A proof of a conjecture of Lu concerning inequality for the Gamma function

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ABSTRACT

Dawei Lu [Dawei Lu, A generated approximation related to Burnside's formula, Journal of Number Theory 136 (2014) 414–422; http://dx.doi.org/10.1016/j.jnt.2013.10.016] proposed a conjecture: for every real number $k>0$, there exists m_1 depending k, such that for every $x \geq m_1$, it holds:

$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}\left(1-\frac{k}{24x}+\left(\frac{k^2}{1152}+\frac{k}{48}\right)\frac{1}{x^2}\right)^{1/k}.$$

He guessed that it is suitable for taking $m_1 = 0.5$. In this paper, we prove the conjecture of Dawei Lu.

Keywords: Rate of convergence, Monotonic function, Gamma function, Burnside Formula

1. Introduction

The big factorials arise in several situations in the mathematics and other branches of science.

Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

is one of the most well-known formulas for approximation of the factorial function.

Burnside's formula [3]:

$$n! \approx \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2}, \tag{1.2}$$

which is more precise than (1.1).

In [5], Dawei provided a polynomial approximation for factorial function starting from (1.2).

$$n! \approx \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2}\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\frac{c_3}{n^3}+\cdots\right)^{1/k}, \tag{1.3}$$

where

$$c_1 = -\frac{k}{24}, c_2 = \frac{k^2}{1152}+\frac{k}{48}, c_3 = -\frac{23k}{2880}-\frac{k^2}{1152}-\frac{k^3}{82944}, \cdots.$$

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Then, Using (1.3), he showed an inequality for the gamma function as follows: for every positive real number *k*, there exists m_1 depending *k*, such that for every $x \ge m_1$, it holds:

$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}\left(1-\frac{k}{24x}+\left(\frac{k^2}{1152}+\frac{k}{48}\right)\frac{1}{x^2}\right)^{1/k}.$$

In particular, he proposed the following conjecture:

Conjecture. It is suitable for taking $m_1 = 0.5$.

The aim of this paper is to prove the conjecture of Dawei Lu concerning inequality for gamma function.

2. Proof of conjecture

To give the proof of Conjecture, we need the following result of Alzer [2] for all $x > 0$ and $n \ge 0$,

$$\ln\Gamma(x+1) = \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{j=1}^{n}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} + (-1)^n R_n(x), \tag{2.1}$$

where $R_n(x)$ is completely monotonic on $(0,\infty)$, B_j is the *j-th* Bernoulli number defined by the power series expansion

$$\frac{x}{e^x-1} = \sum_{l=0}^{\infty} B_l \frac{x^l}{l!} = 1 - \frac{x}{2} + \sum_{l=1}^{\infty} B_{2l}\frac{x^{2l}}{(2l)!}. \tag{2.2}$$

For all $l \ge 1$, $B_{2l+1} = 0$ and the first few Bernoulli numbers are $B_1 = -1/2, B_2 = 1/6$ $B_4 = -1/30$, $B_6 = 1/42$.

From (2.2), we have the following inequalities, for *x*>0.

$$\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}\right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}\right). \tag{2.3}$$

For the right inequality in Theorem, combining (2.3), we need to get

$$\exp\left(\frac{1}{2}+\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}\right) < \left(1+\frac{1}{2x}\right)^{x+\frac{1}{2}}\left(1-\frac{k}{24x}+\left(\frac{k^2}{1152}+\frac{k}{48}\right)\frac{1}{x^2}\right)^{1/k}. \tag{2.4}$$

Inequality (2.4) is equivalent to $g_k(x) > 0$, where

$$g_k(x) = \left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{2x}\right) + \frac{1}{k}\ln\left(1-\frac{k}{24x}+\left(\frac{k^2}{1152}+\frac{k}{48}\right)\frac{1}{x^2}\right) - \left(\frac{1}{2}+\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}\right) \tag{2.5}$$

Let $t = 2x$, then from (2.5)

$$g_k(t)=\left(\frac{t}{2}+\frac{1}{2}\right)\ln\left(1+\frac{1}{t}\right)+\frac{1}{k}\ln\left(1-\frac{k}{12t}+\left(\frac{k^2}{288}+\frac{k}{12}\right)\frac{1}{t^2}\right)-\left(\frac{1}{2}+\frac{1}{6t}-\frac{1}{45t^3}+\frac{1}{39.375t^5}\right). \quad (2.6)$$

