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TRIANGULAR SIMPLIFYING AND RECOVERING: A NOVEL GEOMETRIC APPROACH FOR FOUR COLOR THEOREM

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Abstract. The Four Colour Theorem is one of mathematical problems with a fairly short history. This problem originated from coloring areas on a map, but has been dealt with graph and topological theory. Since the discovery of the problem, there have been many proofs by people interested in this mathematical problem, but in 1976 it was recognized as a proof by computer. The method of proof was to show that many graphs or many patterns can be colored with four colors. This proposed algorithm aims to show that all graphs are satisfied with the four color theorem regardless of the topology and the four color problem has no more non-deterministic polynomial time complexity.

1. Introduction

Considering the time that has passed, a number of articles on the four-color problem is not large. It means that the problem is more eccentric than a trivial one. However articles of this issue until recently are published and computer-aided proof does not seem to be deeply accepted.

It is well known that an area is represented by a vertex, and adjacent information is represented by an edge of graph. A vertex should be set a only one color, and if exist, an edge between two vertices should be unique. A. B. Kempe mentioned the existence of topology of the form 'digin', 'triangle', 'square', and 'pentagon'[5]. Afterwards, P. J. Heawood turned out that Kempe's proof is flawed[4], but the topology discovered by A. B. Kempe is still valid. Later, the concepts of unavoidable sets and reducible are appeared, and the topology corresponding to unavoidable sets has been expanded by Wernicke[8], P. Franklin[3]. Birkhoff, well known as the 'Birkhoff Diamond'[1] contributed to the reducible. Attempts to solve the four-color problem are still ongoing[2], and the history has been organized and announced until recently[7, 6].

The most of studies have focused on topology or patterns of graphs and considered an extension of graph. Therefore, it was necessary to find the unavoidable sets or reducible configurations. However, graph expanding can create a different topology infinitely. This thesis proposes a novel geometric approach. It provides simple procedures includes simplifying and recovering process. Simplifying process makes any graph to one triangle and recovering process recovers the original topology with vertices colors from one triangle.

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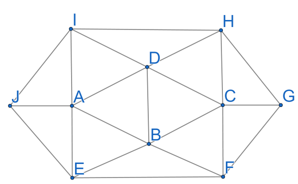
2. Main Ideas

There are two main processes, simplifying and recovering. Simplifying process makes a given graph to a single triangle, and the recovering process recovers the original topology of a graph from single triangle with coloring of vertices.

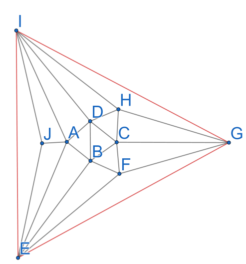
The procedures are below.

- Step 1. Constructing a maximal planar graph by adding edges to a given original graph.
- Step 2. Simplifying until the maximal planar graph becomes a single triangle.
- Step 3. Recovering with assigning colors to the vertices until the single triangle becomes the same maximal planar graph created in Step 1.
- Step 4. Removing the additional edges added in Step 1.

2.1. **Constructing a maximal planar graph.** Given a map, it can be represented as a graph consisting of vertices and edges. After map is converted to a graph, make this graph a triangulated graph and add some edges to create a maximal planar. There are no special rules and the result of the step 1 is maximal planar graph surrounded by triangle. Fig.1a shows an example original graph, and the Fig.1b shows maximal planar graph. The original graph is well-known for a Birkhoff Diamond[1]. The example graph is well triangulated, but it doesn't matter if vertices are added to make it looks nice for arbitrary graphs. This is because the added edges and vertices are all removed in step 4. The red colored edges($\overline{IE}$, $\overline{EG}$, $\overline{GI}$) are added edges in Fig.1b.



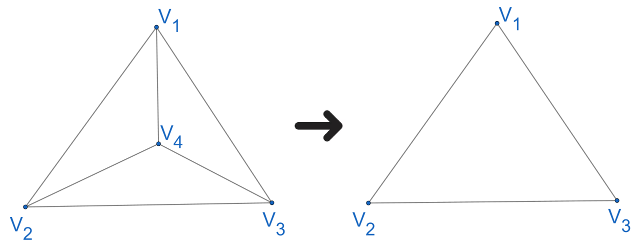
(A) The original graph.



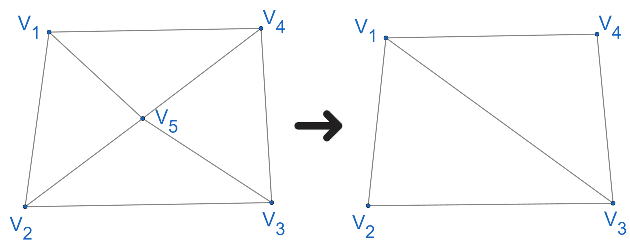
(B) The maximal planar graph.

Figure 1. The original graph and its maximal planar graph.

