Triangular Simplifying and Recovering: A Novel Geometric Approach for Four Color Theorem

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Abstract

The Four Colour Theorem is one of the mathematical problems with a fairly short history. This problem originated from coloring areas on a map, but has been dealt with graph and topological theory. Since the discovery of the problem, there have been many proofs by people interested in this mathematical problem, but in 1976 it was recognized as a proof by computer. The method of proof was to show that many graphs or many patterns can be colored with four colors. This proposed algorithm aims to show that all graphs are satisfied with the four color theorem regardless of the topology and the four color problem has no more non-deterministic polynomial time complexity.

1 Introduction

Considering the time that has passed, the number of articles on the four-color problem is not large. It means that the problem is more eccentric than a trivial one. However the articles of this issue until recently are published and the computer-aided proof does not seem to be deeply accepted.

It is well known that an area is represented by a vertex, and adjacent information is represented by an edge of the graph. A vertex should be set a only one color, and if exist, an edge between two vertices should be unique. A.B. Kempe mentioned the existence of topology of the form 'digin', 'triangle', 'square', and 'pentagon'[Kem79]. Afterwards, P.J. Heawood turned out that Kempe's proof is flawed[Hea90], but the topology discovered by A.B. Kempe is still valid. Later, the concepts of unavoidable sets and reducible are appeared, and the topology corresponding to unavoidable sets has been expanded by Wernicke[Wer04], P. Franklin[Fra22]. Birkhoff, well known as the 'Birkhoff Diamond'[Bir13] contributed to the reducible. Attempts to solve the four-color problem are still ongoing[Cah06], and the history has been organized and announced until recently[Wal04, Nan18].

The most of studies have focused on the topology or patterns of graphs and considered the extension of the graph. Therefore, it was necessary to find the unavoidable sets or reducible configurations. However, graph expanding can create a different topology infinitely. This thesis proposes a novel geometric approach. It provides a simple procedure includes simplifying and recovering process. The simplifying process makes an any graph to an one triangle and the recovering process recovers the original topology of a graph from one triangle with color setting of vertices.

2 Main Ideas

There are two main processes, triangular simplifying and recovering. The simplifying process makes an any graph to an one triangle and the recovering process recovers the original topology of a graph from one triangle with color setting of vertices.

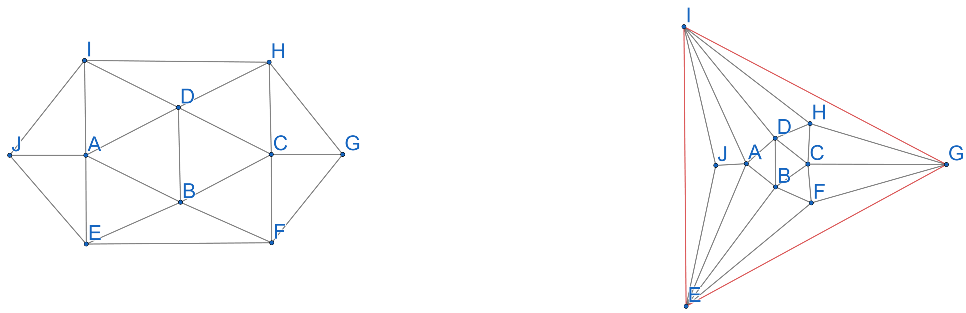
The procedure is below.

- Step 1. Constructing a maximal planar graph by adding edges to a given original graph.
- Step 2. Simplifying until the maximal planar graph becomes a single triangle.
- Step 3. Recovering with assigning colors to the vertices until the single triangle becomes the same maximal planar graph created in Step 1.
- Step 4. Removing the edges added in Step 1.

As a result of performing this procedure, the vertices of original graph contain colors and the colors are within four.

2.1 Constructing a maximal planar graph

Given a map, it can be represented as a graph consisting of vertices and edges. If the map is converted to a graph, make this graph a triangulated graph and add some edges to create a maximal planar. There are no special rules and the result of the step 1 is maximal planar graph surrounded by triangle. The Fig.1 shows the original graph and maximal planar graph. The original graph is well-known for a Birkhoff Diamond[Bir13]. The example graph is well triangulated, but it doesn't matter if vertices are added to make it looks nice for arbitrary graphs. This is because the added edges and vertices are all removed in step 4. The red colored edges($\overline{IE}$, $\overline{EG}$, $\overline{GI}$) in Fig.1b are added and these edges will be removed at step 4.



