Double inequalities related to approximate formulas of Euler-Mascheroni constant with continued fraction

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ABSTRACT

In this paper, we present some new double inequalities starting from the approximate formula for Euler-Mascheroni constant the newly obtained by us.

Keywords: Euler-Mascheroni constant, Generalized Euler-Mascheroni constant, Continued fraction, Double inequality

1. Introduction

Double inequalities associated with approximate formulas of mathematical constants play an important role in solving various scientific and technological problems.

Let $a>0$. Then the generalized Euler–Mascheroni constant $\gamma(a)$ is given by

$$\gamma(a)=\lim_{n\to\infty}\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right). \tag{1.1}$$

When $a=1$, (1.1) represents the classical Euler-Mascheroni constant, that is,

$$\gamma=\gamma(1)=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)=0.5772156649\ 0115328\cdots.$$

Many mathematicians have focused on studying approximate formulas and inequalities associated with the Euler-Mascheroni constant and the generalized Euler-Mascheroni constant; see, for example, [1-3]. We enumerate some main results:

$$\frac{1}{24(n+1)^2}<\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)-\gamma<\frac{1}{24n^2},$$

$$\frac{1}{48(n+1)^3}<\gamma-\sum_{k=1}^{n}\frac{1}{k}+\ln\left(n+\frac{1}{2}+\frac{1}{24n}\right)<\frac{1}{48n^3},$$

$$\frac{\sqrt{6}}{144}\frac{1}{(n+1)^3}<\gamma-\sum_{k=1}^{n-1}\frac{1}{k}-\frac{1}{rn}+\ln n+\frac{a_1}{n+a_2}<\frac{\sqrt{6}}{144}\frac{1}{n^3},\ r=6-2\sqrt{6},a_1=\frac{2-r}{2r},a_2=\frac{r}{6(2-r)},$$

$$\frac{\sqrt{6}}{144}\frac{1}{(n+1)^3}<\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{rn}-\ln n-\frac{a_1}{n+a_2}-\gamma<\frac{\sqrt{6}}{144}\frac{1}{(n-1)^3},\ r=6+2\sqrt{6},a_1=\frac{2-r}{2r},a_2=\frac{r}{6(2-r)},$$

$$\frac{1}{120(n+1)^4}<\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{2n}-\ln n-\frac{b_1}{n^2}-\gamma<\frac{1}{120(n-1)^4},\ b_1=-\frac{1}{12}.$$

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In [4-6], the authors introduced the following double inequalities and the best possible constants.

$$\frac{1}{12(a+n-\alpha_1)^2} \le \gamma(a) - \sum_{k=1}^{n-1} \frac{1}{a+k-1} - \frac{1}{2(a+n-1)} + \ln\frac{a+n-1}{a} < \frac{1}{12(a+n-\beta_1)^2},$$

$$\frac{1}{24(a+n-\alpha_2)^2} \le \sum_{k=1}^{n} \frac{1}{a+k-1} - \ln\frac{a+n-1/2}{a} - \gamma(a) < \frac{1}{24(a+n-\beta_2)^2},$$

$$\frac{1}{48(a+n-\alpha_3)^3} \le \gamma(a) - \sum_{k=1}^{n-1} \frac{1}{a+k-1} - \frac{1}{2(a+n-1)} + \ln\left(\frac{a+n-1/2}{a} - \frac{1}{24(a+n-1)}\right) < \frac{1}{48(a+n-\beta_3)^3},$$

$$\frac{\alpha_4}{(a+n-1)^4} \le \gamma(a) - \sum_{k=1}^{n-1} \frac{1}{a+k-1} - \frac{1}{2(a+n-1)} + \ln\left(\frac{a+n-1/2}{a} - \frac{1}{24(a+n-1)}\right) < \frac{\beta_4}{(a+n-1)^4},$$

In this paper, we provide new inequalities concerning the Euler-Mascheroni constant.

2. Inequalities for the Euler-Mascheroni constant

Bell polynomials play important roles in our derivation, so we give the definition of Bell polynomials and related polynomials and give some properties of them. The exponential partial Bell polynomials are the polynomials $B_{n,k} := B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ in an infinite number of variables $x_1, x_2, \cdots$ defined by power series expansion

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \cdots.$$

Also, an alternative representation of Bell polynomials is

$$B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

the sum extending over all sequences $j_1, j_2, \cdots, j_{n-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \quad \text{and} \quad j_1 + 2j_2 + \cdots + (n-k+1) j_{n-k+1} = n.$$

Related to Bell polynomials are logarithmic type Bell polynomials, or logarithmic polynomials in short, $L_n := L_n(x_1, x_2, \cdots, x_n)$ defined by

$$\ln\left(1 + \sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = \sum_{n=1}^{\infty} L_n \frac{t^n}{n!}.$$

