An Extension to Fermat's Pythagorean Triangle Area Proof, and Fermat's Last Theorem

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Abstract. Pierre de Fermat proved that the area of a Pythagorean triangle is not a square. Here we extend his result for Pythagoran triangles to consider cubic integer areas and higher power areas. We show how each such power k immediately leads to a Fermat's equation $a^k + b^k = c^k$ for integer $k > 2$ and positive integers a, b, and c. Using only elementary results, we show that a Pythagorean triangle area is not a cube. Using non-elemantary results from Darmon and Merel, we can extend Fermat's Pythagorean triangle area result to show that these areas cannot be higher powers either. The results from Darmon and Merel are an alternative to using Andrew Wiles more complex result for Fermat Last Theorem to establish the same result - using the impossibility of the Fermat equations $a^k + b^k = c^k$. Based on equations derived in this paper, we may wonder if some of these elementary results could have been known to Fermat himself. That is, could Fermat's proof that the area of a Pythagorean triangle is not a square have helped him to envision what we have come to know as Fermat equations and Fermat's Last Theorem?

1. INTRODUCTION

A Pythagorean triangle is a right triangle with positive integer sides. In the 17th century, Pierre de Fermat proved that the area of a Pythagorean triangle is not an integer squared. A version of this proof by Stillwell [1] uses the method of infinite descent, which we show in this section and use in the next section.

A Pythagorean triangle with positive integer sides A and B and hypotenuse C obeys the equation $A^2 + B^2 = C^2$. Without loss of generality, the Pythagorean triples A, B, and C are positive integers that can be put into lowest terms. In other words, they are relatively-prime, so no two of the integers have a common factor greater than 1. Primitive triples A, B, and C follow Euclid's formula:

$$A = m^2 - n^2 \tag{1}$$

$$B = 2mn \tag{2}$$

$$C = m^2 + n^2 \tag{3}$$

These formulas show that B is even, so A and C are odd. The positive integers m and n have no common factor: for otherwise, A, B, and C would share the common factor (twice) and a contradiction would result. [2,3] Also, $m > n \geq 1$. Finally, m and n have different parity, so one is even, and the other is odd. Next, Euclid's formulas are used for the area of a Pythagorean triangle.

Fermat proved that the area of a Pythagorean triangle is not an integer squared. [4, p.13] Here we outline the proof shown by Stillwell. [1] Without loss of generality, a Pythagorean triangle given by the equation $A^2 + B^2 = C^2$ has A, B, and C that are primitive Pythagorean triples. The area of the Pythagorean triangle is given by:

$$\frac{AB}{2} = mn(m^2 - n^2) = mn(m + n)(m - n) \tag{4}$$

Since m and n have no common factor, then none of $m, n, m + n, m - n$ have a common factor.

Suppose, for the sake of later contradiction, that the area of a Pythagorean triangle was a square. Then each of $m, n, m + n, m - n$ would itself be a square. Let $m = r^2$, $n = s^2$, $m + n = t^2$, and $m - n = u^2$, for relatively-prime positive integers r, s, t, u (*i.e.,* $u > 0$ since $m > n \geq 1$). Multiplying $m + n$ by $m - n$ gives another square: $(m + n)(m - n) = m^2 - n^2 = t^2u^2$. So $m^2 = (tu)^2 + n^2$ shows that n, tu, and m are another primitive Pythagorean triple. Here it is clear that tu is odd, since $(tu)^2 = A$, which was odd in Eq. (1). Euclid's formula now shows that: $tu = m_1^2 - n_1^2$, $n = 2m_1n_1$, and $m = m_1^2 + n_1^2$, where m_1 and n_1 are relatively-prime positive integers. By simple substitution, the last equation is $r^2 = m_1^2 + n_1^2$. Here n_1, m_1, and r are yet another primitive Pythagorean triple. The area of this Pythagorean triangle is $m_1n_1/2$, or $(2m_1n_1)/4 = n/4 = s^2/4$. So, the area of this Pythagorean triangle, $s^2/4$, is a square that is also smaller than the area of the original Pythagorean triangle, $AB/2 = (tu)^2(2mn)/2 = (tu)^2(r^2s^2)$, which was also a square. But this new triple, with the property that it also has an area that is a square, can then be reapplied in an infinite descent, which is impossible. So, the area for a Pythagorean triangle cannot be a square.