From (2.6), it is easy to obtain

$$g_k'(t)=-\frac{1}{630}\frac{G_k(t)}{t^6(288t^2-24kt+k^2+24k)}, \quad (2.7)$$

where

$$\begin{aligned}G_k(t)&=-90720\ln\left(1+\frac{1}{t}\right)t^8+\left(7560k\ln\left(1+\frac{1}{t}\right)+90720\right)t^7\\&-\left(315\ln\left(1+\frac{1}{t}\right)k^2+7560\left(\ln\left(1+\frac{1}{t}\right)+1\right)k+45360\right)t^6+\left(315k^2+11340k+30240\right)t^5\\&-\left(105k^2+2520k-12096\right)t^4-1008kt^3+(42k^2+1008k-23040)t^2+1920kt-80k^2-1920k.\end{aligned} \quad (2.8)$$

For any $t\ge 1$,

$$\ln\left(1+\frac{1}{t}\right)<\left(\frac{1}{t}-\frac{1}{2t^2}+\frac{1}{3t^3}-\frac{1}{4t^4}+\frac{1}{5t^5}\right). \quad (2.9)$$

The coefficient of $\ln(1+1/t)$ in $G_k(t)$ is as follows:

$$-\left(90720t^8-7560t^7+(315k^2+7560k)t^6\right)=-90720t^6\left(\left(t-\frac{1}{24}\right)^2+(315k^2+7560k-157.5)\right). \quad (2.10)$$

As $t\ge 1, k\ge 1$, (2.10) is negative. Using (2.8), we have

$$G_k(t)>H(t), \quad (2.11)$$

where

$$\begin{aligned}H(t)=&((210k^2+15120k+139104)t^4-(420k^2+21672k+72576)t^3\\&+(483k^2+17640k-92160)t^2+(-252k^2+1632k)t-320k^2-7680k)/4.\end{aligned} \quad (2.12)$$

Also,

$$\begin{aligned}H(t+2)=&\left(\frac{105}{2}k^2+3780k+34776\right)t^4+(315k^2+24822k+260064)t^3\\&+\left(\frac{3003}{4}k^2+62622k+702720\right)t^2+(840k^2+73992k+802944)t\\&+277k^2+33672k+319104\end{aligned} \quad (2.13)$$

is polynomial of degree 4 with all positive coefficients.

Combining (2.8)-(2.13), for every $t\in[2,+\infty)$, $g_k'(t)<0$. Thus, $g_k(t)$ is strictly decreasing on $[2,+\infty)$ with $g_k(\infty)=0$, so for every $t\in[2,+\infty)$, $g_k(t)>0$.

Now, we need to prove for $t \in [1,2]$, $g_k(t) > 0$ because m_1 is 0.5 in conjecture.

First, we prove $g_k(1) > 0$. In fact,

$$g_k(1) = \ln 2 + \frac{1}{k}\ln\left(1 + \frac{k^2}{288}\right) - \frac{211}{315} > 0,$$

since

$$2\left(1 + \frac{k^2}{288}\right)^{\frac{1}{k}} - \exp\frac{211}{315} > 2 - \exp\frac{211}{315} = 0.0461 > 0.$$

Next, we prove $G_k(t) > 0$, for $t \in [1,2]$. From (2.8), we have

$$\begin{aligned} G_k(t) = &-90720\ln\left(1+\frac{1}{t}\right)t^8 + \left(7560k\ln\left(1+\frac{1}{t}\right) + 90720\right)t^7 \\ &- \left(315\ln\left(1+\frac{1}{t}\right)k^2 + 7560\left(\ln\left(1+\frac{1}{t}\right)+1\right)k + 45360\right)t^6 \\ &+ \left(315k^2 + 11340k + 30240\right)t^5 - \left(105k^2 + 2520k - 12096\right)t^4 \\ &- 1008kt^3 + (42k^2 + 1008k - 23040)t^2 + 1920kt - 80k^2 - 1920k. \end{aligned} \tag{2.14}$$