The Logarithmic polynomials can be expressed in Bell polynomials:

$$L_n(x_1, x_2, \cdots, x_n) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}),$$

where

$$L_1 = x_1,\ L_2 = x_2 - x_1^2,\ L_3 = x_3 - 3x_1x_2 + 2x_1^3,\ L_4 = x_4 - 4x_1x_3 + 12x_1^2x_2 - 3x_2^2 - 6x_1^4,\ \cdots.$$

The Bernoulli numbers B_k are defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}..$$

For reader's convenience, we record the first few terms of B_k .

$$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6,} B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \cdots, \ B_{2k+1} = 0 \ \ (k = 1, 2, \cdots).$$

The logarithmic derivative of the gamma function is called the psi or digamma function and is symbolized as $\psi(x)$, that is

$$\psi(x) = \frac{d}{dx}(\ln \Gamma(x)).$$

The psi function ψ has the recursive and asymptotic formulas as follows:

$$\begin{aligned} &\psi(x+1) = \psi(x) + \frac{1}{x}, \\ &\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \ (x \to \infty). \end{aligned} \tag{1.2}$$

We present an approximate formula for the generalized Euler-Mascheroni constant which contains the continued fraction term.

Theorem 1. For any fixed $l, s, b_1, b_2 \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, we have the following sequence convergent to the generalized Euler-Mascheroni constant.

$$\gamma_{n,l}^{s}(a) = \sum_{k=1}^{n-2} \frac{1}{a+k-1} + \frac{1}{b_2(a+n-2)} + \frac{1}{b_1(a+n-1)} - \ln\frac{a+n-1}{a} - \frac{1}{l}\ln\left(1 + \cfrac{c_1}{n+a-1+\ddots+\cfrac{c_s n}{n+a-1}}\right),$$

where

$$c_1 = \frac{l(2b_1 + 2b_2 - 3b_1 b_2)}{2b_1 b_2},$$

$$c_2 = \frac{3l(4b_1^2 + 8b_1 b_2 - 12b_1^2 b_2 + 4b_2^2 - 12b_1 b_2^2 + 9b_1^2 b_2^2) + 2b_1 b_2(12b_1 - 13b_1 b_2)}{12b_1 b_2(-2b_1 - 2b_2 + 3b_1 b_2)},$$

$\cdots$

To obtain double inequalities associated with this approximation formula, the following lemmas are necessary.

Lemma 1. (see [7]) Let $k \ge 1$ and $n \ge 0$ be integers. Then, for all real numbers $x > 0$:

$$S_k(2n; x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1; x), \tag{2.1}$$

where

$$S_k(p;x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left(B_{2i}\prod_{j=1}^{k-1}(2i+j)\right)\frac{1}{x^{2i+k}}.$$

B_i(i=0,1,2,…) are Bernoulli number. It follows from (2.1) that, for $x>0$,

$$\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}-\frac{1}{30x^9}<\psi'(x)<\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7},$$

from which it follows that

$$\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}-\frac{1}{30x^9}<\psi'(x+1)<\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}. \tag{2.2}$$

Lemma 2. (see [7]) It is also known that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}. \tag{2.3}$$

Lemma 3. Let

$$v(x) = \frac{288}{7}\left(-\psi(x+1)+\ln x+\ln\left(1+\frac{1/2}{x-1/12}\right)\right)..$$

Then, for $x \ge 16$,

$$(-v'(x))^3 < 27v^4(x). \tag{2.4}$$

In case of $b_1 = b_2 = 1$, we know that the fastest possible sequence is obtained only for $l = 1$ and $c_1 = 1/2, c_2 = 1/12.$. We have the following inequalities.

First, we give inequalities when $a = b_1 = b_2 = 1$.

Theorem 2. For $b_1 = b_2 = 1$ and all integers $n \ge 1$,

$$\frac{7}{288}\frac{1}{(n+\alpha_1)^3} \le \gamma(1) - \gamma_{n,1}^2(1) < \frac{7}{288}\frac{1}{(n+\beta_1)^3}, \tag{2.5}$$

with the best possible constants

$$\alpha_1 = \frac{1}{\sqrt[3]{\frac{288}{7}\left(-1+\gamma+\ln\frac{17}{11}\right)}} - 1 = 0.247025\cdots, \text{ and } \beta_1 = \frac{137}{630}.$$

The proof of Theorem 2 is similar to Theorem 3.

Second, we give inequalities, when $b_1 = b_2 = 1, a = 2$. In this case, we know that the fastest possible sequence is obtained only for $l = 1$ and $c_1 = 1/2, A_2 = -1/12.$.