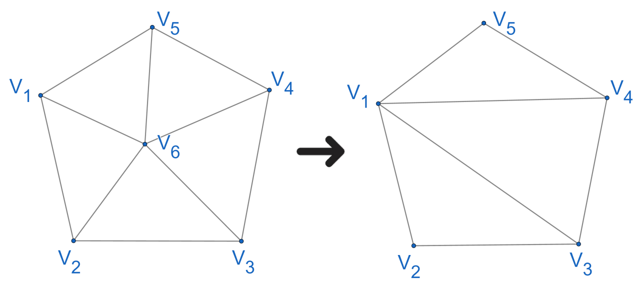
2.2. **Simplifying.** Simplifying is for making a smaller number of triangles graph by removing one vertex and three edges. There are three simplifying types. The first type reforms a W_4 wheel graph or tetrahedral planar to one triangle. The vertex V_4 and connected three edges are removed(see Fig.2a). The second type reforms a W_5 wheel graph to two triangles. The vertex V_5 is removed and making two triangles by connecting V_1 and V_3 vertices or connecting V_2 and V_4 vertices(see Fig.2b). The last type reforms a W_6 wheel graph to three triangles. The vertex V_6 is removed and making three triangles by connecting two edges from any vertices to opposite two vertices(see Fig.2c). Which edges are selected does not affect this algorithm. Simplifying continues until one triangle remains by Corollary 2.1.



(A) Simplifying type-1: W_4 wheel graph to one triangle.



(B) Simplifying type-2: W_5 wheel graph to two triangles.



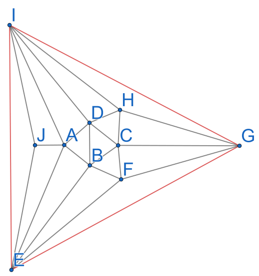
(C) Simplifying type-3: W_6 wheel graph to three triangles.

FIGURE 2. Simplifying W_4, W_5 and W_6 graph.

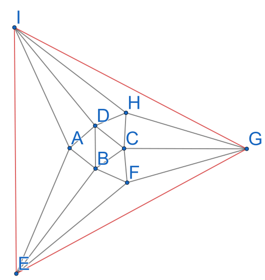
Corollary 2.1. *Every maximal planar graph can be adopt simplifying until one triangle remains.*

Proof. 'Every simple planar graph has a vertex of degree at most five.' is proved by Euler's equation. The maximal planar graph is a kind of planar graph, and a graph after performing simplifying once from a maximal planar graph is still maximal planar. Therefore the maximal planar graph can be adopt simplifying until one triangle remains. □

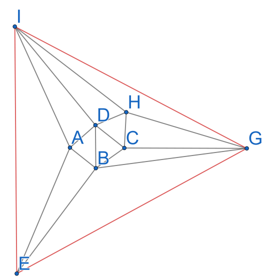
Each phases and the result of applying the simplifying process shown in Fig.3 to the maximal planar graph in Fig.1b. The simplifying sequences have no order but the recovering sequences should be reverse of simplifying. Fig.3b shows the result of first simplifying type(removing vertex J and connected edges) from maximal planar graph, Fig.1b. Fig.3b and Fig.3c show the process of second simplifying type(removing vertex F and selecting edge $\overline{BG}$). Fig.3c and Fig.3d are phases for W_6 graph simplifying(removing vertex D and selecting edge $\overline{AH}$ and $\overline{AC}$).



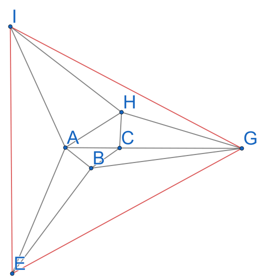
(A) The maximal planar graph.



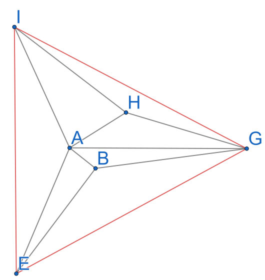
(B) Remove vertex J of the maximal planar graph.



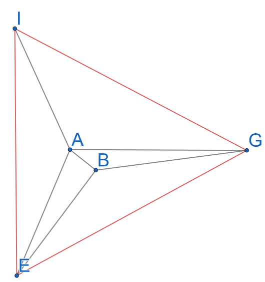
(C) Remove vertex F and the edge $\overline{BG}$ is selected.



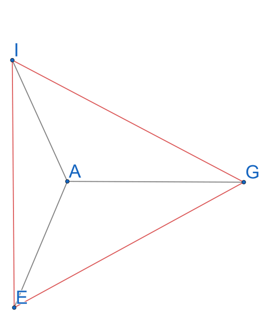
(D) Remove vertex D and the edges $\overline{AH}$ and $\overline{AC}$ are selected.



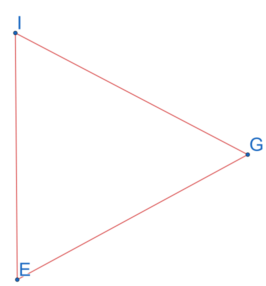
(E) Remove vertex C and the edge $\overline{AG}$ is selected.



(F) Remove vertex H.