(a) The original graph.

(b) The maximal planar graph.

Figure 1: The original graph and its maximal planar graph.

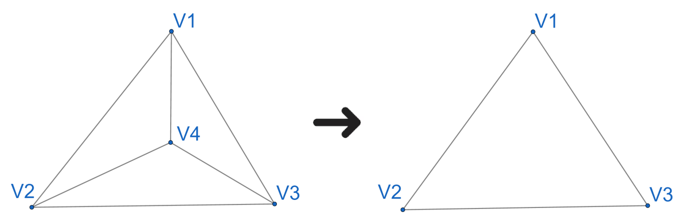
2.2 Simplifying

The simplifying is for making a smaller number of triangles graph by removing one vertex and three edges. There are three simplifying types. The first type reforms a W_4 wheel graph or tetrahedral planar to one triangle. The vertex $V4$ and connected three edges are removed(see Fig.2a). The second type reforms a W_5 wheel graph to two triangles graph. The vertex $V5$ is removed and making two triangles by connecting $V1$ and $V3$ vertices or connecting $V2$ and $V4$ vertices(see Fig.2b). The last type reforms a W_6 wheel graph to three triangles graph. The vertex $V6$ is removed and making three triangles by connecting two edges from any vertices to opposite two vertices(see Fig.2c). Which edges are selected does not affect this algorithm. The simplifying continues until one triangle remains by Corollary 2.0.1.

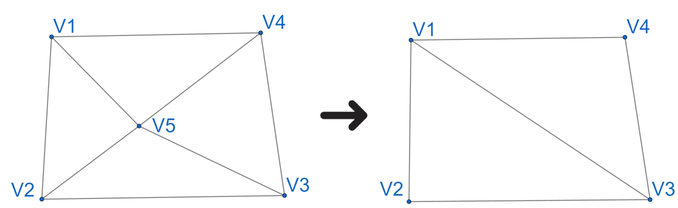
Corollary 2.0.1 *Every maximal planar graph can be adopt the simplifying until one triangle remains.*

'Every simple planar graph has a vertex of degree at most five.' is proved by Euler's equation. Also the maximal planar graph has one or more degree-three, degree-four, degree-five topology. However, the simplifying is defined on W_4(degree-three), W_5(degree-four), and W_6(degree-five) wheel graph at vertices $\geq$ 3. Therefore the maximal planar graph can be adopt the simplifying until one triangle remains.

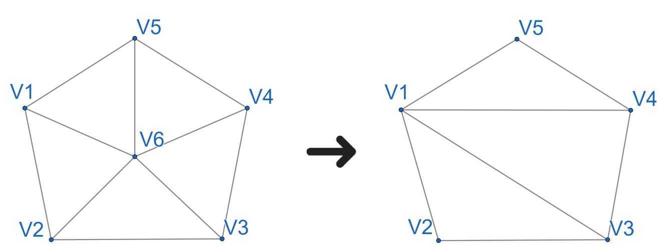
Each phases and the result of applying the simplifying process shown in Fig.3 to the maximal planar graph in Fig.1b. The simplifying sequences have no order but the recovering sequences should be reverse of simplifying. The Fig.3a shows the result of first simplifying type(removing vertex J and connected edges) from maximal planar graph, Fig.1b. The Fig.3a and Fig.3b show the result of second simplifying type(removing vertex F and selecting edge $\overline{BG}$). The Fig.3b and Fig.3c are phases for W_6 graph simplifying.



(a) Simplifying type 1: W_4 wheel graph to triangle.

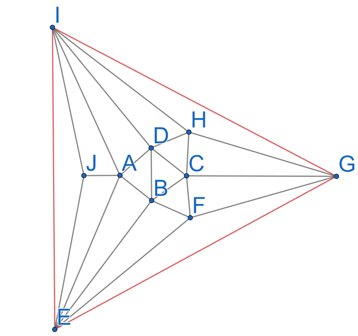


(b) Simplifying type 2: W_5 wheel graph to two triangles.