2. EXTENSION OF THE AREA OF A PYTHAGOREAN TRIANGLE TO CUBES AND HIGHER POWERS.

In the previous section, we considered if the area of a Pythagorean triangle (*i.e.*, Area $\triangle$) was a squared integer. In this section, we consider if the area of a Pythagorean triangle was a cubed integer or higher power. In other words:

$$\text{Area } \triangle = w^k, \text{ for positive integer } w \text{ and integer } k > 2.$$

As before, Pythagorean triple with $A = (m^2 - n^2)$, $B = 2mn$, and $C = m^2 + n^2$ corresponds to a Pythagorean triangle. The area of a Pythagorean triangle is given by Eq. (4), $\frac{AB}{2} = mn(m+n)(m-n)$, and none of $m, n, m+n, m-n$ have a common factor.

Suppose, that Area $\triangle = w^k$, for positive integer w and integer $k \geq 2$. From Eq. (4), then each of $m, n, m+n, m-n$ would itself be a power of k. Let $m = r^k$, $n = s^k$, $m + n = t^k$, and $m - n = u^k$, for relatively-prime positive integers r, s, t, and u. Then Area $\triangle = r^k s^k t^k u^k$.

By simple substitution, $m + n = r^k + s^k = t^k$ and $m - n = r^k - s^k = u^k$, and we have the following two equations, respectively:

$$\boldsymbol{r^k + s^k = t^k} \tag{5}$$

$$\boldsymbol{r^k - s^k = u^k} \tag{6}$$

Each resembles a Fermat equation for integer $k > 2$, where r, s, t, and u are relatively-prime positive integers.

Yet another Fermat equation arises using $A = m^2 - n^2 = (m+n)(m-n) = t^k u^k$, whereby $A = m^2 - n^2 = (r^k)^2 - (s^k)^2 = (r^k + s^k)(r^k - s^k) = (tu)^k$. Note that since A is odd per Eq. (1), then tu is odd. Here the Fermat equation is:

$$\boldsymbol{(tu)^k + (s^2)^k = (r^2)^k} \tag{7}$$

which is not a multiple of Eq. (5) or (6).

Adding Eq. (5) and (6), then:

$$\boldsymbol{t^k + u^k = 2r^k} \tag{8}$$

So, we see that for integer $k > 2$, we have a set of Fermat Equations shown by the last four equations.

We also know from elementary methods [4, p. 34.] that Eq. (8), $t^k + u^k = 2r^k$, has no integer solutions when $k = 3$ and t, u, and r are relatively-prime. Since the case of $k = 3$ is impossible, then we know that $\boldsymbol{r^3 + s^3 \neq t^3}$ and Area $\varDelta \neq w^3$ for positive integer w. So, the area of a Pythagorean triangle cannot be a cube.

3. FURTHER RESULTS.

In the last section, we showed equations that would result from the area of a Pythagorean triangle being a cube or higher power. Specifically, this included Eq. (8), $\boldsymbol{t^k + u^k = 2r^k}$,where $k > 2$. However, a non-elementary paper by Darmon and Merel[5] (see their Main Theorem 1) shows that the equation $\boldsymbol{t^k + u^k = 2r^k}$ "has no non-trivial primitive solutions when" integer $k \geq 3$.

This means that the Eq. (5), $\boldsymbol{r^k + s^k = t^k}$**,** where $k > 2$ cannot exist, since it was used to derive Eq. (8).

So, we can use results from Darmon and Merel, as opposed to Andrew Wiles[6] more elaborate proof of Fermat's Last Theorem, to show that Fermat Equations cannot result from the area of a Pythagorean triangle. Therefore the Area $\Delta \neq w^k$, for positive integer w and integer $k > 2$. This is an extension of Fermat's Pythagorean triangle area proof to cubic integer areas and higher powers.

Although non-elementary results were cited in this section, it is still possible that there are additional elementary results (such as indicated from the Fermat Equations from Section 2 or the Appendix) that remain to be discovered.