(G) Remove vertex B.



(H) Remove vertex A, and final single triangle.

FIGURE 3. Each phases and the result of applying the simplifying for Fig.1b.

2.3. **Recovering.** Recovering process recovers the topology with colored vertices. In this paper, the number 1, 2, 3, and 4 are stand for the colors(1: Red, 2: Green, 3: Blue, 4: Black, these colors are assigned to number and used in this paper). TP(Toggle Path) and TS(Toggle Switching) are useful concept for determining a vertex color in recovering process. TP and TS are originated from Kempe Chain, however TP and TS include some parameters and are defined for more clear color manipulation. Just as there are three types of simplifying, recovering types are also three. Since a color of vertices is determined through the recovering process, this process is more complicate than simplifying. For this reason, each recovering types are occupied in a small section.

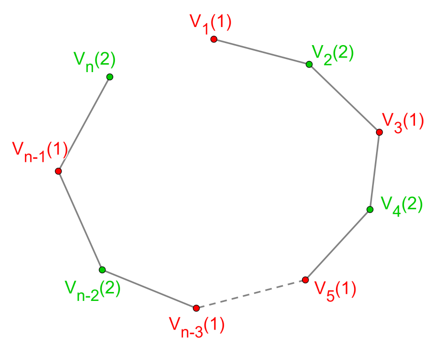
Definition 2.2. Let $V_n(x)$ is that the color of vertex V_n is x.

Definition 2.3. $TP^{x,y}_{V_a,V_b}$ is a path between V_a and V_b, and the colors of vertices are x and y alternatively.

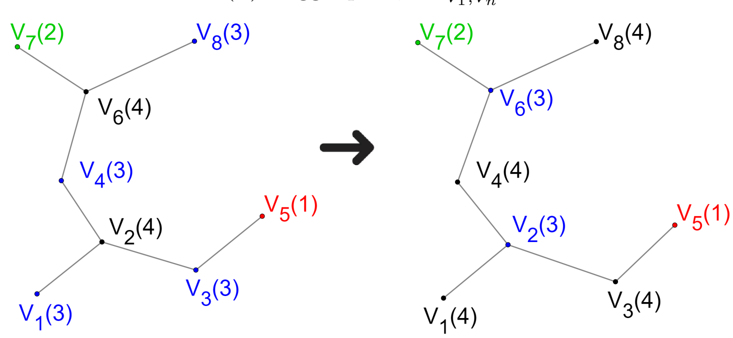
The Fig.4a is an example of $TP^{1,2}_{V_1,V_n}$.

Definition 2.4. Let $TS^{x,y}_{V_a(|V_b)}$ is a vertex color switching between x and y that starts from V_a. '$(|V_b)$' is an optional and if there is, color switching is not propagated to V_b.

The Fig.4b is an example of $TS^{3,4}_{V_1}$.



(A) Toggle path, $TP^{1,2}_{V_1,V_n}$.



(B) Toggle switching, $TS^{3,4}_{V_1}$.

FIGURE 4. An example of toggle path and toggle switching.

2.3.1. *Recovering Type 1: One triangle to W_4 graph.* The first recovering type recovers W_4 graph from one triangle. This is a reverse procedure of Fig.2a. When the vertices colors of triangle($\triangle V_1V_2V_3$) are 1, 2, and 3, the recovered center color of $V4$ is determined to 4(see Fig.5). There is no other choice.

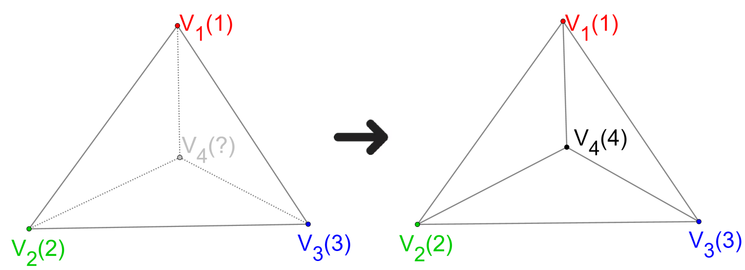


FIGURE 5. The only one case of W_4 graph recovering.

2.3.2. *Recovering Type 2: Two triangles to W_5 graph.* The second recovering type recovers W_5 graph from two triangles. This is a reverse process of Fig.2b.

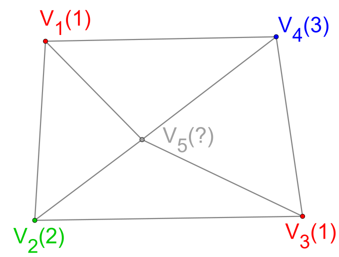
The W_5 graph has two cases for vertices coloring. The first case is the boundary vertices consist with only three colors(see Fig.6a). The second case is a the boundary vertices consist with four colors(see Fig.6b).