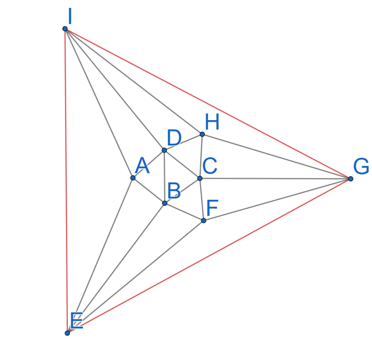


(c) Simplifying type 3: W_6 wheel graph to three triangles.

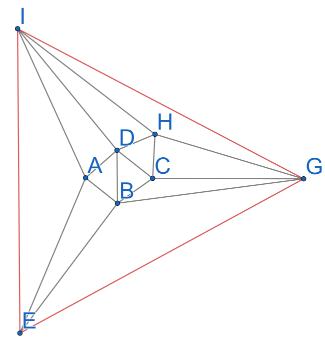
Figure 2: Simplifying W_4, W_5 and W_6 graph.



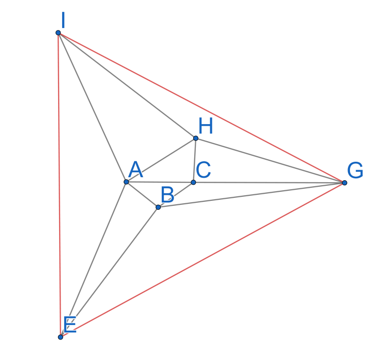
(a) The maximal planar graph.



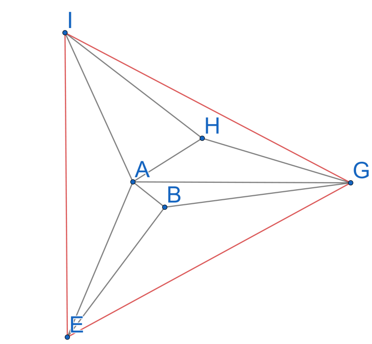
(b) Remove vertex J of the maximal planar graph.



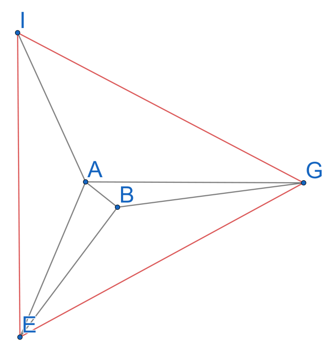
(c) Remove vertex F and the edge $\overline{BG}$ is selected.



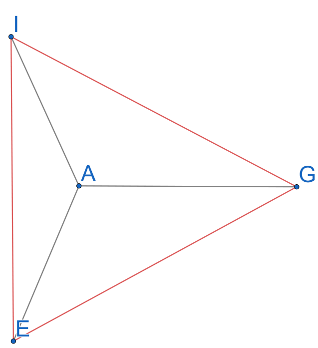
(d) Remove vertex D and the edges $\overline{AH}$ and $\overline{AC}$ are selected.



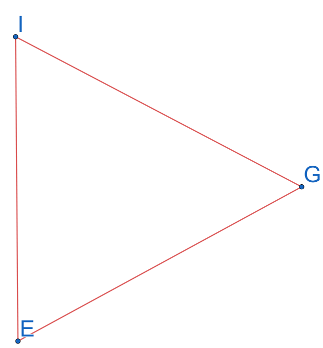
(e) Remove vertex C and the edge $\overline{AG}$ is selected.



(f) Remove vertex H.



(g) Remove vertex B.



(h) Remove vertex A.

Figure 3: Each phases and the result of applying the simplifying for Fig.1b.

2.3 Recovering

The recovering process recovers the topology with vertices contains color. In this paper, the number 1, 2, 3, and 4 are stand for the colors. The TP(Toggle Path) and TS(Toggle Switching) is defined below. These TP and TS are useful concept for determining a vertex color in recovering process. Just as there are three types of simplifying, recovering types are also three. Since the color of the vertices is determined during the recovering, this process is more complicate than the simplifying. For this reason, each recovering types are occupied in a small section.

Definition 2.1 *Let $V_n(x)$ is that the color of vertex V_n is x.*

Definition 2.2 *The $TP_{x,y}$ is a path in which consists of x and y vertex color alternatively.*

The Fig.4 is an example of $TP_{1,2}$.