Theorem 3. For $b_1 = b_2 = 1$ and all integers $n \ge 1$,

$$\frac{5}{288}\frac{1}{(n+1+\alpha_2)^3} \le \gamma_{n,1}^2(2) - \gamma(2) < \frac{5}{288}\frac{1}{(n+1+\beta_2)^3} \tag{2.6}$$

with the best possible constants

$$\alpha_2 = \frac{1}{\sqrt[3]{\frac{288}{5}\left(-\ln 2 - \ln\left(1+\frac{1}{4-1/12}\right)\right) - \gamma + \frac{3}{2}}} - 2 = -0.0231607\cdots \text{ and } \beta_2 = -\frac{17}{450}.$$

Proof. (2.6) is equivalent to the following inequality:

$$\alpha_2 \ge f(n) = \frac{1}{\sqrt[3]{\frac{288}{5}\left(\psi(n+2) - \ln(n+1) - \ln\left(1+\frac{1/2}{n+1-1/12+1/(12(n+1))}\right)\right)}} - (n+1) > \beta_2.$$

We let $v(x) = \frac{288}{5}\left(\psi(x+1) - \ln x - \ln\left(1+\frac{1/2}{x-1/12+1/(12x)}\right)\right)$ in Lemma 3, then we obtain following inequalities. For $x \ge 1$,

$$v(x) > \frac{1}{x^3} + \frac{17}{150x^4} + \frac{19}{3600x^5} - \frac{6737}{30240x^6},$$

$$-v'(x) < \frac{3}{x^4} + \frac{34}{75x^5} + \frac{19}{720x^6} - \frac{6737}{5040x^7} + \frac{503}{103680x^8} + \frac{747761}{388800x^9}. \tag{2.7}$$

Consequently,

$$27v(x)^4 - (-v'(x))^3 = \frac{q(x)}{2257828295933952000000000x^{27}},$$

where

$$\begin{aligned} q(x) = (&80651143275920669766868643071\,3 + 3737478095261359869286206977229(x-3) \\ &+ 7701270516281610339355498650363(x-3)^2 + 9401384497393676854552116830607(x-3)^3 \\ &+ 7610303630824613327019013269408(x-3)^4 + 4318542137335131906112415993616(x-3)^5 \\ &+ 1766194544792250060876182027520(x-3)^6 + 525728969689235608702754562816(x-3)^7 \\ &+ 113292083926222825658322781798\,4(x-3)^8 + 17286119887670868371033788416(x-3)^9 \\ &+ 1781506646749844118904504320(x-3)^{10} + 112777434463186185590145024(x-3)^{11} \\ &+ 3506898369871212812697600(x-3)^{12} + 200269369849341542400\,00(x-3)^{13}). \end{aligned} \tag{2.8}$$

$q(x)$ is polynomial with all positive coefficients, so $q(x) > 0$ for $x \ge 3$.

Therefore, the inequality holds for $x \ge 3$. From (2.8), we get that $f(1) = -0.02316$, $f(2) = -0.03033, \cdots$, we know that $f(n)$ is strictly decreasing. This leads to

$$\lim_{n\to\infty} f(n) < f(n) \le f(1) = \frac{1}{\sqrt[3]{\frac{288}{5}\left(-\ln\frac{118}{47} - \gamma + \frac{3}{2}\right)}} = -0.0231607\cdots.$$

We have that

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} (n+1)\left(\left(1 + \frac{17}{150(n+1)} + o((n+1)^{-1})\right)^{-1/3} - 1\right) = \frac{17}{450}.$$

The proof of Theorem 3 is completed.

References

[1] D. W. Lu, Some new improved classes of convergence to-wards Euler's constant. *Applied Mathematics and Computation.* 243(2014) 24-32.

[2] T. Negoi, A faster convergence to the constant of Euler, *Gazeta Mathematica Seria A* 15(1997) 111-113.

[3] A. Vernescu, A new accelerate convergence to the constant of Euler, *Gazeta Mat., Ser. A, Bucharest* 104 (4) (1999) 273-278 (in Romanian).

[4] A. Sînt˘am˘arian, A generalization of Euler's constant. *Numer. Algorithms* 46(2) (2007), 141–151.

[5] V. Berinde, C. Mortici, New sharp estimates of the generalized Euler–Mascheroni constant. *Math. Inequal. Appl.* 16(1) (2013), 279–288.

[6] T. R. Huang, B.-W. Han, X.-Y. Ma, Y.-M. Chu, Optimal bounds for the generalized Euler–Mascheroni constant, *J. Inequal. Appl.* (2018)2018 118.

[7] C.P.Chen,C. Mortici, Limits and inequalities and associated with the Euler-Mascheroni constant. *Applied Mathematics and Computation.* 219(2013)9755-9761.