(1) The First Case: The V_1 and V_3 contain same color. This case of W_5 recovering is simple. The V_5 is set to 4(see Fig.7a).
(2) The Second Case: When the four vertices contain different colors each(see Fig.6b). TPs should be evaluated.
   (a) $TP^{1,4}_{V_1,V_3}$ is: If there is a $TP^{1,4}_{V_1,V_3}$ between $V_1(1)$ and $V_3(4)$, the color of V_5 can not be set to 1 or 4 by Theorem.2.5, and the color 2 and 3 are blocked by Theorem.2.7. Therefore, the color of V_5 can be set to 2 or 3. If the color 2 is selected for V_5, $TS^{2,3}_{V_2|V_5}$ is occurred, or if the color 2 is selected for V_5, $TS^{2,3}_{V_2|V_5}$ is occurred. Fig.7b shows the result of this case.
   (b) $TP^{2,3}_{V_2,V_4}$ is: In contrary, if there is $TP^{2,3}_{V_2,V_4}$ between $V_2(2)$ and $V_4(3)$, the color of V_5 can be set to 1 or 4, and related TS is occurred.
   (c) Both $TP^{1,4}_{V_1,V_3}$ and $TP^{2,3}_{V_2,V_4}$ are: Both $TP^{1,4}_{V_1,V_3}$ and $TP^{2,3}_{V_2,V_4}$ are not exist by Theorem.2.6.
   (d) No $TP^{1,4}_{V_1,V_3}$ and $TP^{2,3}_{V_2,V_4}$: The color of V_5 can be set to 1, 2, 3, and 4 with related TS.

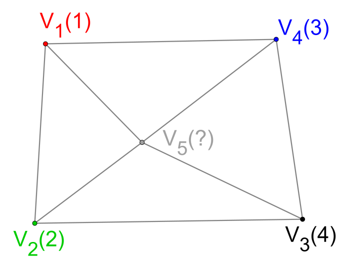
Theorem 2.5. *If there is $TP^{1,2}_{V_a,V_b}$ and a vertex I exists between V_a and V_b, the color of I can not be set to 1 or 2.*

Proof. If the color of I is set to 1 or 2, adjacent vertices have the same color. This is map coloring violation. Even if $TS^{1,2}_{V_a}$ or $TS^{1,2}_{V_b}$ is occurred, the color of I switches 1 and 2 infinitely. Therefore, when there is $TP^{1,2}_{V_a,V_b}$, and a vertex I exists between V_a and V_b, the color of I can not be set to 1 or 2. □

Lemma 2.6. *If there is $TP^{1,2}_{V_a,V_b}$ and a vertex I exists between V_a and V_b, $TP^{1,2}_{V_a,V_b}$ separates inside and outside areas to the color 3 and 4 with I as the center.*



(A) V_1 and V_3 vertices contain same color 1.



(B) The four vertices contain different color each.

FIGURE 6. The two cases of topology with vertices colors of W_5 graph.

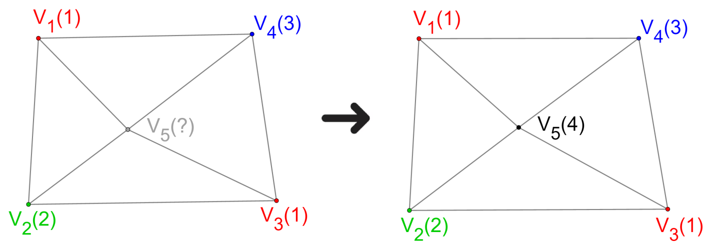
Proof. A path is made up of only colors 1 and 2 like a fence, and I is like a gate. The other colors 3 and 4 are blocked by $TP^{1,2}_{V_a,V_b}$ and to connect 3 or 4 in different area, this path should pass through I. $\square$

Theorem 2.7. *Let $TP^{1,2}_{V_a,V_b}$ is and a vertex I exists between V_a and V_b. If $V_c(3)$ and $V_d(4)$ each belongs to a different area separated by $TP^{1,2}_{V_a,V_b}$, and I exists between V_c and V_d, there is not exist $TP^{3,4}_{V_c,V_d}$ that does not pass through I.*

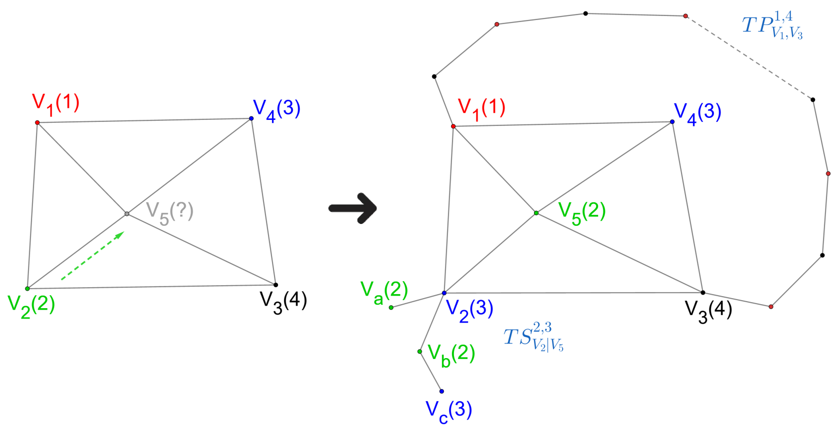
Proof. If there $TP^{1,2}_{V_a,V_b}$ and $TP^{3,4}_{V_c,V_d}$ are and a vertex I exists between V_a, V_b, V_c, and V_d, this is a violation of the planar graph. $\square$

2.3.3. *Recovering Type 3: Three triangles to W_6 graph.* The last recovering type recovers W_6 graph from three triangles. This is a reverse process of Fig.2c. The W_6 graph also has two cases for vertices coloring. The first case is the boundary vertices consist with only three colors(see Fig.8a). The second case is a the boundary vertices consist with four colors(see Fig.8b).