Definition 2.3 *Let $TS_{x,y}$ is a vertex color switching between x and y in a graph or a sub-graph.*

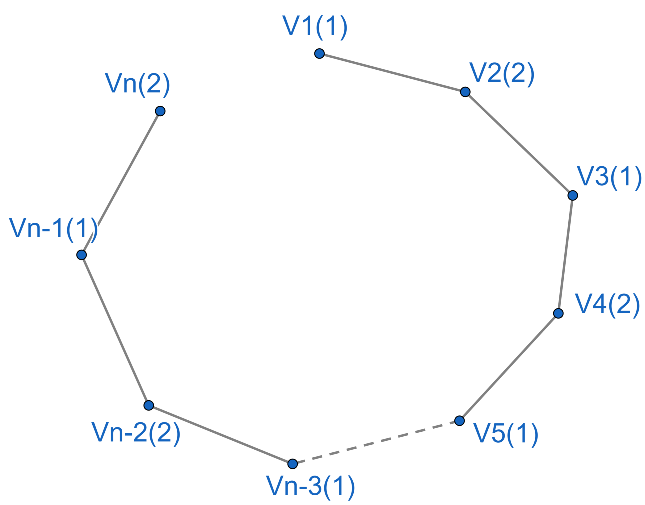


Figure 4: An example of toggle path, $TP_{1,2}$.

2.3.1 Recovering Type 1: One triangle to W_4 graph

The first recovering type recovers W_4 graph from one triangle. This is a reverse procedure of Fig.2a. When the vertices colors of triangle($\triangle V1V2V3$) are 1, 2, and 3, the recovered center color of $V4$ is determined to 4(see Fig.5). There is no other choice.

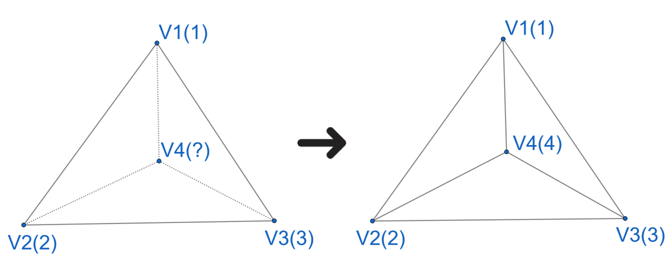


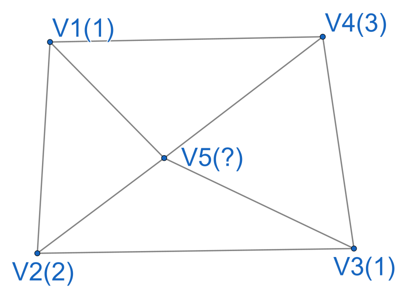
Figure 5: W_4 graph recovering.

2.3.2 Recovering Type 2: Two triangles to W_5 graph

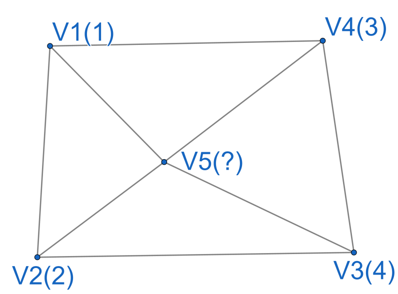
The second recovering type recovers W_5 graph from two triangles graph. This is a reverse process of Fig.2b.

The W_5 graph has two cases for vertices coloring. The first case is the boundary vertices consist with only three colors. The second case is a the boundary vertices consist with four colors.

- The First Case: The $V1$ and $V3$ contain same color(see Fig.6a). This case of W_5 recovering is simple. The $V5$ is set a color number 4.
- The Second Case: The four vertices contain different colors each(see Fig.6b). When the second case in Fig.6b, the TP should be considered.
  - If there is a $TP_{1,4}$ between $V1(1)$ and $V3(4)$, the $V2(2)$ and $V4(3)$ are blocked by $TP_{1,4}$. Therefore the color 2 of $V2$ or the color 4 of $V4$ moves to $V5$. If the color 2 of $V2$ moves to $V5$, the $TS_{2,3}$ is occurred at the sub-graph contains $V2$(see Fig.7).
  - If there are no $TP_{1,3}$ and $TP_{2,4}$, the all color 1, 2, 3 and 4 can move to vertex $V5$ and switch the color of vertices the concerned sub-graph.



