indices, such as the Bianchi identities:

$$R^{\alpha}_{\beta\mu\nu} + R^{\alpha}_{\nu\beta\mu} + R^{\alpha}_{\mu\nu\beta} = 0, \quad (178)$$

$$\text{with: } R^{\alpha}_{\beta\mu\nu} = R^{\mu}_{\nu\alpha\beta} = -R^{\beta}_{\alpha\mu\nu} = -R^{\alpha}_{\beta\nu\mu}. \quad (179)$$

2. ***Contractions:*** Contracting the Riemann tensor allows the derivation of other curvature-related tensors, such as the Ricci tensor and the Ricci scalar, which play vital roles in Einstein's field equations.
The Ricci tensor $R_{\mu\nu}$ is a symmetric rank-2 tensor formed as a contraction of the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$ over one pair of indices. Mathematically, it is expressed as:

$$\begin{aligned} &R^{\alpha}_{\beta\alpha\nu} = g^{\alpha\mu} R_{\alpha\beta\mu\nu} = R_{\beta\nu} \\ &R_{\beta\nu} = \partial_{\rho}\Gamma^{\rho}_{\beta\nu} - \partial_{\nu}\Gamma^{\rho}_{\beta\rho} + \Gamma^{\rho}_{\beta\nu}\Gamma^{\gamma}_{\rho\gamma} - \Gamma^{\rho}_{\beta\gamma}\Gamma^{\gamma}_{\rho\nu} \end{aligned} \quad (180)$$

3. ***Curvature Scalars:*** The Riemann curvature tensor yields curvature

scalars, like the Kretschmann scalar, which provide a scalar measure of curvature magnitude at a point in spacetime.
The Ricci scalar R is a scalar quantity representing the curvature scalar curvature of spacetime. It is obtained by contracting the Ricci tensor with the metric tensor $g^{\mu\nu}$:

$$R = g^{\mu\nu} R_{\mu\nu} \quad (181)$$

The Kretschmann scalar K is a scalar curvature invariant obtained by contracting the Riemann curvature tensor $R^{\alpha}_{\beta\gamma\delta}$ over all four indices, yielding a single scalar quantity. Mathematically, it is expressed as:

$$K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \quad (182)$$

For a Schwarzschild black hole with mass M, it is expressed as:

$$K = \frac{48G^2M^2}{c^4r^6} \quad (183)$$

1.9 Definition of Einstein's Field Equation

Einstein's field equation encapsulates the essence of general relativity by relating the curvature of spacetime (determined by the metric tensor $g_{\mu\nu}$ to the distribution of energy-momentum (expressed by the stress-energy tensor $T_{\mu\nu}$. Mathematically, it takes the form:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (184)$$

Here, $G_{\mu\nu}$ represents the Einstein tensor, G is the gravitational constant.

1.9.1 Mathematical Formulation and Properties

1. ***Einstein Tensor:*** The Einstein tensor $G_{\mu\nu}$ is a contraction of the Ricci tensor $R_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$ minus half of the Ricci scalar R:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (185)$$

In this formulation, the Einstein tensor essentially quantifies the difference between the local geometry described by the Ricci tensor and the geometry that would be expected in a flat, Minkowski spacetime, represented by the term $\frac{1}{2} R g_{\mu\nu}$.

2. ***Stress-Energy Tensor:*** The stress-energy tensor is denoted as $T_{\mu\nu}$, where the indices μ and ν range over four values representing the spacetime coordinates (t, x, y, z). The components of this tensor provide information about the energy density, momentum density, and stresses associated with the matter and energy content of spacetime.

In matrix form, it is defined as:

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}. \quad (186)$$

Each element of the matrix represents a component of energy and momentum flow in spacetime.

The T^{00} component represents energy density, T^{0i} represents energy flux in the i direction, T^{i0} represents momentum density in the i direction, and T^{ij} $(i,j = 1,2,3)$ represent momentum fluxes in various spatial directions.

To derive the matrix elements of the energy-momentum tensor, we start with the conservation of energy-momentum, expressed by the equation:

$$\partial_\mu T^{\mu\nu} = 0. \quad (187)$$

This equation represents the conservation of energy and momentum at a point in spacetime.

For the T^{00} component (energy density), with N being the number density (the number of particles per unit volume):

$$T^{00} = \frac{\gamma m_0 c}{V} \times c = \frac{\gamma m_0 c^2}{V} = \frac{E}{V}, \quad (188)$$
$$T^{00} = N\gamma m_0 c^2 = \gamma \rho c^2,$$

the conservation equation becomes:

$$\partial_t T^{00} + \partial_x T^{01} + \partial_y T^{02} + \partial_z T^{03} = 0. \quad (189)$$

This equation relates the change in energy density to energy fluxes in different directions.

For the T^{0i} components (energy flux per unit area (A) perpendicular to the i direction) and T^{i0} components (momentum density per unit area (A) perpendicular to the i direction):

$$T^{0i} = \frac{\gamma mc}{V} \times u^i = \frac{\gamma mcu^i}{V} = \frac{\gamma mc^2 u^i}{Vc} = \frac{Eu^i}{Vc} = \frac{E(dx^i/dt)}{(dx^i dx^j dx^k)c} = \frac{E/dt}{(dx^j dx^k)c}, \quad (190)$$

$$T^{i0} = \frac{\gamma mu^i}{V} \times c = \frac{\gamma mcu^i}{V} = \frac{\gamma mc^2 u^i}{Vc} = \frac{Eu^i}{Vc} = \frac{E(dx^i/dt)}{(dx^i dx^j dx^k)c} = \frac{E/dt}{(dx^j dx^k)c}, \quad (191)$$

$$T^{0i} = T^{i0} = \frac{(Nu^i At)(\gamma mc^2)}{(Atc)} = N\gamma mcu^i = \rho\gamma cu^i, \quad (192)$$

the conservation equations become:

$$\partial_t T^{0i} + \partial_x T^{1i} + \partial_y T^{2i} + \partial_z T^{3i} = 0, \quad (193)$$
$$\partial_t T^{i0} + \partial_x T^{i1} + \partial_y T^{i2} + \partial_z T^{i3} = 0. \quad (194)$$

These equations relate the change in energy flux and momentum density to other components of the tensor.
For the T^{ij} $(i,j=1,2,3)$ components (momentum fluxes in spatial directions):

$$T^{ii} = \frac{\gamma m u^i}{V} \times u^i = \frac{p^i u^i}{V} = \frac{p^i (dx^i/dt)}{(dx^i dx^j dx^k)} = \frac{p^i/dt}{(dx^j dx^k)}, \quad (195)$$

$$T^{ij} = \frac{\gamma m u^i}{V} \times u^j = \frac{p^i u^j}{V} = \frac{p^i (dx^j/dt)}{(dx^i dx^j dx^k)} = \frac{p^i/dt}{(dx^i dx^k)}, \quad (196)$$

$$T^{ji} = \frac{\gamma m u^j}{V} \times u^i = \frac{p^j u^i}{V} = \frac{p^j (dx^i/dt)}{(dx^i dx^j dx^k)} = \frac{p^j/dt}{(dx^j dx^k)}, \quad (197)$$

$$T^{ji} = \frac{\gamma m u^j}{V} \times u^i = \frac{\gamma m u^i}{V} \times u^j = \frac{p^i u^j}{V} = \frac{p^i (dx^j/dt)}{(dx^i dx^j dx^k)} = \frac{p^i/dt}{(dx^i dx^k)}, \quad (198)$$

$$T^{ji} = \frac{(N u^j A t)(\gamma m u^i)}{(At)} = N \gamma m u^i u^j = \rho \gamma u^i u^j, \quad (199)$$

the conservation equations become:

$$\partial_t T^{ij} + \partial_x T^{ji} + \partial_y T^{ji} + \partial_z T^{ji} = 0. \quad (200)$$

These equations express the conservation of momentum fluxes in different spatial directions.
Gathering the results, we can write the energy-momentum tensor in matrix form as:

$$[T^{\mu\nu}] = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix} = \begin{bmatrix} N\gamma m_0 c^2 & N\gamma m_0 c u^1 & N\gamma m_0 c u^2 & N\gamma m_0 c u^3 \\ N\gamma m_0 u^1 c & N\gamma m_0 u^1 u^1 & N\gamma m_0 u^1 u^2 & N\gamma m_0 u^1 u^3 \\ N\gamma m_0 u^2 c & N\gamma m_0 u^2 u^1 & N\gamma m_0 u^2 u^2 & N\gamma m_0 u^2 u^3 \\ N\gamma m_0 u^3 c & N\gamma m_0 u^3 u^1 & N\gamma m_0 u^3 u^2 & N\gamma m_0 u^3 u^3 \end{bmatrix} \quad (201)$$

$$N\gamma m_0 \begin{bmatrix} c^2 & cu^1 & cu^2 & cu^3 \\ u^1 c & u^1 u^1 & u^1 u^2 & u^1 u^3 \\ u^2 c & u^2 u^1 & u^2 u^2 & u^2 u^3 \\ u^3 c & u^3 u^1 & u^3 u^2 & u^3 u^3 \end{bmatrix} = \gamma \rho \begin{bmatrix} c^2 & cu^1 & cu^2 & cu^3 \\ u^1 c & u^1 u^1 & u^1 u^2 & u^1 u^3 \\ u^2 c & u^2 u^1 & u^2 u^2 & u^2 u^3 \\ u^3 c & u^3 u^1 & u^3 u^2 & u^3 u^3 \end{bmatrix} \quad (202)$$

where each matrix element represents energy and momentum components.

The components of $T_{\mu\nu}$ are determined by various contributions from different sources, such as matter, energy, pressure, and momentum. In a formal sense, the stress-energy tensor for a ideal (or perfect: no internal friction forces or viscosity) fluid is:

$$T_{\mu\nu} = (\rho + \frac{P}{c^2}) u_\mu u_\upsilon - P g_{\mu\upsilon} \quad (203)$$

Here:
- ρ is the energy density of the matter.
- P is the pressure.
- $u_\mu u_\upsilon$ is the four-velocity vector of the matter.
- $g_{\mu\nu}$ is the metric tensor of spacetime, describing its curvature.