(1) The First Case: The colors of V_1, V_3 are same and the colors of V_2, V_4 are same, and only one color of vertex is different(see Fig.8a). This case of W_6 recovering is simple like the first case of W_5 recovering in Fig.6a. The vertex V_6 is set to 4(see Fig.9).



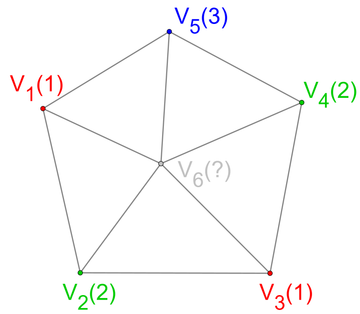
(A) V_1 and V_3 vertices contain same color 1.



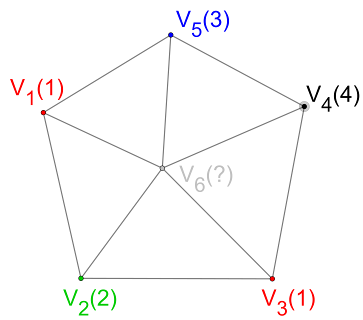
(B) The four vertices contain different color each.

Figure 7. W_5 graph recovering.

(2) The Second Case: As see in Fig.8b, the color of V_1 and V_3 are same. The V_1 color is denoted as $1a$ and V_3 color is denoted as $1b$ for discrimination. The two TPs should be evaluated to set a color of $V6$.
   (a) $TP^{1,4}_{V_1,V_4}$ is: As see in Fig.10a, two paths are possible from $V_1(1a)$ to $V_4(4)$. One path is direct TP from V_1a to V_4, and another path is via V_1b. However, as long as either one exists, color 2 of V_2 and color 3 of V_5 are blocked by $TP^{1,4}_{V_1,V_4}$. Therefore, V_6 can be set to 2 or 3 with related TS. Fig.10a shows the result that color of V_6 is set to 2, and $TS^{2,3}_{V_2|V_6}$ is occurred.
   (b) $TP^{1,3}_{V_3,V_5}$ is: As see in Fig.10b, two paths are possible from $V_3(1b)$ to $V_5(3)$. One path is direct TP from V_1b to V_5, and another path is via V_1a. As in the case of above, as long as either one exists, color 2 of V_2 and color 4 of V_4 are blocked by $TP^{1,3}_{V_3,V_5}$. Therefore, V_6 can be set to 2 or 4 with related TS. Fig.10b shows the result that color of V_6 is set to 2, and $TS^{2,4}_{V_2|V_6}$ is occurred.
   (c) Both $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$ are: Unlike the W_5 recovering process, both $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$ are possible. Fig.10 shows these paths. However color of V_6 can be set to 2, 3, and 4 with related TS. Fig.10 shows



(A) V_1, V_3 vertices color is 1 and V_2, V_4 vertices color is 2.



(B) The four vertices contain different color each.

FIGURE 8. The two cases of topology with vertices colors of W_6 graph.

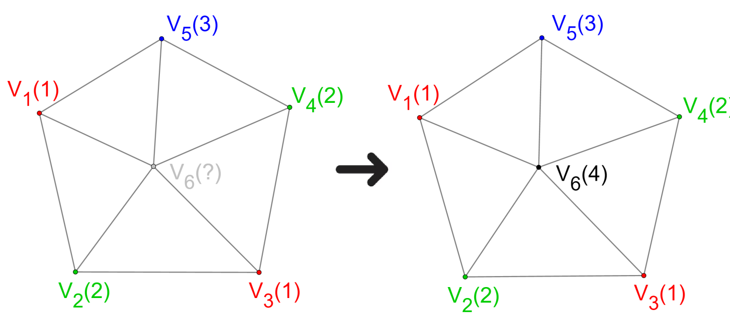
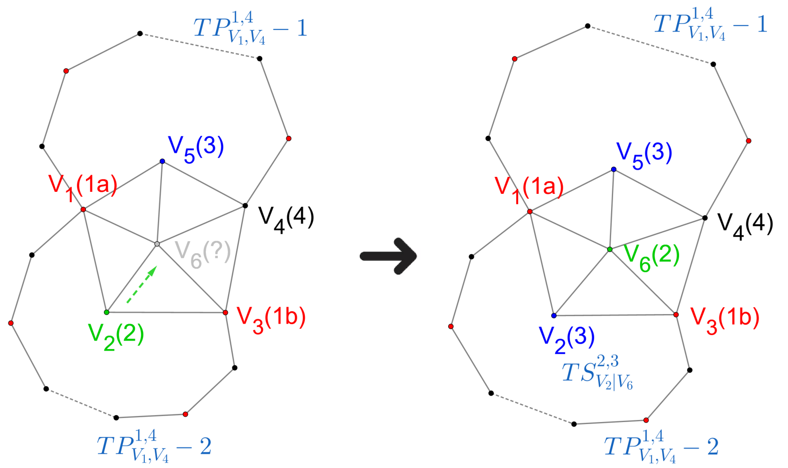


