C001107 “Eureka” Shift, Taylor Shift, Offset, Symmetry Point, and Symmetry in Polynomials.

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1. Introduction.

In this study we show the existence of three types of shifts in polynomial curves that will always result in integer sequences: 1. "Eureka" shift, 2. Taylor shift, and 3. Offset.

Then, we demonstrate that every polynomial equation has a reference point that we call sp - symmetry point.

From the symmetry point of any polynomial sequence of integers we can define two types of symmetry and one type of asymmetry.

At the end, we name and define asymmetry, and the two types of symmetries.

Please, consult the last version of this study at:

https://www.facebook.com/groups/snypo/posts/421837489473196

Please, consult the last version of C000000 Conventions, notations, abbreviations, glossary, and references at:

https://www.facebook.com/groups/snypo/posts/653023753021234/.

Please, consult some threads at:
https://www.mersenneforum.org/showthread.php?p=618837#post618837.

2. The "Eureka shift".

Given the polynomial where the index y, the degree d, and the coefficients a_n are integers:

$$Yd[y] = a_d y^d + a_{d-1} y^{d-1} + a_{d-2} y^{d-2} + \cdots + a_4 y^4 + a_3 y^3 + a y^2 + by + c$$

The curve shift is always continuous and can move the polynomial curve at any position parallel to a chosen axis. There is no rotation of the curve when we do a shift.

If we shift the polynomial curve along the X-axis, we get:

$$Yd[y] + m = a_d y^d + a_{d-1} y^{d-1} + a_{d-2} y^{d-2} + \cdots + a_4 y^4 + a_3 y^3 + a y^2 + by + c + m$$

Now, $c' = c + m$, and:

$$Yd[y] + m = a_d y^d + a_{d-1} y^{d-1} + a_{d-2} y^{d-2} + \cdots + a_4 y^4 + a_3 y^3 + a y^2 + by + c'$$

We propose to call this polynomial curve shift with integer steps $m = integer$ as "Eureka shift". This way we can get a kind of "Eureka" polynomial sequences. We created the name "Eureka Shift" inspired by Neil Sloane video available online at https://www.youtube.com/watch?v=6X2D497is6Y.

3. The Taylor shift.

If we shift the polynomial curve

$$Y[y] = a_d y^d + a_{d-1} y^{d-1} + a_{d-2} y^{d-2} + \cdots + a_4 y^4 + a_3 y^3 + a y^2 + by + c$$

along the Y-axis, we get a new expression:

$$Y[y+h] = a_d(y+h)^d + a_{d-1}(y+h)^{d-1} + a_{d-2}(y+h)^{d-2} + \cdots + a_4(y+h)^4 + a_3(y+h)^3 + a(y+h)^2 + b(y+h) + c$$

This shift h of the polynomial curve along the Y-axis can be any Real value.
We can analyze this Y-axis shift using the Taylor expansion at the point $y = h$.

$$Y[y] = \sum_{n=0}^{d} \frac{Y^{[n]}[h]}{n!}(y-h)^n$$

$$Y[y] = Y[h] + Y^{[1]}[h](y-h) + \frac{Y^{[2]}[h]}{2!}(y-h)^2 + \frac{Y^{[3]}[h]}{3!}(y-h)^3 + \cdots + \frac{Y^{[d-3]}[h]}{(d-3)!}(y-h)^{(d-3)} + \frac{Y^{[d-2]}[h]}{(d-2)!}(y-h)^{(d-2)} + \frac{Y^{[d-1]}[h]}{(d-1)!}(y-h)^{(d-1)} + \frac{Y^{[d]}[h]}{d!}(y-h)^d$$

Let us write the polynomial above reversing the direction of its terms:

$$Y[y] = \frac{Y^{[d]}[h]}{d!}(y-h)^d + \frac{Y^{[d-1]}[h]}{(d-1)!}(y-h)^{(d-1)} + \frac{Y^{[d-2]}[h]}{(d-2)!}(y-h)^{(d-2)} + \frac{Y^{[d-3]}[h]}{(d-3)!}(y-h)^{(d-3)} + \cdots + \frac{Y^{[3]}[h]}{3!}(y-h)^3 + \frac{Y^{[2]}[h]}{2!}(y-h)^2 + Y^{[1]}[h](y-h) + Y[h]$$