In this equation, the first term on the right-hand side accounts for the energy and momentum carried by the matter, where the components of the four-velocity $u_\mu u_\upsilon$ ensure the correct scaling.
The second term, $P g_{\mu\upsilon}$, corresponds to the pressure contribution, and it takes into account the isotropic pressure acting on the matter, the energy-momentum tensor in this case becomes:

$$[T^{\mu\nu}] = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \quad (204)$$

1.9.2 Physical Implications

1. ***Curvature-Gravity Relationship:*** Einstein's field equation establishes that matter and energy distribution (as described by $T_{\mu\nu}$) dictates the curvature of spacetime (as determined by $G_{\mu\nu}$).
2. ***Geodesic Motion:*** Particles move along geodesic paths determined by the curvature of spacetime, reflecting the influence of gravity.
The geodesic equation describes the paths that objects follow in curved spacetime. It is given by:

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \quad (205)$$

where x^α are the spacetime coordinates and λ is an affine parameter

along the geodesic. The Equivalence Principle, a cornerstone of GR, asserts the local indistinguishability between acceleration due to gravity and acceleration in an inertial frame.
3. ***Gravitational Waves:*** The field equation predicts the existence of gravitational waves, ripples in spacetime curvature that propagate at the speed of light.

1.9.3 Definition of Einstein's Field Equation with Cosmological Constant

Einstein's field equation, when augmented with the cosmological constant term Λ, accounts for an additional contribution to the curvature of spacetime. Mathematically, it is expressed as:

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu} \quad (206)$$

The cosmological constant term Λ acts as a repulsive force on large scales, leading to the observed accelerated expansion of the universe. The cosmological constant is often associated with dark energy, an enigmatic form of energy that drives the accelerated expansion of the cosmos.

2 Definition of Minkowski Metric

In the context of special relativity, the Minkowski metric describes spacetime intervals ds^2 in a flat spacetime with four coordinates (ct,x,y,z):

$$ds^2=-c^2dt^2+dx^2+dy^2+dz^2 \quad (207)$$

Here, the signature $(-+++)$ signifies one timelike dimension and three spacelike dimensions.

2.1 Metric Components and Properties

1. ***Temporal Component*** g_{tt}***:*** This component involves a factor of $-c^2$

and represents the time component of the spacetime interval.
2. ***Spatial Components*** g_{xx}***,*** g_{yy}***, and*** g_{zz}***:*** These components are positive and represent the spatial components of the spacetime interval.
3. ***Invariant Interval:*** The Minkowski metric's structure ensures that the spacetime interval ds^2 is invariant under Lorentz transformations, a fundamental property in special relativity.

2.2 Gravitational Time Dilation and Length Contraction

In the presence of a gravitational field, the Lorentz factor accounts for the time dilation effects caused by differences in gravitational potential. As an object moves closer to a massive body, its time slows down relative to an observer in a weaker gravitational field.
Additionally, the Lorentz factor contributes to length contraction effects, where the dimensions of an object are perceived to be smaller when it moves at high speeds or in strong gravitational fields.

2.2 Definition of Newtonian Gravitational Metric

In classical physics, the Newtonian gravitational metric characterizes the geometry of spacetime influenced by a static gravitational field. In Cartesian coordinates (t,x,y,z), the metric can be expressed as:

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)dt^2 + dx^2 + dy^2 + dz^2 \quad (208)$$

Here, Φ represents the gravitational potential.

$$\Phi = -\frac{GM}{r} \quad (209)$$

In Newtonian gravity, the gravitational field is denoted by the vector field "g." This field represents the gravitational acceleration at different points within it. The vector field "g" is derived from the negative gradient of the gravitational potential:

$$\vec{g} = -\vec{\nabla}\Phi = -\partial_x \Phi \vec{e_x} - \partial_y \Phi \vec{e_y} - \partial_z \Phi \vec{e_z} \quad (210)$$

2.2.1 Metric Components and Properties

1. ***Temporal Component g_{tt} :*** This component is affected by the gravitational potential Φ and represents the time part of the spacetime interval.
2. ***Spatial Components g_{xx} , g_{yy} , and g_{zz} :*** These components remain unchanged from the flat spacetime metric and represent spatial dimensions.

2.2.2 Physical Implications

1. ***Curved Spacetime:*** The Newtonian gravitational metric introduces curvature in spacetime due to the presence of a gravitational field, reflecting the way masses influence the geometry of their surroundings.
2. ***Gravitational Time Dilation:*** The term $1+2\Phi/c^2$ in the temporal component g_{tt} leads to gravitational time dilation, where clocks in a stronger gravitational field tick more slowly.
3. Newtonian Gravitational Potential: The gravitational potential Φ in the metric is related to the Newtonian gravitational potential energy and determines the gravitational forces acting on test particles.

2.3 Definition of Schwarzschild Metric

The Schwarzschild metric describes the geometry of spacetime outside a spherically symmetric massive object in vacuum. In Schwarzschild coordinates (t,r,θ,ϕ), the metric takes the form:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (211)$$

where G is the gravitational constant, M is the mass of the object, θ and ϕ are the angular coordinates.

2.3.1 Metric Coefficients and Properties

1. ***Temporal Component*** g_{tt}***:*** This component is associated with time dilation effects and represents the gravitational time dilation near a massive object. As r approaches $2GM/c^2$, time dilation becomes infinitely large, marking the event horizon of a non-rotating black hole.
2. ***Radial Component*** g_{rr}***:*** This component accounts for the spatial curvature in the radial direction. It diverges at $r = 2GM/c^2$, indicating the presence of a singularity at the center of the black hole.
3. ***Angular Components*** $g_{\theta\theta}$ ***and*** $g_{\phi\phi}$***:*** These components reflect the curvature in the angular directions and show that the geometry remains spherical.

2.3.2 Physical Implications

1. ***Event Horizon:*** The radius $r = 2GM/c^2$ corresponds to the event horizon—the boundary beyond which nothing, not even light, can escape the gravitational pull of the black hole.
2. ***Singularity:*** The singularity at $r = 0$ signifies an infinitely dense point at the center of the black hole, where classical physics breaks down.
3. ***Gravitational Redshift:*** The time dilation term g_{tt} leads to gravitational redshift, causing light emitted from near the event horizon to appear redshifted to distant observers.

2.4 Newtonian gravity

Postulated by Isaac Newton, it considers gravity as an instantaneous force acting at a distance between masses. Classical gravity encompasses the broader concept of gravity that transcends the specific details of any given theory. It describes the mutual attraction between masses that results in an attractive force. Mathematically, the gravitational force F between two masses m_1 and m_2 separated by a distance r is given by Newton's law of universal gravitation:

$$F = G\frac{m_1 m_2}{r^2} \quad (212)$$

2.4.1 Properties of Newtonian Gravity

1. ***Inverse Square Law:*** The gravitational force diminishes with the square of the distance between masses, resulting in weaker interactions at larger distances.
2. ***Superposition Principle:*** The gravitational force between multiple masses can be determined by summing up the individual forces between pairs of masses.
3. ***Action and Reaction:*** Newton's third law dictates that for every gravitational force exerted by one mass on another, an equal and opposite force is exerted on the first mass.

2.5 Definition of Gravitational Poisson Equation

The gravitational Poisson equation links the distribution of mass or energy density ρ to the gravitational potential Φ in a given region of spacetime. Mathematically, it is expressed as:

$$\nabla^2\Phi = 4\pi G\rho \quad (213)$$

Here, ∇^2 represents the Laplacian operator, G is the gravitational constant.

2.5.1 Mathematical Formulation and Properties

1. ***Laplacian Operator:*** The Laplacian operator ∇^2 is defined as the divergence of the gradient of a scalar function. In Cartesian coordinates, it is given by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (214)$$

2. ***Physical Significance:*** The Poisson equation relates the curvature of the gravitational potential Φ to the distribution of mass density ρ.

Defining the gravitational force between a gravitating test particle with mass m and a spherical body with mass M whose position is at the origin of a spherical coordinate system, so we have:

$$\begin{aligned} F &= -\frac{GmM}{r^2} e_r \\ g &= \frac{F}{m} = -\frac{GM}{r^2} e_r \end{aligned} \quad (215)$$

And be the flow of the gravitational field as the integral of the area over a closed surface S surrounding a spherical mass M, we have then:

$$Flux = \int_S \boldsymbol{g} \cdot \boldsymbol{n} \, dA \quad (216)$$

Where n is a unit vector normal to the surface and pointing outwards and dA is an infinitesimal element of the area, making this surface to be the surface of the mass M whose radius is R, so we conclude that:

$$\begin{aligned} \int_S \boldsymbol{g} \cdot \boldsymbol{n} \, dA &= \int_S \left(-\frac{GM}{R^2} e_r\right) e_r \, dA \\ &= \int_S -\frac{GM}{R^2} \, dA \\ &= \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} -\frac{GM}{R^2} R^2 \sin\theta \, d\theta \, d\phi \\ &= \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} -GM \sin\theta \, d\theta \, d\phi \\ &= \int_{\theta=0}^{\theta=\pi} -2\pi GM \sin\theta \, d\theta \\ &= -4\pi GM \end{aligned} \quad (217)$$

By the divergence theorem we have:

$$\int_S \boldsymbol{g} \cdot \boldsymbol{n}\, dA = \int_\Omega \nabla \cdot \boldsymbol{g}\, dV \quad (218)$$

Where $\boldsymbol{\Omega}$ is the region of space occupied by the gravitating object and dV is the infinitesimal volume, making a substitution on the left side we then have:

$$\int_\Omega \nabla \cdot \boldsymbol{g}\, dV = -4\pi GM \quad (219)$$

To eliminate integration on both sides, we express M as a density in the volume integral, so:

$$\int_\Omega \nabla \cdot \boldsymbol{g}\, dV = -4\pi G \int_\Omega \rho\, dV \quad (220)$$

So finally we have:

$$\begin{aligned}
&\nabla \cdot \boldsymbol{g} = -4\pi G\rho \\
&\nabla \cdot (-\nabla \Phi) = -4\pi G\rho \quad (221) \\
&\nabla^2 \Phi = -4\pi G\rho
\end{aligned}$$

2.6 Definition of Weak Limit in General Relativity

In the context of general relativity, the weak limit refers to scenarios where the gravitational field is weak enough that perturbative methods can be employed to approximate its effects. This typically occurs in regions of spacetime where gravitational potentials and velocities are much smaller than the speed of light c. Mathematically, this involves expanding the metric tensor $g_{\mu\nu}$ and solving equations to first order in terms of gravitational potentials[88][89][90][91].