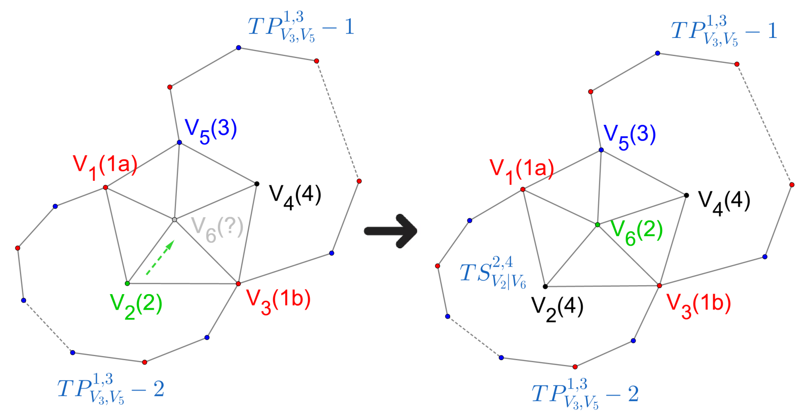
FIGURE 9. The first case of W_6 graph recovering.

the result of the color of V_6 is set to 2 with $TS^{2,3}_{V_2,V_6}$(see Fig.11a) or $TS^{2,4}_{V_2,V_6}$(see Fig.11b).

(d) No $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$: The color of V_6 is set to 1(the color that appears twice), and the colors of V_1 and V_3 are set to blank(see Fig.12a). Because of $TP^{1,3}_{V_3,V_5}$ is not exist, the color of V_3 can be set to 3 and $TS^{1,3}_{V_3|V_6}$ is occurred(see Fig.12b, the color of V_1 can be set to 4 and



(A) There is $TP^{1,4}_{V_1,V_4}$, the color of V_6 is set to 2 and $TS^{2,3}_{V_2|V_6}$ is occurred.

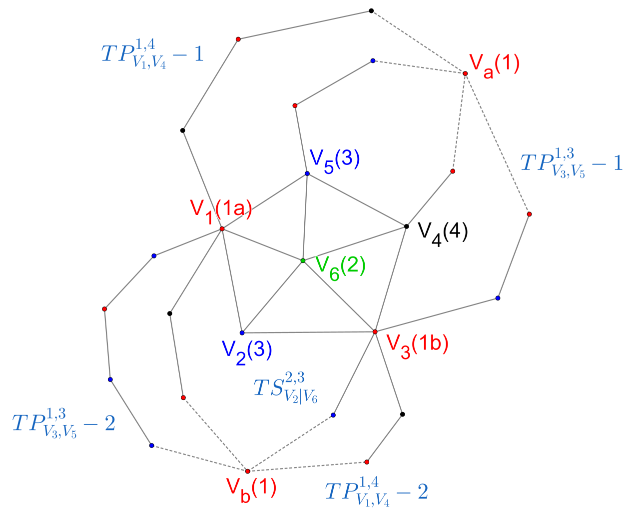


(B) There is $TP^{1,3}_{V_3,V_5}$, the color of V_6 is set to 2 and $TS^{2,4}_{V_2|V_6}$ is occurred.

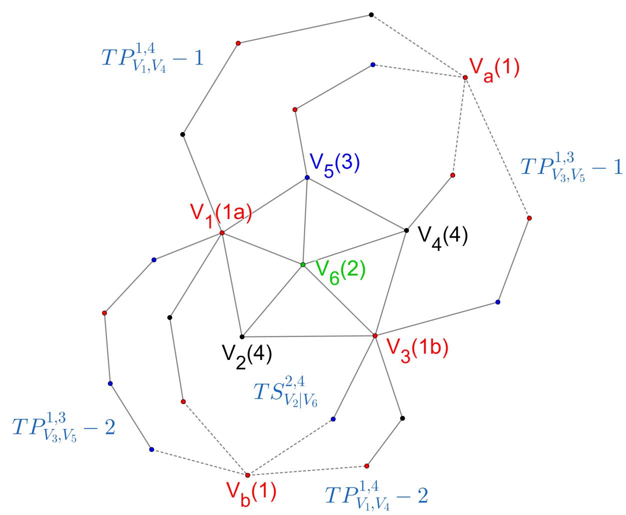
FIGURE 10. W_6 graph recovering when $TP^{1,4}_{V_1,V_4}$ or $TP^{1,3}_{V_3,V_5}$ is exist.