Differentiating (2.14), we get that

$$\begin{aligned} -G_k'(t) = &725760\ln\left(1+\frac{1}{t}\right)t^8 + \left(725760\ln\left(1+\frac{1}{t}\right) - 52920k\ln\left(1+\frac{1}{t}\right) - 725760\right)t^7 \\ &+ \left(52920k - 7560k\ln\left(1+\frac{1}{t}\right) + 1890k^2\ln\left(1+\frac{1}{t}\right) - 362880\right)t^6 \\ &+ \left(45360k\ln\left(1+\frac{1}{t}\right) - 18900k + 1890k^2\ln\left(1+\frac{1}{t}\right) - 1890k^2 + 120960\right)t^5 \\ &+ (-1155k^2 - 46620k - 199584)t^4 + (420k^2 + 13104k - 48384)t^3 \\ &+ (-84k^2 + 1008k + 46080)t^2 + (-84k^2 - 3936k + 46080)t - 1920k)/(t+1). \end{aligned} \tag{2.15}$$

The coefficient of $\ln(1+1/t)$ in (2.15) is as follows:

$$(725760t^3 - (52920k - 725760)t^2 - (7560k - 1890k^2)t + 1890k^2 + 45360k)t^5. \tag{2.16}$$

For $t \in [1,2]$, (2.16) is positive, since

$$\begin{aligned} &(725760t^3 - (52920k - 725760)t^2 - (7560k - 1890k^2)t + 1890k^2 + 45360k) \\ &\geq (3780k^2 - 75600k + 1451520) = 3780(k-10)^2 + 1073520 > 0. \end{aligned} \tag{2.17}$$

For any $t \geq 1$, we have

$$\ln\left(1+\frac{1}{t}\right)<\left(\frac{1}{t}-\frac{1}{2t^2}+\frac{1}{3t^3}-\frac{1}{4t^4}+\frac{1}{5t^5}-\frac{1}{6t^6}+\frac{1}{7t^7}-\frac{1}{8t^8}+\frac{1}{9t^9}-\frac{1}{10t^{10}}+\frac{1}{11t^{11}}\right). \quad (2.18)$$

Thus, $G_k^{'}(t)>J(t)$, where

$$\begin{aligned} J(t)=&\big((9240k^2+665280k+6120576)t^{10}+(-4620k^2-49896k+3725568)t^9\\ &+(-3234k^2-327096k-3091968)t^8+(7854k^2+350592k-1267200)t^7\\ &+(-2772k^2-37488k-570240)t^6+(1980k^2+89100k+443520)t^5\\ &+(-1485k^2-67980k-354816)t^4+(1155k^2+53592k+290304)t^3+(-924k^2-43344k\\ &-2903040)t^2+(756k^2+229824k)t-7560k^2-181440k\big)/(44t^7+44t^6). \end{aligned} \quad (2.19)$$

Also, we have

$$\begin{aligned} J(t+1)=&\big((9240k^2+665280k+6120576)t^{10}+(87780k^2+6602904k+64931328)t^9\\ &+(370986k^2+29161440k+305864064)t^8+(924462k^2+75771168k+842586624)t^7\\ &+(1513974k^2+128775504k+1502252928)t^6+(1715538k^2+150272892k+1809067392)t^5\\ &+(1373625k^2+122611104k+1487261952)t^4+(774081k^2+69518592k+821816064)t^3\\ &+(295713k^2+26383320k+288080640)t^2+(69627k^2+6293676k+53571456)t+390k^2\\ &+681144k+2392704\big)/(44t^7+352t^6+1188t^5+2200t^4+2420t^3+1584t^2+572t+88)>0. \end{aligned} \quad (2.20)$$

For $t\in[1,2]$, we have $G_k^{'}(t)>0$. Also,

$$G_k(1)=-46.3414k^2+1260k+1773.7,\quad G_k(2)=313.8234k^2+32788k+361524.6>0.$$

We consider two cases according to a sign of $G_k(1)$.

(i) $G_k(1)>0$ for $k\le 28$.

In this case, $t\in[1,2], g_k^{'}(t)<0,$ since $G_k(t)>0$, for $t\in[1,2]$. Thus, we have $g_k(t)>0$.

(ii) $G_k(1)<0$ for $k>28$

For $t\in[1,2],\ G_k^{'}(t)>0$, and $G_k(2)>0$, then

$$\exists t_1\in[1,2], t\in[1,t_1), G_k(t)<0, t\in(t_1,2], G_k(t)>0.$$

Thus, $t\in[1,t_1), g_k^{'}(t)>0,$ and $t\in(t_1,2], g_k^{'}(t)<0.$ From $g_k(1)>0,$ for $t\in[1,2],\ g_k(t)>0.$ Hence, $g_k(t)>0$ for $t\ge 1$, and this implies that it is suitable for taking $m_1=0.5$.

We complete proof of conjecture.

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