So,

$$Y[y+h] = \frac{Y^{[d]}[h]}{d!}y^d + \frac{Y^{[d-1]}[h]}{(d-1)!}y^{(d-1)} + \frac{Y^{[d-2]}[h]}{(d-2)!}y^{(d-2)} + \frac{Y^{[d-3]}[h]}{(d-3)!}y^{(d-3)} + \cdots + \frac{Y^{[3]}[h]}{3!}y^3 + \frac{Y^{[2]}[h]}{2!}y^2 + Y^{[1]}[h]y + Y[h]$$

Let us say,

$$Y[y+h] = \mathring{a}_d y^d + \mathring{a}_{d-1} y^{d-1} + \mathring{a}_{d-2} y^{d-2} + \cdots + \mathring{a}_3 y^3 + \mathring{a} y^2 + \mathring{b} y + \mathring{c}$$

Or,

$$Y[y+h] = \mathring{a}_d y^d + \mathring{a}_{d-1} y^{d-1} + \mathring{a}_{d-2} y^{d-2} + \cdots + \mathring{a}_3 y^3 + \mathring{a}_2 y^2 + \mathring{a}_1 y + \mathring{a}_0$$

Then,

$$\mathring{a}_d = \frac{Y^{[d]}[h]}{d!}$$
$$\mathring{a}_{d-1} = \frac{Y^{[d-1]}[h]}{(d-1)!}$$
$$\mathring{a}_{d-2} = \frac{Y^{[d-2]}[h]}{(d-2)!}$$
$$\ldots$$
$$\mathring{a}_2 = \frac{Y^{[2]}[h]}{2!}$$
$$\mathring{a}_1 = Y^{[1]}[h]$$
$$\mathring{a}_0 = Y[h]$$

Because,

$$Y[y] = a_d y^d + a_{d-1} y^{d-1} + a_{d-2} y^{d-2} + \cdots + a_3 y^3 + a_2 y^2 + a_1 y + a_0$$

Then,

$$a_0^{\circ} = Y[h] = a_d h^d + a_{d-1}h^{d-1} + a_{d-2}h^{d-2} + \cdots + a_3h^3 + a_2h^2 + a_1h + a_0$$

$$a_1^{\circ} = Y^{[1]}d[h] = da_dh^{d-1} + (d-1)a_{d-1}h^{d-2} + (d-2)a_{d-2}h^{d-3} + \cdots + 3a_3h^2 + 2a_2h + a_1$$

$$a_2^{\circ} = \frac{Y^{[2]}[h]}{2!} = \frac{d(d-1)a_dh^{d-2} + (d-1)(d-2)a_{d-1}h^{d-3} + \cdots + 3.2.\,a_3h + 2a_2}{2!}$$

Before we continue, please note that for any polynomial of the 2nd degree or higher, the 2nd derivative, or higher will always be an even number for the integers. All the next derivatives will continue to be an even number for the integers.

...

$$a_{d-3}^{\circ} = \frac{Y^{[d-3]}[h]}{(d-3)!}$$

$$= \left(\frac{d(d-1)(d-2)}{3!}a_dh^3 + \frac{(d-1)(d-2)}{2}a_{d-1}h^2 + (d-2)a_{d-2}h\right) + \frac{(d-3)!\,a_{d-3}}{(d-3)!}$$

$$= \frac{d(d-1)(d-2)}{3!}a_dh^3 + \frac{(d-1)(d-2)}{2}a_{d-1}h^2 + (d-2)a_{d-2}h + a_{d-3}$$

$$a_{d-2}^{\circ} = \frac{Y^{[d-2]}[h]}{(d-2)!} = \frac{\left(\frac{d!\,a_dh}{2} + (d-1)!\,a_{d-1}\right)h + (d-2)!\,a_{d-2}}{(d-2)!}$$

$$= \frac{d(d-1)}{2}a_dh^2 + (d-1)a_{d-1}h + a_{d-2}$$

$$a_{d-1}^{\circ} = \frac{Y^{[d-1]}[h]}{(d-1)!} = \frac{d!\,a_dh + (d-1)!\,a_{d-1}}{(d-1)!} = da_dh + a_{d-1}$$

$$a_d^{\circ} = \frac{Y^{[d]}[h]}{d!} = \frac{d!\,a_d}{d!} = a_d$$

4. Offset in polynomials.

Let's define offset in a polynomial $Y[y]$ as the Taylor shift $Y[y+h]$ with $h = integer$.

5. Symmetry Point (sp) in polynomials

Let's define the symmetry point (sp) of a polynomial curve as the point on the polynomial curve that divides the curve into 2 parts as symmetrically as possible.

In these studies, the coordinates of a symmetry point in the XY plane are x_{sp} and y_{sp}. Also, we denote a symmetry point as being the point $sp = (x_{sp}, y_{sp})$.