(a) $V1$ and $V3$ vertices contain same color 1.



(b) The four vertices contain different color each.

Figure 6: W_5 graph recovering.

Theorem 2.1 *If there is a $TP_{1,2}$, let the start vertex is $S(1)$, the end vertex is $E(2)$, and a vertex I between $S(1)$ and $E(2)$, the color of I can not set 1 or 2.*

If the color of I is set 1 or 2, the $TS_{1,2}$ is occurred. However, there is a $TP_{1,2}$ between $S(1)$ and $E(2)$, the color of I switches 1 and 2 infinitely. The color of I cannot be determined, and the color of I can not set 1 or 2.

Theorem 2.2 *If there is a $TP_{1,2}$, let the start vertex is $S(1)$, the end vertex is $E(2)$. and a vertex I between $S(1)$ and $E(2)$, the $TP_{1,2}$ separates the inner and outer of color 1 and 2 with I is center.*

If the I is not a unique vertex for other colors of vertices connection, there is an edge for other colors of vertices without through I. However, this is a violation of the planar graph.

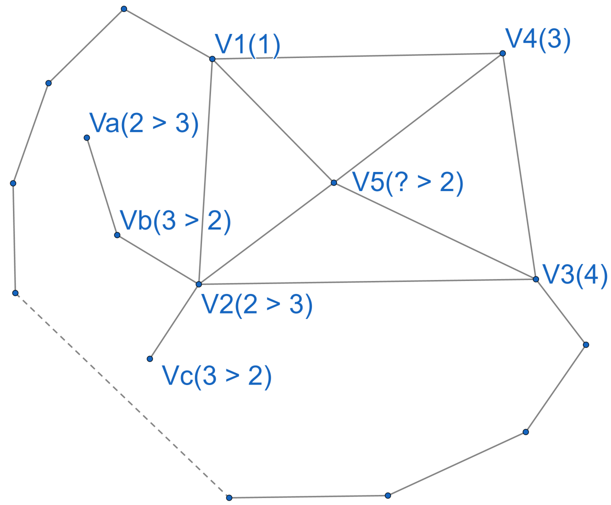


Figure 7: When a $TP_{1,4}$ is and color 2 of vertex $V2$ moves to vertex $V5$.

2.3.3 Recovering Type 3: Three triangles to W_6 graph

The last recovering type recovers W_6 graph from three triangles graph. This is a reverse process of Fig.2c. The W_6 graph also has two cases for vertices coloring.

- The First Case: The color of $V1$, $V3$ is same and the color of $V2$, $V4$ is same, and only one color of vertex is different(see Fig.8a). The first case of W_6 recovering in Fig.8a is simple like a Fig.6a. The vertex $V6$ is set a color number 4.

- The Second Case: The second case in Fig.8b is more complex than W_5 graph of Fig.6b. As see in Fig.8b, the color of $V1$ and $V3$ is 1. The $V1$ color is denoted as $1a$ and $V3$ color is denoted as $1b$ for discrimination. The two TP have to be considered for set a color of $V6$.

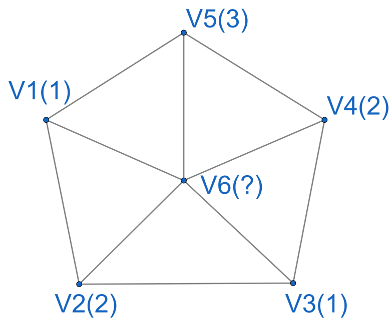
  – If there is a only $TP_{1,3}$, path from $V1(1a)$ to $V4(3)$, the $V2(2)$ and $V5(4)$ are blocked by $TP_{1,3}$. Therefore the color 2 of $V2$ or the color 4 of $V5$ moves to $V6$. When the color 2 of $V2$ moves to $V5$, the $TS_{2,4}$ is occurred at the sub-graph contains $V2$. In contrary, if there is a only $TP_{1,4}$, path from $V3(1b)$ to $V5(4)$, the $V2(2)$ and $V4(3)$ are blocked by $TP_{1,4}$. In this case, the color 2 of $V2$ or color 3 of $V4$ moves to $V6$ and the concerned color of TS is occurred.