Einstein's remarkable prediction was that the curvature terms in the spatial parts of spacetime could be observed through the bending of light around a massive object. Light's trajectory is represented as ± 1 on a spacetime diagram, signifying its equal movement in space and time. In his analysis of the weak field expression, Einstein determined that the spatial components exhibited curvature of precisely equal magnitude but opposite direction.

$$\Delta s^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\Delta t)^2 - \left(1 + \frac{2GM}{c^2 r}\right)\left[\Delta x^2 + \Delta y^2 + \Delta z^2\right] \quad (222)$$

In Newtonian gravity, the coefficient $(1 - 2GM/c^2 r)$ preceding $(c\Delta t)^2$ predicts light bending around stars. In the framework of general relativity, the coefficient $(1 + 2GM/c^2 r)$ preceding $[\Delta x^2 + \Delta y^2 + \Delta z^2]$ predicts a two-fold increase in this bending effect.

2.6.1 Mathematical Formulation and Properties

1. ***Perturbation Theory:*** Perturbative methods involve decomposing the metric tensor $g_{\mu\nu}$ into a background metric $\eta_{\mu\nu}$ representing a flat spacetime and a small perturbation $h_{\mu\nu}$ representing the weak gravitational field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (223)$$

2. ***Linearized Einstein Equations:*** In the weak limit, the Einstein field equations are linearized, resulting in equations that can be solved iteratively to obtain solutions that approximate the weak gravitational field's effects.

We begin with the general Schwarzschild space-time metric, which describes a spherically symmetric gravitational field around a mass M:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (224)$$

Here, r is radial distance, θ is polar angle, and ϕ is azimuthal angle.

In the Newtonian limit, we consider velocities $v \ll c$ and weak gravitational fields, leading us to work with the Taylor expansion of the metric around v/c and $GM/(c^2 r)$:

$$1-\frac{2GM}{c^2r}\approx 1+\frac{2GM}{c^2r},$$
$$\left(1-\frac{2GM}{c^2r}\right)^{-1}\approx 1+\frac{2GM}{c^2r}. \quad (225)$$

Substituting the approximations into the metric, we get:

$$ds^2\approx-\left(1+\frac{2GM}{c^2r}\right)dt^2+\left(1+\frac{2GM}{c^2r}\right)dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right). \quad (226)$$

Calculate the components of the metric tensor $g_{\mu\nu}$ from the approximated metric:

$$g_{00}=-\left(1+\frac{2GM}{c^2r}\right),\ g_{11}=1+\frac{2GM}{c^2r},\ g_{22}=r^2,\ g_{33}=r^2\sin^2\theta. \quad (227)$$

Calculate the curvature tensor $R^{\alpha}_{\beta\mu\nu}$ using the affine connection compatible with $g_{\mu\nu}$. However, since we're interested only in the R_{00} component in a diagonal space-time, we use the following approximation:

$$R_{00}\approx-\frac{1}{2}\nabla^2 g_{00}, \quad (228)$$

where ∇^2 is the three-dimensional Laplacian.

Calculate the second derivative of g_{00} with respect to r and obtain the Laplacian:

$$\nabla^2 g_{00}=-\frac{4GM}{c^2r^3}. \quad (229)$$

Substitute the result into the approximate expression for R_{00}:

$$R_{00}\approx\frac{2GM}{c^2r^3}. \quad (230)$$

This resembles the $4\pi G\rho$ component of the Einstein equation in General Relativity, where ρ is the mass density. However, in the Newtonian limit, ρ is the mass density due to a stationary mass distribution.

Comparing R_{00} with the $4\pi G\rho$ component of the Einstein equation, we obtain:

$$\frac{2GM}{c^2 r^3} \approx 4\pi G\rho, \ (231)$$

which leads us to the gravitational Poisson equation:

$$\nabla^2\Phi = 4\pi G\rho. \ (232)$$

2.7 Differential Forms

Formally, a differential form is a concept arising from multivariable calculus and algebraic geometry, serving as a generalization of the notion of a differential of a function[92][93][94][95][96].
At its core, a differential form can be defined as a collection of smoothly varying "pieces" of information, each of which quantifies how a particular quantity changes as one moves along different directions within a given space. This notion is particularly useful when dealing with spaces that are curved or possess intricate geometrical structures, as traditional methods of calculus often fall short in such contexts.

Mathematically, a differential form is built from differentials of coordinate functions. Suppose we have a smooth manifold—a space that locally resembles Euclidean space—and at each point, we can associate a set of coordinates. A differential form of degree k is an expression of the form:

$$\omega = \sum_{i1<i2<\ldots<ik} \omega_{i1,i2,\ldots,ik}(x) dx^{i1} \wedge dx^{i2} \wedge \ldots \wedge dx^{ik}, \ (233)$$

where x represents the coordinates, dx^i are the differentials of the coordinate functions, and $\wedge$ denotes the wedge product, which captures the antisymmetric nature of the form.

Differential forms bring a host of advantages to mathematical analysis. For instance, they provide an elegant way to express and manipulate concepts like integration, differentiation, and orientation. The exterior derivative operator, denoted by d, allows us to naturally extend the concept of differentiation to differential forms. This operator also gives rise to the notion of closed forms (forms for which $d\omega = 0$) and exact forms (forms that are the exterior derivative of another form).

2.8 Stokes' theorem

Stokes' theorem, a fundamental result in differential geometry and calculus, beautifully encapsulates the relationship between integration and differentiation for differential forms. It states that the integral of an exact form over a closed manifold is equal to the integral of its derivative over the boundary of the manifold.

It extends the concept of the Fundamental Theorem of Calculus to higher dimensions, establishing a connection between the flux of a vector field across a manifold's boundary and the circulation of the same vector field along the manifold's interior. Formally stated, Stokes' Theorem can be expressed in mathematical terms as follows:

Let M be an oriented, smooth, compact n-dimensional manifold with boundary ∂M, and let $\mathbf{F}$ be a smooth vector field defined on an open region containing M. Then, Stokes' Theorem asserts that the flux of $\mathbf{F}$ across M is equal to the circulation of $\mathbf{F}$ around the boundary ∂M:

$$\int_M \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial M} \mathbf{F} \cdot d\mathbf{r} \quad (234)$$

Here, $\nabla \times \mathbf{F}$ represents the curl of the vector field $\mathbf{F}$, $d\mathbf{S}$ is the outward-pointing differential surface area element on M, $\oint_{\partial M}$ denotes the line integral along the boundary curve ∂M, and $d\mathbf{r}$ signifies the differential displacement vector along the boundary curve.

In a more detailed breakdown, Stokes' Theorem essentially bridges the gap between differential forms and integration. It links the differential form $\nabla \times \mathbf{F} \cdot d\mathbf{S}$, which represents the circulation density across the manifold's infinitesimal surfaces, with the differential form $\mathbf{F} \cdot d\mathbf{r}$, which quantifies the tangential component of the vector field along the boundary curve. The theorem asserts that the total circulation density over the manifold's interior equals the total circulation along its boundary.

2.8.1 Generalized Stokes' Theorem

The Generalized Stokes' Theorem, also known as the Fundamental Theorem of Calculus for Differential Forms, relates the integral of an n-form $d\omega$ over a manifold M to the boundary $(n-1)$-form ω over the boundary of M. Mathematically, it can be stated as:

$$\int_M d\omega = \int_{\partial M} \omega \quad (235)$$

where M is an n-dimensional oriented manifold with boundary ∂M, d is the exterior derivative, ω is an $(n-1)$-form, and the integral is taken over appropriate domains.

2.9 Lagrangian

The Lagrangian, denoted as L, is a function that encapsulates the dynamics of a physical system in terms of generalized coordinates q_i and their time derivatives $\dot{q}_i$. Mathematically, it is defined as:

$$L = L(q_i, \dot{q}_i, t) \quad (236)$$

Where q_i represents the generalized coordinates that describe the configuration of the system, $\dot{q}_i$ are their corresponding time derivatives, and t is time. The Lagrangian contains information about the kinetic and potential energies of the system, as well as any external forces acting on it.

$$L = T_{(kinetic\ energy)} - V_{(potential\, energy)} \quad (237)$$

3 Action

The action, denoted as S, is a fundamental quantity that summarizes the behavior of a system over a given time interval. It is defined as the integral of the Lagrangian over time:

$$S = \int_{t1}^{t2} L(q_i, \dot{q}_i, t) dt \quad (238)$$

Where t_1 and t_2 represent the initial and final times of the interval. The action is a measure of how the system's configurations and velocities evolve over time, and it takes into account the entire history of the system between the initial and final times.

The principle of least action, also known as Hamilton's principle, states that the actual motion of a physical system is the one for which the action is minimized (or maximized) among all possible paths connecting the initial and final configurations. This principle leads to the Euler-Lagrange equations, which are the equations of motion for the system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (239)$$

These equations describe how the system's generalized coordinates q_i evolve with time, taking into account the forces and constraints described by the Lagrangian.

3.1 Square Root of the Determinant of Metric Tensor $\sqrt{-g}$

For a D-dimensional spacetime with coordinates x^i, the metric tensor is $g_{ij} = g_{ij}(x^k)$, and its determinant is $g = \det(g_{ij})$.