4. FINAL REMARKS

There is a striking similarity between Fermat's proof that the Pythagorean triangle has no square area and the Fermat equations arising from considering higher powers. We might wonder if this is how Fermat first envisioned what came to be known as Fermat's Last Theorem. Could he have had some further insight into the elementary equations shown here?

APPENDIX (Note the Appendix can be drastically changed or removed)

This appendix is included for the interested reader. It shows some other results developed during this paper that could lead to additional elementary results for Fermat's Last Theorem, perhaps by an infinite descent argument.

We first note some relationships of the right triangle, shown by DiDomenico [7, p.77] that will be used. Every triangle has an incircle with an inradius R, such as shown in Figure 2 for the right triangle with sides $A = X + R$ and $B = R + Z$, and hypotenuse $C = X + Z$.

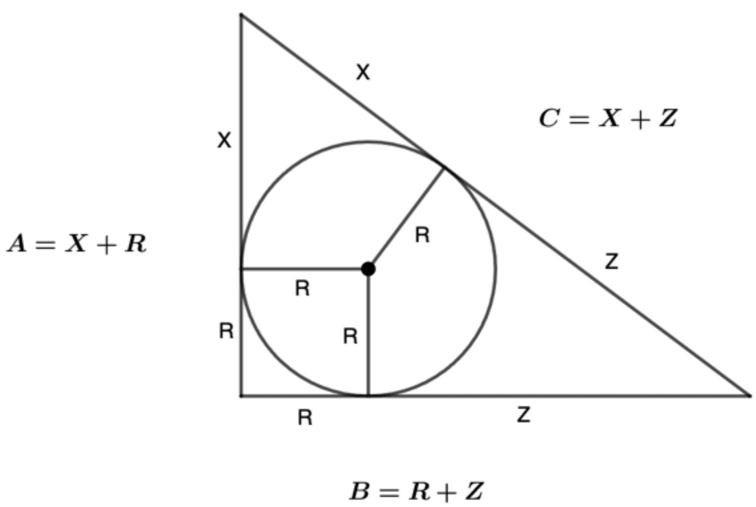

Figure 2: Right triangle with sides $A = X + R$ and $B = R + Z$, and hypotenuse $C = X + Z$. The triangle has an incircle with inradius R.

The sides of the triangle can be used to determine the inradius $R = \frac{(X+R)+(R+Z)-(X+Z)}{2} = (A + B - C)/2$. DiDomenico showed that the right triangle's area is $\frac{AB}{2} = XZ$.

To establish an elementary proof of Fermat's Last Theorem, it would be sufficient to show that $k = p$, an odd prime, would show that the Fermat equation, $a^k + b^k = c^k$, was *not* possible for integer $k > 2$ and positive integers a, b, and c. This is because it has been proven for exponent 4, and any other exponent could be rewritten with odd prime exponent p, such that $(a^w)^p + (b^w)^p = (c^w)^p$, for positive integer w. Therefore, we consider a Pythagorean triangle to have Area $\varDelta = w^p$, for positive integer w and odd prime p.

As discussed in previous papers [8,9], Fermat's equation $a^p + b^p = c^p$ for odd prime p would result in an acute triangle with sides a, b, and c. The acute triangle's area, when squared, would be a positive integer (note that this does say that the area itself is a square). In addition, it was shown that the Fermat equation $a^p + b^p = c^p$ could be rewritten with $a = x + y$, $b = y + z$, and $c = x + z$ as:

$$(x + y)^p + (y + z)^p = (x + z)^p \tag{9}$$

Here, x, y, and z are positive integers, and $y = (a + b - c)/2$ is divisible by odd prime p. The Pythagorean triangle with Area $\varDelta = w^p$ was previously shown to have sides $A = (tu)^p$, B, C such that $A^2 + B^2 = C^2$. Equation (7) show that $(tu)^p + (s^2)^p = (r^2)^p$, which corresponds to an acute triangle with a squared area that is an integer (discussed in the previous paragraph) with integer sides tu, s^2, and r^2.