$TS^{1,4}_{V_4|V_6}$ is possible, but the color of V_5 is selected for V_3). The result is shown in Fig.12c. After that, $TP^{2,3}_{V_2,V_5}$ should be evaluated to set a color of V_1. If there is $TP^{2,3}_{V_2,V_5}$, this path blocks color 1 and 4, the color of V_1 should be set to 4 and $TS^{1,4}_{V_1,V_6}$ is occurred(see Fig.12d, Fig.12e, and Fig.12f). However, if $TP^{2,3}_{V_2,V_5}$ is not exist, the color of V_1 should be set to 3 and the $TS^{2,3}_{V_5,V_1}$ is occurred(see Fig.13).

2.4. **Removing the additional edges.** Removing the additional edges in step 1 is final step. When this process is done, the original graph with vertex color is remained from maximal planar graph.



(A) There are $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$, the color of V_6 is set to 2 and $TS^{2,3}_{V_2|V_6}$ is occurred.



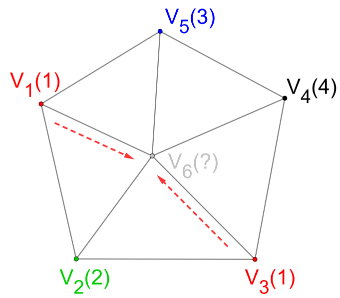
(B) There are $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$, the color of V_6 is set to 2 and $TS^{2,4}_{V_2|V_6}$ is occurred.

FIGURE 11. W_6 graph recovering when $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$ are exist.

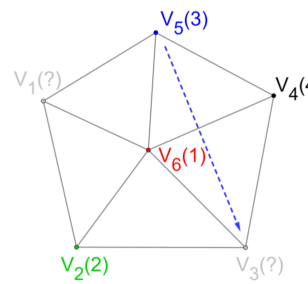
3. RESULTS

Recovering process shows the colors of vertices for each reverse process in Fig.3 from one triangle to maximal planar graph.

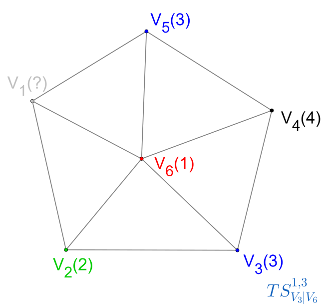
- In Fig.14b, A is recovered in the center of triangle $\triangle IEG$. Since the colors of triangle are 1, 2, and 3, A color is deterministic. This is type-1 recovering.



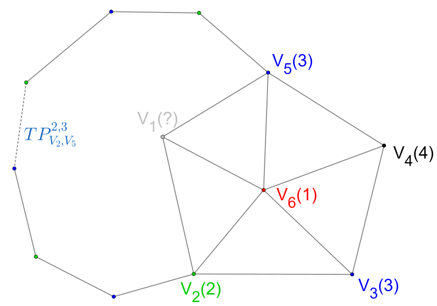
(A) The color of V_6 is set to 1, and V_1 and V_6 are set to blank.



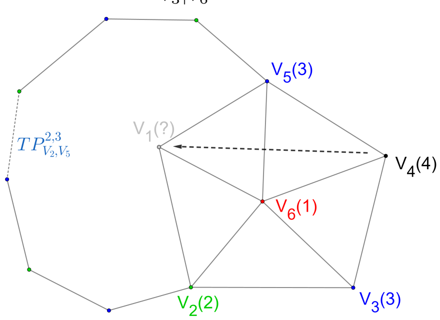
(B) The color of V_3 can be set to 3 because of there is not $TP^{1,3}_{V_3,V_5}$.



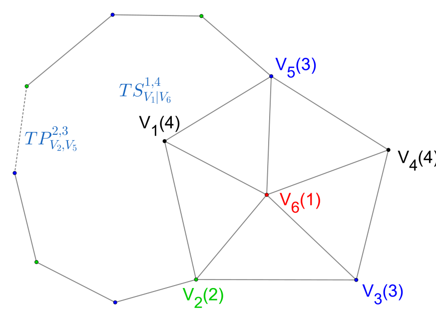
(C) $TS^{1,3}_{V_3|V_6}$ is occurred.



(D) $TP^{2,3}_{V_2,V_5}$ should be evaluated.



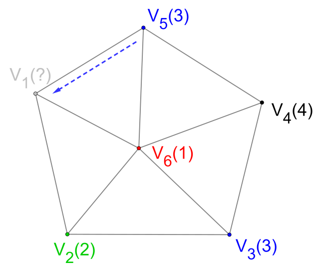
(E) The color of V_1 should be set to 4, because of $TP^{2,3}_{V_2,V_5}$ is exist.