3.1.1 Jacobian

The Jacobian J quantifies the scaling factor in coordinate transformations. Given coordinates x^i and y^a, the Jacobian is defined as:

$$J = \det\left(\frac{\partial y^a}{\partial x^i}\right) \quad (240)$$

3.1.2 Relationship between $\sqrt{-g}$ and J

The relationship becomes apparent when examining how volume elements change under coordinate transformations in curved spacetimes. Consider a change of coordinates $x^i \to y^a$, where J accounts for the scaling factor between differential changes in x^i and y^a. Simultaneously, $\sqrt{-g}$ captures the local curvature effects on volume scaling due to the metric tensor's determinant.

The critical relationship is given by:

$$\sqrt{-g} = |J| \quad (241)$$

Here, $|J|$ represents the magnitude of the Jacobian. This equation signifies that the square root of the negative metric tensor's determinant corresponds to the absolute value of the Jacobian. This relationship ensures that when transforming coordinates, the geometric effects of curvature are combined with the effects of the coordinate scaling, maintaining the accurate calculation of volume elements.

3.2 The Einstein-Hilbert action

This action forms the basis for Einstein's equations of gravitation, which describe how matter and energy influence the curvature of spacetime. Mathematically, the Einstein-Hilbert action is expressed as follows:

$$S_{EH} = \frac{c^4}{16\pi G}\int R\sqrt{-g}\, d^4x \quad (242)$$

Here, R is the scalar curvature of spacetime, g is the determinant of the metric tensor that characterizes the geometry of spacetime, d^4x represents the volume element in spacetime.

The action S_{EH} encapsulates the gravitational dynamics by quantifying the curvature of spacetime induced by the distribution of matter and energy. The principle of least action, derived from variational calculus, states that physical systems evolve in a way that minimizes the action. Applying this principle to the Einstein-Hilbert action leads to the famous Einstein's field equations:

$$
\begin{aligned}
&S=\int(\lambda R+\mathcal{L}_M)\sqrt{-g}\,dx^4\\
&\int\left(\frac{\delta\left((\lambda R+\mathcal{L}_M)\sqrt{-g}\right)}{\delta g_{\mu\upsilon}}\right)\delta g_{\mu\upsilon}dx^4=0\\
&\sqrt{-g}\frac{\delta\mathcal{L}_M}{\delta g_{\mu\upsilon}}+\lambda\sqrt{-g}\frac{\delta R}{\delta g_{\mu\upsilon}}+(\mathcal{L}_M+\lambda R)\frac{\delta\sqrt{-g}}{\delta g_{\mu\upsilon}}=0 \qquad (243)\\
&\frac{\delta R}{\delta g_{\mu\upsilon}}+\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g_{\mu\upsilon}}=-\frac{1}{\lambda}\left(\frac{1}{\sqrt{-g}}\mathcal{L}_M\frac{\delta\sqrt{-g}}{\delta g_{\mu\upsilon}}+\frac{\delta\mathcal{L}_M}{\delta g_{\mu\upsilon}}\right)\\
&R_{\mu\upsilon}-\frac{1}{2}Rg_{\mu\upsilon}=\frac{1}{2\lambda}T_{\mu\upsilon}
\end{aligned}
$$

Where $\frac{\delta R}{\delta g_{\mu\upsilon}}=R_{\mu\upsilon},\ \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g_{\mu\upsilon}}=-\frac{1}{2}g_{\mu\upsilon},\ \lambda=\frac{c^4}{16\pi G}.$

Could Planck units hold the key to quantum gravity and unification?

The equations of standard quantum mechanics can be derived from the dimensional analysis of Planck units, starting from this idea we can do the same to find the equations of quantum gravity. We start with the momentum operator from quantum mechanics, this operator is nothing more than the Comptom wavelength in disguise:

$$
\begin{aligned}
&\lambda=\frac{h}{mc}\\
&\frac{i\lambda}{4\pi}=\frac{ih}{4\pi mc}\\
&\frac{\partial x}{\partial}=\frac{i\hbar}{2mc} \qquad (244)\\
&2mc=i\hbar\frac{\partial}{\partial x}
\end{aligned}
$$

We can make the following arrangement of this operator:

$$\frac{\partial}{\partial x}=\frac{2mc}{i\hbar} \quad (245)$$

The magic happens when we scale that term to the Planck scale and square it:

$$\begin{aligned}
&\sqrt{\frac{c^3}{\hbar G}}=\frac{2mc}{i\hbar} \\
&\frac{c^3}{\hbar G}=-\frac{4m^2c^2}{\hbar^2} \quad (246) \\
&c^2=-\frac{4Gm^2c}{\hbar}
\end{aligned}$$

Considering $m=m_p$ as the modified Planck mass, we have:

$$c^2=-4G\frac{i}{2}\sqrt{\frac{\hbar c}{G}}\frac{i}{2}\sqrt{\frac{\hbar c}{G}}\frac{c}{\hbar} \quad (247)$$

By rearranging we arrive at the most familiar form of the gravitational potential:

$$\begin{aligned}
c^2 &=-2Gi\sqrt{\frac{\hbar c}{G}}\frac{i}{2}\sqrt{\frac{c^3}{\hbar G}} \\
&=-2G\frac{i}{2}\sqrt{\frac{\hbar c}{G}}i\sqrt{\frac{c^3}{\hbar G}} \quad (248) \\
&=\frac{2Gm}{i\partial x}
\end{aligned}$$

Equating the Schwarzschild radius with the Comptom wavelength we make the following comparison:

Standard theory mass	Non-standard theory mass
$\frac{2Gm}{c^2}=\frac{\hbar}{mc}$ $m^2=\frac{\hbar c}{2G}$ $m=\sqrt{\frac{1}{2}\frac{\hbar c}{G}}$	$\frac{2Gm}{ic^2}=\frac{i\hbar}{2mc}$ $m^2=\frac{i^2\hbar c}{4G}$ $m=\frac{i}{2}\sqrt{\frac{\hbar c}{G}}$

Now in the same way we use the Compton frequency to designate the temporal part, starting from:

$$\begin{aligned}
&\frac{1}{c}\lambda=\frac{1}{c}\frac{h}{mc}\\
&\frac{1}{f}=\frac{h}{mc^2}\\
&\frac{i}{4\pi f}=\frac{ih}{4\pi mc^2}\\
&\frac{\partial t}{\partial}=\frac{i\hbar}{2mc^2}\\
&2mc^2=i\hbar\frac{\partial}{\partial t}
\end{aligned}\qquad(249)$$

Scaling to Planck units and squaring:

$$\begin{aligned}
&\frac{\partial}{\partial t}=\frac{2mc^2}{i\hbar}\\
&\sqrt{\frac{c^5}{\hbar G}}=\frac{2mc^2}{i\hbar}\\
&\frac{c^5}{\hbar G}=-\frac{4m^2c^4}{\hbar^2}\\
&\frac{c^5}{\hbar G}=-\frac{4c^4}{\hbar^2}\left(\frac{i}{2}\sqrt{\frac{\hbar c}{G}}\right)^2\\
&\frac{c^5}{\hbar G}=-\frac{4c^4}{\hbar^2}\frac{i^2}{4}\frac{\hbar c}{G}\\
&\frac{c^5}{\hbar G}=\frac{c^5}{\hbar G}
\end{aligned}\qquad(250)$$

An important observation to point out is how the Compton wavelength is present in the mass particle equations of standard quantum mechanics, this gives us a very important clue to build the quantum gravity equations, using the reduced form of the wavelength we have the three main equations, the first is the relativistic Klein-Gordon equation for a free particle:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \left(\frac{mc}{\hbar}\right)^2\psi \quad (251)$$

The Dirac equation:

$$i\gamma^{\mu}\partial_{\mu}\psi - \left(\frac{mc}{\hbar}\right)\psi = 0 \quad (252)$$

In the Schrödinger equation it is more difficult to visualize this directly, so let's make some modifications, using the equation for an electron in a hydrogen-like atom:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}\psi \quad (253)$$

Dividing both sides by hc and rewriting in terms of the fine structure constant we get:

$$\frac{i}{c}\frac{\partial}{\partial t}\psi = -\frac{1}{2}\left(\frac{\hbar}{mc}\right)\nabla^2\psi - \frac{\alpha Z}{r}\psi \quad (254)$$

Using the momentum operator as a basis we easily arrive at the Schrödinger equation for a free particle:

$$\begin{aligned}
&i\hbar\frac{\partial}{\partial x} = 2mc \\
&\frac{i\hbar}{2m}\frac{\partial}{\partial x} = c \\
&\left(\frac{i\hbar}{2m}\frac{\partial}{\partial x}\right)i\hbar\frac{\partial}{\partial x} = (c)i\hbar\frac{\partial}{\partial x} \quad (255) \\
&-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = i\hbar c\frac{\partial}{\partial x} \\
&-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = i\hbar\frac{\partial}{\partial t}
\end{aligned}$$

We then have the central idea of this article in the following relation:

Modified Compton wavelength	Gravitational wavelength
$\frac{i\hbar}{2mc}$	$\frac{2Gm}{ic^2}$

This suggests that we can unify quantum mechanics with gravity on the Planck scale, for example, starting with the quantum scale length:

$$\frac{\partial}{\partial x} = \frac{2mc}{i\hbar} \quad (256)$$

Assuming the quantities are scaled in Planck units and multiplying both sides by the Planck length squared we get the gravitational wavelength:

$$\begin{aligned} \frac{\partial}{\partial x}\frac{\partial x^2}{\partial^2} &= \frac{2mc}{i\hbar}\left(\frac{\hbar G}{c^3}\right) \\ \frac{\partial x}{\partial} &= \frac{2Gm}{ic^2} \end{aligned} \quad (257)$$

This is the key piece we need, from which we can obtain the quantum gravity operators:

Quantum Mechanics	Quantum Gravity?
$2mc = i\hbar\frac{\partial}{\partial x}$	$\frac{c^2}{2m} = -iG\frac{\partial}{\partial x}$
$2mc^2 = i\hbar\frac{\partial}{\partial t}$	$\frac{c^3}{2m} = -iG\frac{\partial}{\partial t}$