  – Unlike the second case of W_5 recovering, the $TP_{1,3}$(between $V1(1a)$ and $V4(3)$) and the $TP_{1,4}$(between $V3(1b)$ and $V5(4)$) are not exclusive. If the $TP_{1,3}$ and $TP_{1,4}$ are exist both(see Fig.9), the color 2 of $V2$, the color 3 of $V4$, and the color 4 of $V5$ are able to move to $V6$. When the color 2 of $V2$ moves to $V6$, the $TS_{2,3}$ or the $TS_{2,4}$ are possible at $V2$. When the color 3 of $V4$ moves to $V6$, the $TS_{2,3}$ is possible at $V4$, when the color 4 of $V5$ moves to $V6$, the $TS_{2,4}$ is possible at $V5$.

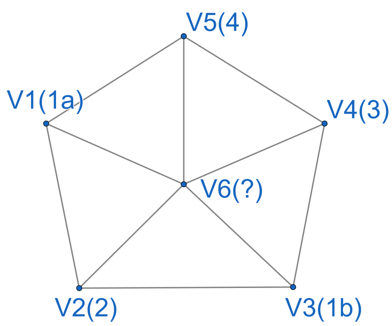
  – If there are no TPs for blocking color 2 of $V2$, set the color 1 to $V6$ from $V1(1a)$ and $V3(1b)$, and then the $TS_{1,3}$ is occurred at the sub-graph contains $V1$ or the $TS_{1,4}$ is occurred at the sub-graph and the sub-graph contains $V3$. If the $TS_{1,3}$ is occurred, $V3$ has no vertex color and check the $TP_{2,3}$ for vertex $V3$. In this step, if there is a $TP_{2,3}$, the $TS_{1,4}$ is occurred at $V3$ or if there is not a $TP_{2,3}$, the $TS_{1,4}$ is occurred at $V3$.

2.4 Removing the additional edges

The removing the additional edges in step 1 is final step of this procedure. When this process is done, the original graph with vertex color number is remained from maximal planar graph.



(a) $V1$, $V3$ vertices color is 1 and $V2$, $V4$ vertices color is 2.



(b) The four vertices contain different color each.

Figure 8: W_6 graph recovering.

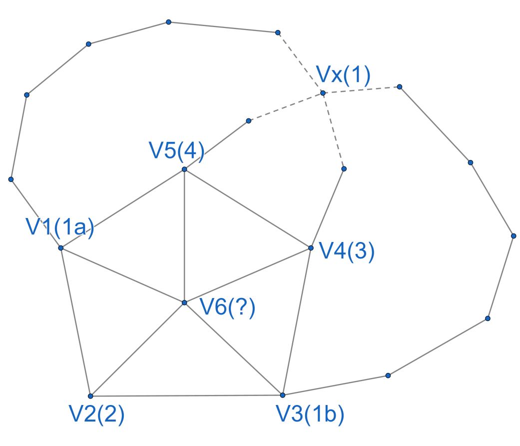


Figure 9: When a $TP_{1,3}$ and $TP_{1,4}$ are and color 2 of $V2$ moves to $V6$.

3 Results

The recovering process shows the vertices colors for each reverse process in Fig.3 from one triangle to maximal planar graph.

- In Fig.10b, the A is recovered in the center of triangle $\triangle IEG$. Since the vertices of triangle colors are 1, 2, and 3, the A color is deterministic. This is type1 recovering.
- In Fig.10c, the B is recovered in the center of triangle $\triangle AEG$. As in Fig.10b, the recovered vertex color is deterministic.
- The Fig.10d shows also W_4 graph recovering, and the recovered vertex color is deterministic.
- In Fig.10e, the C is recovered, but all four colors are assigned yet in the surrounding vertices. Therefore some color of vertex is moved from surrounding vertices and the colors are switching continuously. However, there is a $TP_{1,2}$, the edge sequences are $\overline{BE}$, $\overline{EI}$, and $\overline{IH}$. If the C color set 1 or 2, the path will be a circular path, and the vertices color switched between 1 and 2 eternally. On the other hand, since this $TP_{1,2}$ blocks connection between A and G, C can be set to either color of A or color of G. In this example, the color 4 of A is selected for C, and the color of A switches from 4 to 3. If A has any connection to another vertices with color 3, it will be switched from 3 to 4 in W_5 graph.
- In Fig.10f, the D is recovered. This is W_6 graph recovering. The D also, the all four colors assigned to the surrounding vertices already. Therefore, as illustrated in Fig.9, two paths($SP_{1_I,4}$, $SP_{1_B,2}$) should be considered. However, as in Fig.10e, the path $SP_{1,2}$ exist between H and B, color 3 or 4 is candidate for D. In this example, color 3 is selected color of D, and the color of A switches from 3 to 4.
- The explanation about Fig.10g and Fig.10h are omitted.

