Quick Tiling

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Abstract

In the first part, we tile the plane with k-gons for natural numbers k which have the rest three if we devide it by four. The proof is by pictures.
In a second part, we extend the result to all natural numbers larger than two. The foundation is the tiling of the plane by rectangles or hexagons. We use at most two different tiles for the covering.

We think that it is useful to repeat the definition of a *simple polygon*.
A simple polygon consists of k different points $(x_1, y_1), (x_2, y_2), \ldots (x_{k-1}, y_{k-1}), (x_k, y_k)$ called *vertices*, and the sets which connect the vertices called *edges*, where $k > 2$. The edges are straight lines. We define that it is homeomorphic to a circle. We demand that there are no three consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ for $1 \leq i \leq k-2$. Also we demand that the points $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ and $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ are not collinear.
We call this just described simple polygon a *k-gon*.

Theorem 1. Let k be a natural number such that $k \equiv 3 \mod 4$. There exists for all k a tiling of $\mathbb{R}^2$ with k-gons.

The set of natural numbers of k is $\{3, 7, 11, 15, \ldots\}$.

Proof. We prefer to show some pictures instead of a written proof. In Figure 1 we deal with the case $k = 7$. We show two tiles. Note that the tiles are equal. The square has a sidelength 2, the rectangle has measures of 1×2. In the next picture 'Figure 2' we show one tile of the case $k = 11$.
The other cases work in the same manner.
The theorem is proven. □

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Figure 1:

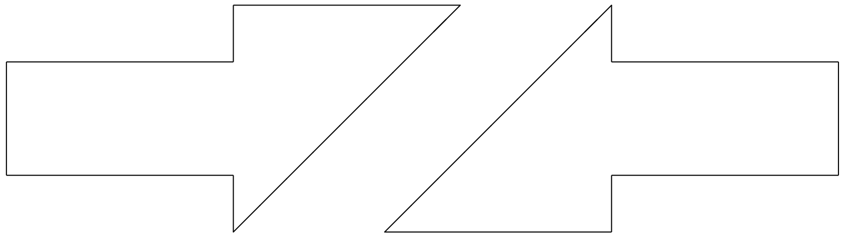
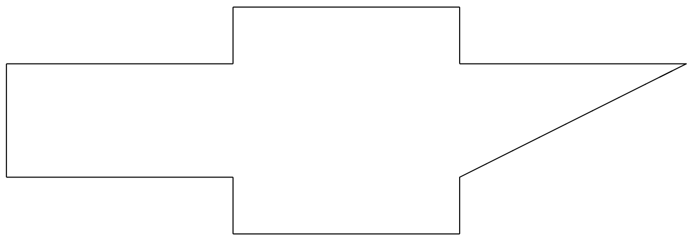


Figure 2:



In the second part, we get straight to the point.

Theorem 2. Let k be a natural number larger than two. There is a tiling of the plane with k-gons.

Remark 1. Theorem 2 includes Theorem 1.

Proof. This theorem is well-known. Please see [1], p. 11.
A second proof is yielded now: The cases $k = 3$ and $k = 4$ and $k = 6$ are known. For $k = 5$ please see the picture 'Figure 3'. There we show two possibilities. For $k = 7$ see Figure 4. Note that the first rectangle consists of two non-convex 7-gons, while the second rectangle consists of one convex 7-gon and one non-convex 7-gon. Further note that for even numbers larger than 4 only the idea of the right-hand rectangle of Figure 4 works.
The other cases work in the same manner. The theorem is shown.
There is another proof. Please see Figure 5. There we show the cases k = 6 and k = 7, respectively. The left hexagon works also for larger even numbers, while the right hexagon works for larger odd numbers.

□

Figure 3:

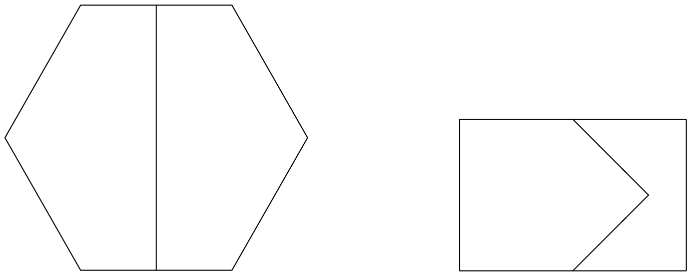
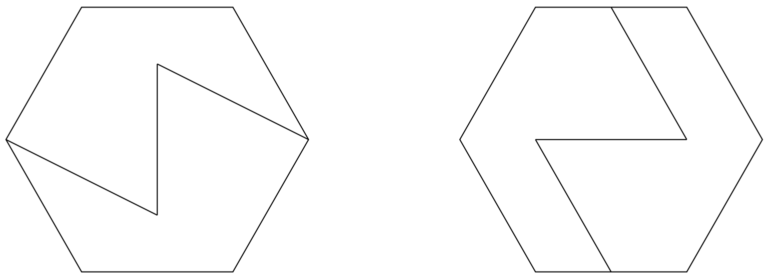


Figure 4:

Figure 5:



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