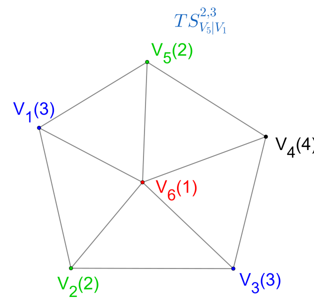
(F) $TS^{1,4}_{V_1|V_6}$ is occurred.

FIGURE 12. W_6 graph recovering when $TP^{1,4}_{V_1,V_4}$ and $TP^{1,3}_{V_3,V_5}$ are not exist, and $TP^{2,3}_{V_2,V_5}$ is exist.

- In Fig.14c, B is recovered in the center of triangle $\triangle AEG$. As in Fig.14b, the color of recovered vertex is deterministic.
- Fig.14d shows also W_4 graph recovering, and color of recovered vertex is deterministic.
- In Fig.14e, C is recovered, but all four colors are assigned yet in surrounding vertices. Therefore some color of vertex is moved from surrounding vertices and the colors are switching continuously. However, there is $TP^{1,2}_{B,H}$, the



(A) When $TP^{2,3}_{V_2,V_5}$ is not exist, the color of V_1 should be set to 3.



(B) $TS^{2,3}_{V_5|V_1}$ is occurred.

FIGURE 13. W_6 graph recovering when $TP^{1,4}_{V_1,V_4}$, $TP^{1,3}_{V_3,V_5}$, and $TP^{2,3}_{V_2,V_5}$ are not exist.

edge sequences are $\overline{BE}$, $\overline{EI}$, and $\overline{IH}$. If C color is set to 1 or 2, this path will be a circular, and the color of vertices switched between 1 and 2 eternally. On the other hand, since this $TP^{1,2}_{B_H}$ blocks connection between A and G, C can be set to either color of A or G. In this example, the color 4 of A is selected for C, and the color of A switches from 4 to 3. If A has any connection to another vertices with color 3, it will be switched from 3 to 4 in W_5 graph.

- In Fig.14f, D is recovered. This is W_6 graph recovering. D also, all four colors assigned to surrounding vertices already. Therefore, as illustrated in Fig.9, two paths($SP^{1,4}_{C,I}$, $SP^{1,2}_{B,H}$) should be considered. However, as in Fig.14e, $SP^{1,2}_{B,H}$ is exist and color 3 or 4 is candidate for D. In this example, color 3 is selected for D, and the color of A switches from 3 to 4.
- Explanation about Fig.14g and Fig.14h are omitted.

4. Discussion

This article is a draft.

5. Conclusion

This article is a draft.

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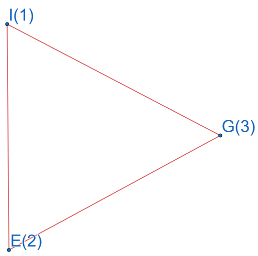
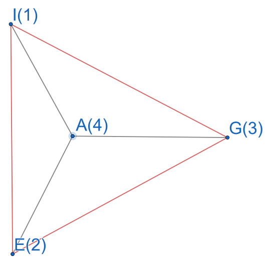
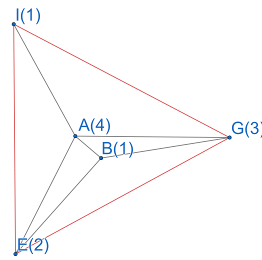


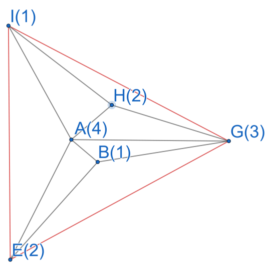
(A) I, E and G contain colors 1, 2 and 3 each.



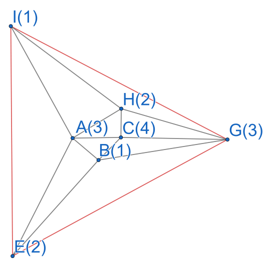
(B) A is recovered with color 4.



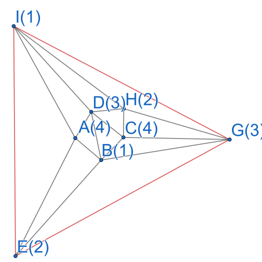
(C) B is recovered with color 1.



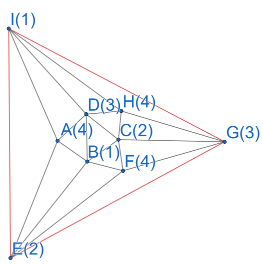
(D) H is recovered with color 2.



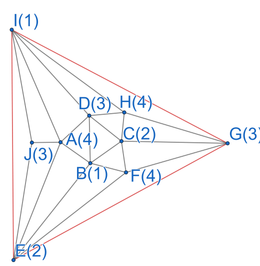
(E) C is recovered with color 4, and A color is switched to 3.



(F) D is recovered with color 3, and A color is switched to 4.



(G) F is recovered with color 4, and C color is switched to 2.



(H) J is recovered with color 3.

FIGURE 14. Each phases and the result of recovering for Fig.3g.

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