4 Conclusion

The four color theorem has continued to lack of proof, but it is also recognized as a natural that does not need proof anymore. However, I am sure that anyone who encounters the this theorem feels anxious, as if they are stepping into an air. This paper was written to free oneself from such anxiety and to fill in a missing stepping stone on the staircase of human progress that might lead to a different branch.

References

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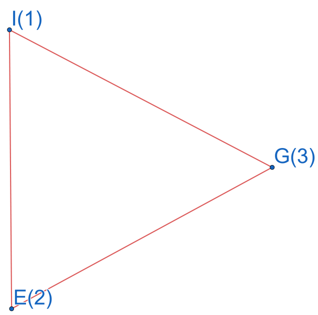
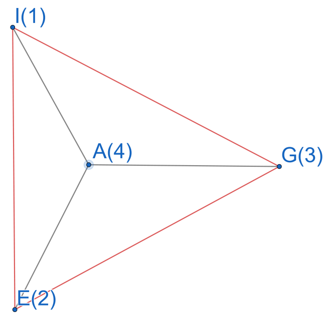
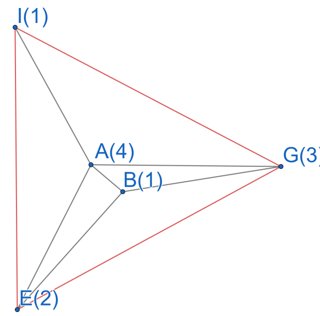


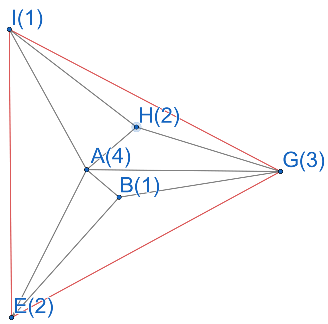
(a) The I, E and G contain colors 1, 2 and 3 each.



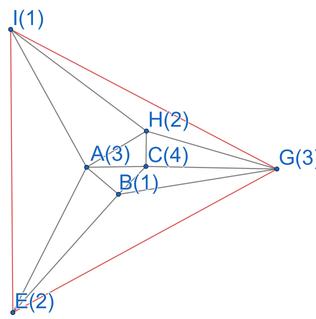
(b) The A is recovered with color 4.



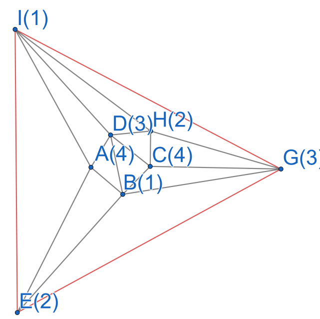
(c) The B is recovered with color 1.



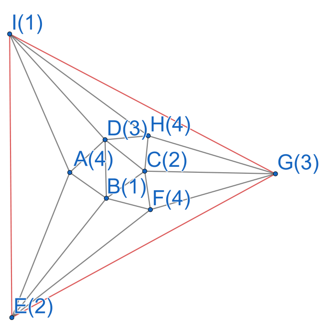
(d) The H is recovered with color 2.



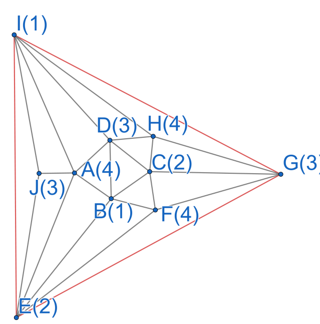
(e) The C is recovered with color 4, and the A color is switched to 3.



(f) The D is recovered with color 3, and the A color is switched to 4.



(g) The F is recovered with color 4, and the C color is switched to 2.



(h) The J is recovered with color 3.

Figure 10: Each phases and the result of applying the recovering for Fig.3g.