Figure 3 illustrates this:

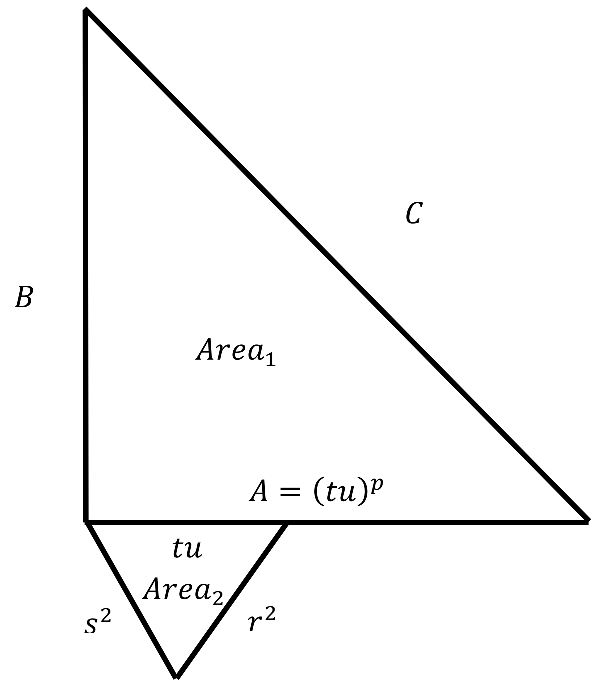

Figure 3: Illustration of a Pythagorean triangle with positive integer sides $A = (tu)^p$, B, C and $Area_1 \in \mathbb{Z}^+$; Acute triangle with positive integer sides tu, s^2, r^2 and $Area_2^2 \in \mathbb{Z}^+$.
(Not to scale) p is an odd prime.

Using relationships for a right triangle, then inradius $R = \frac{A+B-C}{2} = \frac{(tu)^p+(2mn)-(m^2+n^2)}{2} = \frac{(tu)^p+(2r^p s^p)-(r^{2p}+s^{2p})}{2}$. Equation (*7*) can be substituted for the first term in the numerator to give: $R = \frac{(r^{2p}-s^{2p})+(2r^p s^p)-(r^{2p}+s^{2p})}{2} = s^p(r^p - s^p)$. So, Eq. (*6*) shows that the inradius is:

$$R = (su)^p \tag{10}$$

This means that the inradius of the right triangle is a power of p.

Also, since $A = X + R = (tu)^p = X + (su)^p$, then $X = (tu)^p - (su)^p$. Then, from Eq. (5), $X = u^p(t^p - s^p) = u^p r^p$, so:

$$X = (ur)^p \tag{11}$$

The Area $\Delta = r^p s^p t^p u^p = XZ$, per DiDomenico, so $Z = \frac{r^p s^p t^p u^p}{X} = \frac{r^p s^p t^p u^p}{u^p r^p} = s^p t^p$:

$$Z = (st)^p \tag{12}$$

Since the hypotenuse of the right triangle is $C = X + Z$, then we have that $C = (ur)^p + (st)^p$.

There are other results that arise from considering when the sides of a Pythagorean Triangle are not divisible by odd prime $\boldsymbol{p}$**.** In this section, we show that Eq. (6), $r^p = u^p + s^p$, which clearly has smaller primitive solutions than Eq. (5) , $r^p + s^p = t^p$, still does not have the smallest possible solutions for a Fermat equation with odd prime p, when relatively-prime positive integers r, s, t, and u are not divisible by p.

We begin with the following known elementary result [4, pp. 100-101] that are adapted to a notation useful to this paper:

Theorem 1:

Let p be an odd prime where $d^p + e^p = f^p$, with relatively-prime positive integers d, e, and f, **which are not divisible by $\boldsymbol{p}$**. Then:

$$f - e = d_1^p, \quad d^p/(f-e) = d_2^p, \qquad d = d_1 d_2$$

$$f - d = e_1^p, \quad e^p/(f-d) = e_2^p, \qquad e = e_1 e_2$$

$$d + e = f_1^p, \quad f^p/(d+e) = f_2^p, \qquad f = f_1 f_2$$

where $d_1, d_2, e_1, e_2, f_1, f_2$ are pairwise relatively-prime positive integers, and d_2, e_2, f_2 are odd and greater than 1. Also, it is noted in the text that $d_1^p + e_1^p \neq f_1^p$.