Let's extract the Schrödinger equation for the free particle from the gravitational wavelength, so we have:

$$\begin{aligned} \frac{\partial x}{\partial} &= \frac{2Gm}{ic^2} \\ \frac{i}{2m} &= \frac{G}{c^2}\frac{\partial}{\partial x} \end{aligned} \quad (258)$$

Multiplying both sides by l_p^{-2}:

$$\begin{aligned}\frac{i}{2m}\frac{\partial^2}{\partial x^2} &= \frac{G}{c^2}\frac{\partial}{\partial x}\left(\frac{c^3}{\hbar G}\right) \\ \frac{i}{2m}\frac{\partial^2}{\partial x^2} &= \frac{c}{\hbar}\frac{\partial}{\partial x} \\ \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial t}\end{aligned} \tag{259}$$

Finally we multiply both sides by $i\hbar\psi$, so we get:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = i\hbar\frac{\partial}{\partial t}\psi \tag{260}$$

The complete equation can be extracted from the relativity line element, so we start with:

$$\begin{aligned}\partial s^2 &= c^2\partial t^2 - \partial x^2 - \partial y^2 - \partial z^2 + \frac{\hbar G}{c^3} \\ \frac{\partial s^2}{c^2\partial t^2} &= 1 - \left(\frac{\partial x^2}{c^2\partial t^2} - \frac{\partial y^2}{c^2\partial t^2} - \frac{\partial z^2}{c^2\partial t^2}\right) + \frac{\hbar G}{c^3}\frac{\partial^2}{c^2\partial t^2} \\ \frac{\partial s^2}{c^2\partial t^2} &= 1 - \frac{v^2}{c^2} + \frac{\hbar G}{c^5}\frac{c^5}{\hbar G} \\ \frac{\partial s^2}{c^2\partial t^2} &= 1 - \frac{v^2}{c^2} + 1 \\ \frac{\partial s^2}{c^2\partial t^2} &= 2 - \frac{v^2}{c^2}\end{aligned} \tag{261}$$

Making $v^2 = i2Gm/\partial x$, we have:

$$\begin{aligned}\frac{\partial s^2}{c^2\partial t^2} &= 2 - i\frac{2Gm}{c^2\partial x} \\ \frac{1}{2}\frac{\partial s^2}{c^2\partial t^2} &= 1 - i\frac{Gm}{c^2\partial x} \\ \frac{1}{2}\frac{\partial s^2}{c^2\partial t^2} - 1 &= -i\frac{Gm}{c^2\partial x}\end{aligned} \tag{262}$$

Multiplying everything by l_p^2, and considering $\partial s = l_p$, we are left with:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial s^2}{c^2\partial t^2}\left(\frac{c^3}{\hbar G}\right)-\left(\frac{c^3}{\hbar G}\right)=-i\frac{Gm}{c^2\partial x}\left(\frac{c^3}{\hbar G}\right)\\
&\frac{1}{2}\frac{\partial^2}{c^2\partial t^2}-\frac{\partial^2}{\partial x^2}=-i\frac{mc}{\hbar}\frac{\partial}{\partial x}\\
&\frac{1}{2}\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x^2}=-i\frac{mc}{\hbar}\frac{\partial}{\partial x}\\
&\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}-\frac{i\hbar}{m}\frac{\partial^2}{\partial x^2}=c\frac{\partial}{\partial x}
\end{aligned}
\qquad (263)
$$

Finally we multiply everything by $i\hbar\psi$:

$$
\begin{aligned}
&-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi+\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2}\psi=i\hbar\frac{\partial}{\partial t}\psi\\
&-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi+V(x)\psi=i\hbar\frac{\partial}{\partial t}\psi
\end{aligned}
\qquad (264)
$$

The expression in (262) is nothing more than the classical lagrangian for a gravitational potential, ignoring the imaginary term, rearranging and multiplying by mc^2 we have:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial s^2}{c^2\partial t^2}+\frac{Gm}{c^2\partial x}=1\\
&\left(\frac{1}{2}\frac{v^2}{c^2}+\frac{Gm}{c^2\partial x}\right)mc^2=(1)mc^2\\
&\frac{1}{2}mv^2+\frac{Gm^2}{\partial x}=mc^2
\end{aligned}
\qquad (265)
$$

Or solving for $v=c$ we get back to the expression (262):

$$
\begin{aligned}
&\frac{1}{2}\frac{v^2}{c^2}-1=-\frac{Gm}{c^2\partial x}\\
&\frac{1}{2}v^2-c^2=-\frac{Gm}{\partial x}\\
&-\frac{1}{2}c^2=-\frac{Gm}{\partial x}
\end{aligned}
\qquad (266)
$$

Comparing gravity operators with the expression in (266) it is evident that gravity is dual to quantum momentum, a particle with a vector k that crosses a Planck area forms a volume $k\cdot l_p^2$, this is simply Stokes' theorem for a flow k in an area l_p^2. The opposite is also valid for a flow $k*$ in an area $1/l_p^2$. From there we get:

$$\iint\limits_S d\omega = \int\limits_{\partial S} \omega$$
$$\frac{2mc}{i\hbar}\frac{\partial x^2}{\partial^2} = \frac{2Gm}{ic^2} \quad (267)$$
$$\frac{ic^2}{2Gm}\frac{\partial x^2}{\partial^2} = \frac{i\hbar}{2mc}$$

And analogously we have the commutation relations between position $\hat{x}$ and $\hat{p}*$ momentum:

$$\begin{aligned}
[\hat{p}*,\hat{x}]\Psi &= (\hat{p}*\hat{x} - \hat{x}\hat{p}*)\Psi \\
&= -iG\frac{\partial}{\partial x}\hat{x}\Psi + iG\hat{x}\frac{\partial}{\partial x}\Psi \\
&= -iG\Psi\frac{\partial}{\partial x}\hat{x} - iG\hat{x}\frac{\partial}{\partial x}\Psi + \hat{x}iG\frac{\partial}{\partial x}\Psi \\
&= -iG\Psi\frac{\partial}{\partial x}\hat{x} \\
&= -iG\Psi
\end{aligned}
\qquad
\begin{aligned}
[\hat{x},\hat{p}*]\psi &= (\hat{x}\hat{p}* - \hat{p}*\hat{x})\psi \\
&= -iG\hat{x}\frac{\partial}{\partial x}\psi + iG\frac{\partial}{\partial x}\hat{x}\psi \\
&= -iG\hat{x}\frac{\partial}{\partial x}\psi + iG\psi\frac{\partial}{\partial x}\hat{x} + iG\hat{x}\frac{\partial}{\partial x}\psi \\
&= iG\psi\frac{\partial}{\partial x}\hat{x} \\
&= iG\psi
\end{aligned}$$

$$(268)$$

Using the dual operators we get the similar version of Schrödinger's equation for the free particle:

$$\begin{aligned}
\frac{i}{G}\frac{\partial t}{\partial}\psi &= -\frac{c}{2G^2 m}\frac{\partial x^2}{\partial^2}\psi \\
-iG\frac{\partial}{\partial t}\psi* &= -\frac{2G^2 m}{c}\frac{\partial^2}{\partial x^2}\psi*
\end{aligned} \quad (269)$$

And also the similar version of the free particle Dirac equation:

$$\left(\frac{i}{G}\gamma_v\partial^v - \frac{2m}{c^2}\right)\psi = 0 \quad (270)$$

It seems more prudent to treat the dual operators as n-forms, so:

$$\frac{i}{G}\frac{\partial x}{\partial}=\frac{2m}{c^2} \quad (271)$$

We can also derive the equation that combines the standard operators with the dual operators, so we have:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi &= \frac{-\hbar^2}{2m}\nabla^2\psi \\ \frac{\partial}{\partial t}\psi &= i\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi \qquad (272) \\ \frac{\partial}{\partial t}c^2\psi &= i\frac{\hbar}{2m}\frac{\partial^2}{\partial t^2}\psi \end{aligned}$$

$$\begin{aligned} c^2\psi &= i\frac{\hbar}{2m}\frac{\partial}{\partial t}\psi \\ -i\frac{2Gm}{\partial x}\psi &= i\frac{\hbar}{2m}\frac{\partial}{\partial t}\psi \\ -iGm^2\frac{\partial}{\partial x}\psi &= \frac{i\hbar}{4}\frac{\partial}{\partial t}\psi \end{aligned}$$

A variation of the above equation is:

$$\begin{aligned} 2mc^2\psi &= i\hbar\frac{\partial}{\partial t}\psi \\ 2m\frac{2Gm}{i\partial x}\psi &= i\hbar\frac{\partial}{\partial t}\psi \quad (273) \\ -i\frac{4Gm^2}{\partial x}\psi &= i\hbar\frac{\partial}{\partial t}\psi \end{aligned}$$

Multiplying both sides by $1/c$ we get the momentum:

$$-i\frac{4Gm^2}{c\partial x}\psi = i\hbar\frac{\partial}{\partial x}\psi \quad (274)$$

Rearranging according to the standard equation we have:

$$\begin{aligned} -i\frac{4Gm^2}{\partial x}\psi &= \frac{\hat{p}^2}{2m}\psi \\ -i\frac{4Gm^2}{\partial x}\psi &= \frac{1}{2m}\left(-i\frac{4Gm^2}{c\partial x}\right)^2\psi \end{aligned} \qquad (275)$$

$$
\begin{aligned}
-i\frac{4Gm^2}{\partial x}\psi &= -\frac{16G^2m^4}{2mc^2}\frac{\partial^2}{\partial x^2}\psi \\
-i\frac{4Gm^2}{\partial x}\psi &= -\frac{8G^2m^3}{c^2}\frac{\partial^2}{\partial x^2}\psi
\end{aligned}
\quad (276)
$$

Quantum Mechanics	Dual Quantum Mechanics
$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi$	$\frac{i}{G}\frac{\partial t}{\partial}\psi = -\frac{c}{2G^2m}\nabla^{-2}\psi + V(x)*\psi$
$\gamma^{\mu}\partial_{\mu}\psi = \frac{mc}{i\hbar}\psi$	$\gamma_{\upsilon}\partial^{\upsilon}\psi = \frac{Gm}{ic^2}\psi$
$\eta^{\mu\upsilon}\partial_{\mu}\partial_{\upsilon}\psi = -\frac{m^2c^2}{\hbar^2}\psi$	$\eta_{\mu\upsilon}\partial^{\mu}\partial^{\upsilon}\psi = -\frac{G^2m^2}{c^4}\psi$

The Dirac Delta Function

The Dirac delta function, often denoted as $\delta(x)$, is a fundamental mathematical concept widely used in various areas of mathematics and physics to represent localized distributions and impulses. While not a conventional function in the classical sense, the Dirac delta function plays a crucial role in modeling point sources and distributions with infinite density at a single point.

The Dirac delta function is formally defined by its integral properties, notably:

$$\int_{-\infty}^{\infty}\delta(x)dx = 1, \quad (277)$$

and for any suitably smooth function $f(x)$:

$$\int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx = f(x_0). \quad (278)$$

The Dirac delta function can also be described using the limit of a sequence of functions. A common example is:

$$\delta(x) = \lim_{a \to 0} \frac{1}{\sqrt{\pi a}} e^{-\frac{x^2}{a^2}}. \quad (279)$$

When used in integrals, the Dirac delta function effectively "picks out" the value of the function it's multiplied with at the point where the argument of the delta function is zero. This property makes it a powerful tool in solving differential equations and representing physical phenomena.

In more formal terms, the Dirac delta function can be defined as follows:

$$\delta(x - x_0) = \begin{cases} \infty, & \text{if } x = x_0, \\ 0, & \text{if } x \neq x_0. \end{cases} \quad (280)$$

with the constraint:

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1. \quad (281)$$

The Dirac delta function's properties extend beyond single dimensions; for multiple dimensions, the Dirac delta function $\delta(\mathbf{r} - \mathbf{r}_0)$ operates in a similar manner, localizing distributions in space.

Applications of the Dirac delta function abound in fields like quantum mechanics, signal processing, and general relativity. In quantum mechanics, it's used to describe wavefunctions at specific points, and in signal processing, it models impulses in continuous-time signals.

Deriving gravitation

Finally we can find approximations of the weak field using the model proposed so far. Let's start by deriving the gravitational version of Poisson's equation from the Green's function associated with the Laplace operator in three-dimensional space[97].

We start by introducing the Green's function $G(\mathbf{r}, \mathbf{r'})$, which satisfies the Laplace equation in $\mathbb{R}^3$:

$$\nabla^2 G(\mathbf{r},\mathbf{r'}) = -\delta(\mathbf{r}-\mathbf{r'}), \quad (282)$$

where $\delta(\mathbf{r}-\mathbf{r'})$ is the three-dimensional Dirac delta function. The solution for $G(\mathbf{r},\mathbf{r'})$ can be written as:

$$G(\mathbf{r},\mathbf{r'}) = \frac{1}{|\mathbf{r}-\mathbf{r'}|}. \quad (283)$$

Now, the gravitational potential $\boldsymbol{\Phi}$ generated by a mass density $\rho(\mathbf{r'})$ is given by:

$$\boldsymbol{\Phi}(\mathbf{r}) = -G\int \frac{\rho(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} d^3r'. \quad (284)$$

Let's calculate the Laplacian of the potential $\boldsymbol{\Phi}$:

$$\nabla^2 \boldsymbol{\Phi}(\mathbf{r}) = -G\int \rho(\mathbf{r'}) \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r'}|}\right) d^3r'. \quad (285)$$

Calculate the Laplacian of the Green's function:

$$\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r'}|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r'}). \quad (286)$$

Replace the previous result into the equation from (285):

$$\nabla^2 \boldsymbol{\Phi}(\mathbf{r}) = 4\pi G\int \rho(\mathbf{r'})\delta(\mathbf{r}-\mathbf{r'})\, d^3r' = 4\pi G\rho(\mathbf{r}). \quad (287)$$

We arrive at the gravitational Poisson equation:

$$\nabla^2 \boldsymbol{\Phi}(\mathbf{r}) = 4\pi G\rho(\mathbf{r}). \quad (288)$$

This equation relates the gravitational potential $\boldsymbol{\Phi}$ to the mass density ρ, describing how the mass distribution generates the gravitational field around it.

Now let's turn the non-relativistic quantum equation of the free particle into a gravitational equation, starting with:

$$i\hbar\frac{\partial}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \quad (289)$$

Converting the right hand differential operator into Planck units we get:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}&=-\frac{\hbar^2}{2m}\left(\frac{c^3}{\hbar G}\right) \\ i\hbar\frac{\partial}{\partial t}&=-\frac{\hbar c^3}{2Gm} \end{aligned} \quad (290)$$

Using the relation $dx=cdt$, and rearranging:

$$\begin{aligned} i\hbar c\frac{\partial}{\partial x}&=-\frac{\hbar c^3}{2Gm} \\ i\frac{\partial}{\partial x}&=-\frac{c^2}{2Gm} \\ -i\frac{\partial}{\partial x}&=\frac{c^2}{2Gm} \\ \frac{\partial}{\partial x}&=\frac{ic^2}{2Gm} \\ c^2&=\frac{2Gm}{i\partial x} \end{aligned} \quad (291)$$

Using the Sciama potential concept we can associate ϕ with c^2, then in modified form and ignoring the imaginary unit:

$$\begin{aligned} -\frac{c^2}{2}&=-\frac{Gm}{\partial x} \\ -\frac{c^2}{2}&=\phi \end{aligned} \quad (292)$$

Through the dimensional analysis of Planck's units we then associate $1/|\boldsymbol{r}-\boldsymbol{r'}|$ with $\partial/\partial x$ and $\delta(\boldsymbol{r}-\boldsymbol{r'})$ with $(\partial/\partial x)^3$, the modified form of (286) is:

$$\begin{aligned} \nabla^2\frac{\partial}{\partial x}&=-4\pi\frac{\partial^3}{\partial x^3} \\ \left(-4\pi\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}&=-4\pi\frac{\partial^3}{\partial x^3} \end{aligned} \quad (293)$$

Multiplying both sides of (292) by ∇^2 we get:

$$\begin{aligned}&\nabla^2\left(-\frac{c^2}{2}\right)=-4\pi\frac{\partial^2}{\partial x^2}\frac{Gm}{\partial x}\\&\nabla^2\phi=4\pi G\rho\end{aligned}\quad(294)$$

Using the $k*$ operator we can make the following approximation:

$$\begin{aligned}&\frac{i}{G}\frac{\partial x}{\partial}=\frac{2m}{c^2}\\&\frac{i}{G}\frac{\partial x}{\partial}c^4=2mc^2\\&i4\pi\frac{\partial^2}{\partial x^2}c^4=8\pi\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x}Gmc^2\\&i4\pi\frac{\partial^2}{\partial x^2}=\frac{8\pi G}{c^4}\rho c^2\end{aligned}\quad(295)$$

What is quite familiar with the Einstein field equation, the most complete form can be calculated from the line element ds^2, so:

$$\begin{aligned}&ds^2=c^2dt^2-dx^2-dy^2-dz^2+\frac{\hbar G}{c^3}\\&\frac{ds^2}{c^2dt^2}=1-\frac{(dx^2-dy^2-dz^2)}{c^2dt^2}+\frac{\hbar G}{c^3}\frac{d^2}{c^2dt^2}\\&\frac{ds^2}{c^2dt^2}=2-\frac{v^2}{c^2}\\&\frac{ds^2}{c^2dt^2}=2-\frac{2Gm}{c^2\partial x}\\&\frac{1}{2}\frac{ds^2}{c^2dt^2}-1=-\frac{Gm}{c^2\partial x}\\&\left(\frac{1}{2}\frac{ds^2}{c^2dt^2}-1\right)c^2=-\frac{Gm}{\partial x}\end{aligned}\quad(296)$$

Multiplying both sides by $-2\nabla^2c^2$ we get:

$$\begin{aligned}&-\left(\frac{1}{2}\frac{ds^2}{c^2dt^2}-1\right)2\nabla^2c^4=2\nabla^2\frac{Gm}{\partial x}c^2\\&(2\nabla^2)-\frac{1}{2}\left(2\nabla^2\frac{ds^2}{c^2dt^2}\right)=\frac{8\pi G}{c^4}\rho c^2\end{aligned}\quad(297)$$

Or otherwise we have:

$$
\begin{aligned}
&\frac{1}{2}\frac{ds^2}{c^2dt^2}c^2-c^2=-\frac{Gm}{\partial x}\\
&\frac{1}{2}\frac{ds^2}{c^2dt^2}\frac{2Gm}{\partial x}-\frac{2Gm}{\partial x}=-\frac{Gm}{\partial x}\\
&\frac{1}{2}\left(\frac{ds^2}{c^2dt^2}\right)\nabla^2\left(\frac{2Gm}{\partial x}\right)c^2-\nabla^2\left(\frac{2Gm}{\partial x}\right)c^2=-\nabla^2\left(\frac{Gm}{\partial x}\right)c^2\\
&\frac{1}{2}(g_{00})(8\pi G\rho c^2)-(8\pi G\rho c^2)=-4\pi G\rho c^2\\
&\frac{8\pi G}{c^4}\left[(\rho c^2)-\frac{1}{2}(g_{00})(\rho c^2)\right]=\frac{4\pi G\rho c^2}{c^4}\\
&\varkappa\left[T_{00}-\frac{1}{2}g_{00}T\right]=\frac{1}{2}\varkappa\rho c^2\simeq R_{00}
\end{aligned}
\qquad (298)
$$

And using that we arrive at the gravitation operator:

$$
\begin{aligned}
&R_{00}\simeq\frac{4\pi G\rho}{c^2}\\
&R_{00}\simeq\frac{4\pi Gm}{c^2}\frac{\partial^3}{\partial x^3}\\
&R_{00}\simeq 2\pi\left(\frac{2Gm}{c^2}\right)\frac{\partial^3}{\partial x^3}\\
&R_{00}\simeq 2\pi\left(\frac{\partial_x}{\partial}\right)\frac{\partial^3}{\partial x^3}\\
&R_{00}\simeq 2\pi\frac{\partial^2}{\partial x^2}\\
&2\pi\frac{\partial^2}{\partial x^2}\frac{\partial x^3}{\partial^3}\simeq 2\pi\left(\frac{2Gm}{c^2}\right)\\
&\frac{\partial x}{\partial}\simeq\frac{2Gm}{c^2}
\end{aligned}
\qquad (299)
$$