We now use Theorem 1, **where we assume that none of $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, and $\boldsymbol{u}$ are divisible by $\boldsymbol{p}$.**

Eq. (5):

$$t^p - s^p = r^p, \qquad t - s = r_1^p, \quad r^p/(t-s) = r_2^p, \qquad r = r_1 r_2$$

$t^p - r^p = s^p, \qquad t - r = s_1^p, \quad s^p/(t-r) = s_2^p, \qquad s = s_1 s_2$

$r^p + s^p = t^p, \qquad r + s = t_1^p, \quad t^p/(r+s) = t_2^p, \qquad t = t_1 t_2$

where $r_1, r_2, s_1, s_2, t_1, t_2$ are pairwise relatively-prime positive integers, and r_2, s_2, t_2 are odd and greater than 1.

Eq. (6):

$r^p - s^p = u^p, \qquad r - s = u_1^p, \quad u^p/(r-s) = u_2^p, \qquad u = u_1 u_2$

$r^p - u^p = s^p, \qquad r - u = s_3^p, \quad s^p/(r-u) = s_4^p, \qquad s = s_3 s_4$

$u^p + s^p = r^p, \qquad u + s = r_3^p, \quad r^p/(u+s) = r_4^p, \qquad r = r_3 r_4$

where $u_1, u_2, s_3, s_4, r_3, r_4$ are pairwise relatively-prime positive integers, and u_2, s_4, r_4 are odd and greater than 1. Note that we do not assume that $s_1 = s_3$, $s_2 = s_4$, $r_1 = r_3$, and $r_2 = r_4$, as discussed further below.

Using these equations for $(u + s) + (r - u) = (r + s)$, then:

$$r_3^p + s_3^p = t_1^p \tag{13}$$

Also, $(t - s) - (t - r) = (r - s)$, so:

$$r_1^p - s_1^p = u_1^p \tag{14}$$

Suppose earlier that it was assumed that $s_1 = s_3$, $s_2 = s_4$, $r_1 = r_3$, and $r_2 = r_4$. Then, Eq. (13) would become $r_1^p + s_1^p = t_1^p$, which would be a contradiction of $r_1^p + s_1^p \neq t_1^p$ from Theorem 1 for Eq. (5), $r^p + s^p = t^p$.

It is evident from Eq. (13) that $t_1 > 1$. The closest that t_1 and s_3 could be is $t_1 = s_3 + 1$, in which case $t_1^p - s_3^p = (s_3 + 1)^p - s_3^p = r_3^p > 1$. So, $r_3 > 1$. The same argument shows that $s_3 > 1$. A similar argument can be applied to Eq. (14) to show that u_1, r_1, s_1 are all greater than 1. So, we have that $r_1, r_2, s_1, s_2, t_1, t_2$ are pairwise relatively-prime positive integers all greater than 1. And u_1, u_2, s_3, s_4, r_3, r_4 are pairwise relatively-prime positive integers all greater than 1.

Eq. (13), $r_3^p + s_3^p = t_1^p$, provides for smaller primitive solutions than Eq. (5), $r^p + s^p = t^p$, which can be rewritten as $(r_3 r_4)^p + (s_3 s_4)^p = (t_1 t_2)^p$. Since both equations are valid, then we might suspect that they can be related by $r_4 = s_4 = t_2 = 1$, but this is not the case, since they are all greater than 1.

Also, Eq. (14), $r_1^p - s_1^p = u_1^p$, provides for smaller primitive solutions than Eq. (6), $r^p - s^p = u^p$, which can be rewritten as $(r_1 r_2)^p + (s_1 s_2)^p = (u_1 u_2)^p$. Since both of these equations are valid, then we might suspect that they can be related by $r_2 = s_2 = u_2 = 1$, but this is not the case again, since they are all greater than 1. It is possible that some additional elementary results for Fermat's Last Theorem, or even an infinite descent argument, could still be determined, maybe by a reader of this paper.

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Note that if the area of a Pythagorean triangle is a cube or higher power, then Fermat's Last Theorem is false. The contrapositive is that if Fermat's Last Theorem is true, then the area of a Pythagorean triangle is not a cube or higher

power. Andrew Wiles proved Fermat's Last Theorem by non-elementary means, so the last sentence must be true, as we showed in this paper using results from Darmon and Merel instead.

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