The geodesic equation, describing the motion of a free particle on curved spacetime with metric g, can be obtained from the action principle.

$$S[x,\dot{x}]=-mc\int dt\sqrt{-g_{\mu\upsilon}\dot{x}^{\mu}\dot{x}^{\upsilon}} \quad (300)$$

And the metric is:

$$g = \begin{pmatrix} -1 - \frac{2\phi(x)}{c^2} - \frac{\hbar G}{c^5 dt^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (301)$$

the action becomes:

$$S[x,\dot{x}] = -mc \int dt \sqrt{c^2 + 2\phi(x) - \frac{dx^2}{dt^2} + \frac{\hbar G}{c^3 dt^2}} \quad (302)$$

Expansion of the root we have:

$$\begin{aligned} S[x,\dot{x}] &= -mc \int dt \left(\sqrt{c^2} + \frac{1}{2\sqrt{c^2}} \left(2\phi(x) - \frac{dx^2}{dt^2} + \frac{\hbar G}{c^3 dt^2} \right) \right) \\ &= \int dt \left(-mc^2 + \frac{1}{2} mv^2 - m\phi(x) + m\frac{\hbar G}{c^3 dt^2} \right) \\ &= \int dt \left(-mc^2 + \frac{1}{2} mv^2 - m\phi(x) + mc^2 \right) \\ &= \int dt \left(\frac{1}{2} mv^2 - m\phi(x) \right) \end{aligned} \quad (303)$$

which is the newtonian action for the gravitational potential.

Modifying the Einstein–Hilbert action with the Sciama potential we get:

$$\begin{aligned} S &= \frac{c^4}{16\pi G} \int R\sqrt{-g}\, d^4x \\ &= \frac{1}{16\pi G} \left(\frac{2Gm}{i\partial x} \right)^2 \int R\sqrt{-g}\, d^4x \\ &= -\frac{1}{16\pi G} \frac{4G^2 m^2}{\partial x^2} \int R\sqrt{-g}\, d^4x \\ &= -\frac{Gm^2}{4\pi} \frac{\partial^2}{\partial x^2} \int R\sqrt{-g}\, d^4x \end{aligned} \quad (304)$$

Expanding the Planck mass term:

$$
\begin{aligned}
&= -\frac{G}{4\pi}\left(\frac{i}{2}\sqrt{\frac{\hbar c}{G}}\right)^2 \frac{\partial^2}{\partial x^2}\int R\sqrt{-g}\,d^4x \\
&= \frac{\hbar c}{16\pi}\frac{\partial^2}{\partial x^2}\int R\sqrt{-g}\,d^4x
\end{aligned} \quad (305)
$$

Replacing $\partial^2/\partial x^2$ with l_p^{-2}:

$$
\begin{aligned}
&= \frac{\hbar c}{16\pi}\left(\frac{c^3}{\hbar G}\right)\int R\sqrt{-g}\,d^4x \\
&= \frac{c^4}{16\pi G}\int R\sqrt{-g}\,d^4x
\end{aligned} \quad (306)
$$

And we return the original expression.

The cosmological constant

The cosmological constant Λ, introduced by Albert Einstein in his field equations of general relativity[98], has been a subject of intense study and debate in cosmology since its inception. It represents a constant energy density that is associated with space itself, often referred to as “dark energy.” The cosmological constant plays a crucial role in our understanding of the large-scale structure and expansion of the universe.

Einstein originally introduced the cosmological constant in his equations as a means to maintain a static universe, a view held at the time. His field equations, with the cosmological constant included, can be expressed as:

$$
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (307)
$$

However, when Edwin Hubble’s observational evidence showed that the universe was expanding[99], Einstein famously referred to the cosmological constant as his “greatest blunder” and removed it from his equations. It wasn’t until later observations, particularly those related to the accelerating expansion of the universe, that the concept of dark energy[100] was reintroduced to explain these phenomena. The cosmological constant, represented by Λ, was once again included in the equations as a term associated with this dark energy.

Recent cosmological observations, such as those from the Cosmic Microwave Background (CMB) and Type Ia supernovae[101], have provided strong evidence for the existence of dark energy and the non-zero value of the cosmological constant. The current consensus is that approximately 68% of the total energy content of the universe is in the form of dark energy and that the current rate of expansion is around 67.37 km/s/Mpc, with Λ being the simplest explanation for its behavior.

The most widely accepted framework for describing our universe's expansion is the Lambda Cold Dark Matter (ΛCDM) model. In this model, the Hubble parameter (H_0) is used to characterize the current rate of expansion. The Friedmann equation, modified to include the cosmological constant, can be expressed as:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \quad (308)$$

Where:
- H is the Hubble parameter.
- $\dot{a}/a$ represents the relative rate of expansion.
- G is the gravitational constant.
- ρ is the energy density of matter.
- k is the curvature of space.
- c is the speed of light.

Let's derive the Newtonian limit from Einstein's equation, including the cosmological constant. We will begin with the complete equation of general relativity, which includes the cosmological constant Λ. The full Einstein field equation is:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (309)$$

We will make an approximation for the Newtonian regime, where velocities are much smaller than the speed of light (c) and gravitational fields are weak. This allows us to use an approximately flat metric, where $g_{\mu\nu} \approx \eta_{\mu\nu}$, with $\eta_{\mu\nu}$ being the standard Minkowski metric.

Now, we assume that the gravitational field is weak, which means we can write $g_{00} \approx -1 + 2\phi/c^2$ and $g_{ij} \approx \delta_{ij}$, where ϕ is the gravitational potential and δ_{ij} is the Kronecker delta (flat space approximation).

We substitute the approximate metric and the energy-momentum tensor into the Einstein field equation:

$$\left(\partial^2\phi - \frac{1}{c^2}\nabla^2\phi\right) - \Lambda\left(-1 + 2\phi/c^2\right) = 8\pi G T_{00} \quad (310)$$

Let's consider only the T_{00} term, which describes the energy density since we are mainly interested in Newtonian gravity. We assume that the energy density is dominated by mass density ρ, so $T_{00} \approx \rho c^2$.

Finally, we obtain the modified Newtonian Poisson equation that describes Newtonian gravity with the cosmological constant:

$$\nabla^2\phi = 4\pi G\rho + \frac{\Lambda c^2}{3}, \quad (311)$$

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} + \frac{c^2}{3}\Lambda r. \quad (312)$$

This is the field equation for modified Newtonian gravity in the presence of the cosmological constant. Please note that the second term on the right is the contribution of the cosmological constant to gravity on cosmological scales, such as the expansion of the universe.

Then eq. (307) can be rewritten as: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - T_{\mu\nu}^{vacuum})$,

with: $T_{\mu\nu}^{vacuum} = \frac{c^4\Lambda}{8\pi G}g_{\mu\nu}$, we would conclude that the energy density of the vacuum is[102]:

$$\rho_{vac}c^2 = \frac{c^4\Lambda}{8\pi G}, \quad (313)$$

therefore:

$$\Lambda = \frac{8\pi G}{c^2}\rho_{vac}. \quad (314)$$

Astrophysical data suggests the possibility of a nonzero cosmological constant, with supporting details available in K.A. Olive *et al.* [Particle Data Group Collaboration], *Review of Particle Physics*, Chin. Phys. C **38**, 090001 (2014). The review contains relevant entries[103]:

$$\begin{aligned} &\frac{c^2}{3H_0^2} = 6.3 \pm 0.2 \times 10^{51} m^2, \\ &\Omega_\Lambda = 0.685^{+0.017}_{-0.016}, \end{aligned} \quad (315)$$

where H_0 is the present day Hubble parameter, the critical density today is given by $\rho_{c,0} = 3H_0^2/(8\pi G)$, hence:

$$\Omega_\Lambda = \frac{\rho_{vac}}{\rho_{c,0}} = \frac{\Lambda c^2}{3H_0^2}. \quad (316)$$

Organizing everything then we have:

$$\Lambda = \frac{3H_0^2}{c^2}\Omega_\Lambda = (1.09 \pm 0.04) \times 10^{-52} m^{-2}. \quad (317)$$

Using (317), we obtain:

$$\rho_{vac} = \frac{\Lambda c^2}{8\pi G} = \frac{(1.1\times 10^{-52} m^{-2})(3\times 10^8 m s^{-1})^2}{8\pi(6.673\times 10^{-11} m^3 kg^{-1} s^{-2})} = 5.9\times 10^{-27} kg m^{-3}. \quad (318)$$

So the numerical value of the vacuum energy is:

$$\rho_{vac}c^2 = 5.31 \times 10^{-10} J = 3.32\, GeV\, m^{-3}. \quad (319)$$

Converting to the appropriate units used in quantum mechanics we have:

$$\rho_{vac} c^2 = \frac{(2.24 \times 10^{-3}\, eV)^4}{(\hbar c)^3}. \quad (320)$$

These values play a crucial role in our understanding of the cosmos and have significant implications for the ultimate fate of the universe. They suggest that the expansion of the universe is accelerating, driven by the mysterious and dominant force of dark energy.

The cosmological constant problem

The crux of the cosmological constant problem[104] lies in the observed accelerated expansion of the universe, confirmed by various cosmological observations. This expansion contradicts the initially intended purpose of the cosmological constant in Einstein's equations, which was to maintain a static universe. To match the observed cosmic acceleration, Λ must have a particular value. However, the theoretical predictions of the vacuum energy density from quantum field theory are dramatically larger, on the order of the Planck scale, than what is consistent with observations.

This glaring disparity between the predicted vacuum energy density and its observed value constitutes the heart of the cosmological constant problem. It challenges our understanding of fundamental physics and the nature of empty space.

In quantum mechanics, the vacuum energy is not zero due to quantum fluctuations. This is because quantum fields behave like an infinite collection of harmonic oscillators, and the ground state energy of such oscillators is non-zero (Zero-point energy):

$$\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar(2\pi v) = \frac{1}{2}h\frac{c}{\lambda} = \frac{1}{2}hf. \quad (321)$$

While the sum of these ground state energies would theoretically be infinite, in practice, it's expected to be cut off at some energy scale, such as the Planck scale, due to our lack of knowledge about the fundamental theory at higher energies. Therefore, the "expected" value of the cosmological constant, which is related to vacuum energy, is uncertain and may be cut off at the Planck scale as a reasonable approximation.

According to quantum mechanics it was "predicted" that the vacuum energy density due to vacuum fluctuations should be approximately:

$$\begin{aligned}\rho_{vac}^{QM}c^2 &\sim \frac{m_p c^2}{l_p^3} = \sqrt{\frac{\hbar c}{G}}\left(\sqrt{\frac{c^3}{\hbar G}}\right)^3 c^2 = \sqrt{\frac{\hbar c^5}{G}}\sqrt{\frac{c^3}{\hbar G}}\left(\frac{c^3}{\hbar G}\right) = \frac{c^7}{\hbar G^2} \\ &= \frac{\left(\frac{\hbar c}{G}\right)^2 c^8}{(\hbar c)^3} = \frac{(m_p c^2)^4}{(\hbar c)^3}.\end{aligned} \tag{322}$$

This gives in numerical values:

$$\rho_{vac}^{QM}c^2 \sim \frac{(1.22\times 10^{19}\,GeV)^4}{(\hbar c)^3} = \frac{(1.22\times 10^{28}\,eV)^4}{(\hbar c)^3}. \tag{323}$$

Comparing this with the vacuum energy observed in (320):

$$\frac{\rho_{vac}c^2}{\rho_{vac}^{QM}c^2} = \left(\frac{2.24\times 10^{-3}\,eV}{1.22\times 10^{28}\,eV}\right)^4 = 1.13\times 10^{-123}. \tag{324}$$

The vacuum energy density predicted in physics is incredibly greater than that observed, about 10^{123} times greater, this is known as the worst prediction in the history of physics.

Speculations and conclusion

Using Planck units, the unification of quantum mechanics with gravity on such scales becomes evident. We observe that gravity is dualized with quantum momentum and vice versa. Gravity appears to operate in a space of forms or cotangents, while momentum operates in a space of the vector or tangent type. The Compton wavelength is the fundamental term in the main equations of quantum mechanics, its dual being the gravitational wavelength, equivalent to the Schwarzschild radius of General Relativity. We demonstrate that it is possible to derive the momentum and energy operators of gravity through the dimensional analysis of Planck units, from which we also derive the standard quantum operators. The space in which gravity acts appears to be covariant with the contravariant space of quantum mechanics. In the same way that modern theories try to explain quantum gravity using strings or spin foams, when analyzing the duality between quantum momentum and gravity we can deduce that space-time is composed of structures similar to lattices. Hypothetically these lattices would act together to form spacetime, they would be three-dimensional surfaces on a four-dimensional manifold, suggesting that particles could behave as wave functions in these networks at the Planck scale. The curvature of these lattices is determined by the metric, in Minkowski space these networks are flat. In the representation of a "particle" of the fermion type, the lattice wave function of the energy-momentum vector would have a negative curvature, whose flow would affect the gravitational lattice in a covariant way, resulting in a positive curvature, which we know as the manifestation of gravity. The particle mass can be thought of as the rate of longitudinal curvature of this lattice wave function, while the transverse curvatures correspond to the wave functions of the gauge bosons. When we analyze the symmetry of these networks, we can notice the existence of an opposite configuration in which the existence of a universe similar to ours is conceivable. In the modified special relativity we saw that when reaching the speed of light the relativistic mass does not reach infinity due to a correction in the Lorentz factor, which is the inclusion of the Planck mass squared, as it was said, the objective of this modification is to avoid that quantities lose their physical meaning. This modification gives rise to a strange possibility that there exists outside the light cone a different manifestation of mass/energy, this matter is known as tachyon type, or exotic matter. There seems to be a correlation between these lattices and Minkowski's hyperbolic geometry, the world lines inside the light cone are time-like while the lines outside are space-like, the lines in the cone are null or light-like, the causal configuration has a one-sheet hyperboloid for time (space intervals) and the two-sheet hyperboloid for space (time intervals), the opposite configuration is acausal or tachyon-like, this same symmetry can be seen in the lattice hypersurface geometry. Using the concept of modified relativistic momentum we have the following limit:

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2} + m_p^2}},$$
$$\lim_{v \to c} p = \lim_{v \to c} \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar c}{G}}} = \sqrt{\frac{G}{\hbar c}}\, m_0 c, \quad (325)$$

The inverse of the Planck mass multiplied by c becomes the Planck tachyon mass ($\bar{m}_p$):

$$\sqrt{\frac{Gc}{\hbar}}\, m_0 = \bar{m}_p m_0, \quad (326)$$

Considering m_0 as the Planck mass and reversing the equation we arrive at the following relativistic moment:

$$\bar{m}_p \sqrt{\frac{\hbar c}{G}}$$
$$\bar{m}_p c \sqrt{\frac{\hbar}{Gc}} \quad (327)$$
$$\frac{\bar{m}_p v}{i\sqrt{1 - \frac{v^2}{c^2} + \frac{Gc}{\hbar}}} \rightarrow \frac{\bar{m}_0 v}{i\sqrt{1 - \frac{v^2}{c^2} + \bar{m}_p^2}}$$

We conclude that this leads to the following symmetry:

Inside the cone of light (where we are)	Outside the cone of light	In both light cones
$m_p = \sqrt{\frac{\hbar c}{G}}$	$\bar{m}_p = \sqrt{\frac{Gc}{\hbar}}$	$m_p \times \bar{m}_p = c$

The main difference lies in the fact that the constant G would take the place of $\hbar$, inverting their roles. In this scenario, $\hbar$ would act as a gravitational constant, while G would play a role in quantum mechanics, a rather peculiar situation. Furthermore, it is important to note that this universe configuration would be of the tachyon type.

Mass in our universe	Mass in the hypothetical tachyon-like universe
$m = \frac{h}{\lambda c}$	$\bar{m} = \frac{G}{\lambda c}$
$m* = \frac{\lambda c^2}{G}$	$\bar{m}* = \frac{\lambda c^2}{h}$

The question that arises is whether this concept can offer any solution to the problem of dark matter/energy?

Using equations (n1) and (n2) and isolating Λ in both, we have the following equalities:

$$\Lambda = \frac{3}{c^2}\nabla^2\phi - \frac{12\pi G\rho}{c^2}, \quad (328)$$

$$\Lambda = \frac{3}{c^2}\frac{d^2}{dt^2} + \frac{3Gm}{c^2 r^3}, \quad (329)$$

Applying the transformations we used throughout this article to (329) we arrive at:

$$\begin{aligned} \Lambda &= 3\left(\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) \\ \Lambda &= \frac{9}{2}\left(\frac{\partial^2}{\partial x^2}\right). \end{aligned} \quad (330)$$

By dimensional analysis we can consider the operator $(\partial/\partial x)$ as the inverse of a quantity of length $(1/L)$ in meters, in non-rigoristic terms, this can be done by reversing the limit so that the infinitesimal dx approaches Δx, considering that the length is the diameter of the observable universe, like this we have:

$$\frac{9}{2}\left(\frac{1}{8.8 \times 10^{26} m}\right)^2 = 5.8109504132 \times 10^{-54} m^{-2}. \quad (331)$$

Which is a reasonable approximation to the cosmological constant.

Using the inverse of the Planck length or the Planck length itself we get a slightly smaller number:

$$\frac{9}{2}\left(\frac{1}{l_p^{-1}}\right)^2 = \frac{9}{2}(l_p)^2 = 1.1755261361\times10^{-69}m^{-2}. \quad (332)$$

It is still not clear the role of exotic matter in the value of the cosmological constant, we can guess using some approximations like:

$$\frac{8\pi G}{c^4}\frac{\bar{m}_p}{m_p}c^2(l_p)^3 = 2.1242333472\times10^{-54}. \quad (333)$$

Using the gravitational potential of exotic matter divided by the square of the speed of light we get:

$$\frac{8\pi G}{c^4}\left(\frac{2\hbar\bar{m}_p^2}{l_p\cdot c^2}\right) = \frac{8\pi G}{c^4}(2\bar{m}_p) = 5.7209466811\times10^{-27}. \quad (334)$$

Multiplying by $8\pi G/c^2$, we arrived at:

$$\left(\frac{8\pi G}{c^4}\right)^2(2\bar{m}_p c^2) = 1.0677561405\times10^{-52}. \quad (335)$$

Comparing with the result measured in (317) this seems to be a good approximation of Λ.

The three fundamental constants can be used to describe the supposed Planck scale lattice structure of the universe. Considering inside the light cone, in this context, $\hbar$ would represent the time-like momentum-energy lattice, while G would describe the space-like gravitational momentum-energy lattice. In turn, the constant c could be associated with an interaction with gauge bosons, which originate invariants such as mass and charge.

Length in our universe	$\lambda = \frac{h}{mc}$	$\lambda * \lambda = \frac{hG}{c^3}$
	$\lambda * = \frac{Gm}{c^2}$	
Length in the hypothetical tachyon-like universe	$\bar{\lambda} = \frac{G}{\bar{m}c}$	$\bar{\lambda} * \bar{\lambda} = \frac{hG}{c^3}$
	$\bar{\lambda} * = \frac{h\bar{m}}{c^2}$	

The ideas presented in this article may seem premature or obscure to some, and, going further, may be considered pseudoscience by more dogmatic academics. Furthermore, the energy scales involved are completely inaccessible to current measurement devices. The hope lies in improving these ideas, both theoretically and experimentally, in order to provide concrete answers to the challenges faced by modern physics.

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