To define the formula for the coordinates (x_{sp}, y_{sp}) of the symmetry point, we will use the results obtained from the Taylor shift.

5.1. Symmetry point coordinates

Notice that, if we want to have a symmetric polynomial, we just may do:

$$Y[y] = a_d y^d$$

This means that any Taylor shift in this curve results in:

$$Y[y+h] = a_d y^d + d a_d h y^{d-1} + \frac{d(d-1)}{2} a_d h^2 y^{d-2} + \frac{d(d-1)(d-2)}{3!} a_d h^3 y^{d-3} + \cdots + \frac{d(d-1)(d-2)}{3!} a_d h^{d-3} y^3 + \frac{d(d-1)}{2} a_d h^{d-2} y^2 + d a_d h^{d-1} y^1 + a_d h^d$$

These coefficients when integers are the Pascal´s triangle coefficients.

This is an important hint that we can get all coefficients of the Taylor shift from Pascal's triangle. We will show how Shaw and Traub method for the Taylor shift is fully based on Pascal´s triangle.

If $h = integer$ the two expressions above represent the same sequence of integers. What changes is only the displacement of the same curve along the Y-axis.

Then, if there is a symmetry point in each of the two curves, this point of symmetry is in the same position with respect to the integer elements of the sequence or with respect to the infinite points of the polynomial curve.

Because,

$$Y[y+h] = a_d (y+h)^d + a_{d-1}(y+h)^{d-1} + a_{d-2}(y+h)^{d-2} + \cdots + a_3 (y+h)^3 + a(y+h)^2 + b(y+h) + c$$

Or,

$$Y[y] = a_d y^d + a_{d-1} y^{d-1} + a_{d-2} y^{d-2} + \cdots + a_4 y^4 + a_3 y^3 + a y^2 + b y + c$$

Let us define $sp_1 = (x_{sp1}, y_{sp1})$ as the point of symmetry of $Y[y]$ and $sp_2 = (x_{sp} \ , y_{sp2})$ as the point of symmetry of $Y[y+h]$.

To get an equation for the symmetry point in any polynomial, then two things must occur when we perform a Taylor shift from $Y[y]$ with $sp_1 = (x_{sp1}, y_{sp1})$ to $Y[y+h]$ with $sp_2 = (x_{sp2}, y_{sp2})$:

$$x_{sp2} = x_{sp1}$$
$$y_{sp2} = y_{sp} \ + h$$

Then, we must define the Y-coordinate of the symmetry point as being the value of y when:

$$Y^{[d-1]}[y] = \frac{d^{d-1}}{dy^{d-1}}(Y[y]) = 0$$
$$Y^{[d-1]}[y] = d!\, a_d y + (d-1)!\, a_{d-1}$$

And for $y = y_{sp}$:

$$d!\, a_d y_{sp} + (d-1)!\, a_{d-1} = 0$$
$$y_{sp} = -\frac{(d-1)!\, a_{d-1}}{d!\, a_d}$$
$$y_{sp} = -\frac{a_{d-1}}{d a_d}$$

And,

$$x_{sp} = Y[y_{sp}] = Y\left[-\frac{a_{d-1}}{d a_d}\right]$$

So,

$$symmetry\ point = sp = (x_{sp}, y_{sp}) = \left(Y\left[-\frac{a_{d-1}}{da_d}\right], -\frac{a_{d-1}}{da_d}\right)$$

When the polynomial generates a symmetric integer sequence, then the symmetry point coincides with one of the inflection points and/or with one of the Real roots.

When the polynomial generates an asymmetric integer sequence, then the point of symmetry will be closer to the most central inflection point of the curve and/or the most central Real root, if any.

6. Taylor shift, symmetry point (sp) and Pascal´s triangle

Let's better understand the dynamics of the symmetry point, the symmetry of the Taylor shift equation, and the symmetry in Pascal's triangle.

6.1. Taylor shift for 1st degree polynomials

$$Y1[y] = by + c$$
$$Y^{[1]}1[y] = b$$

Then,

$$Y1[y+h] = Y^{[1]}[h]y + Y[h]$$
$$Y1[y+h] = by + bh + c$$

Doing $h = f = integer$, we have

$bº = b$	$b = bº$
$cº = bh + c$	$c = -bh + cº$

Symmetry point:

$$y_{sp} = -\frac{a_{d-1}}{da_d} = -\frac{c}{b}$$
$$x_{sp} = Y1\left[-\frac{c}{b}\right] = b\left(-\frac{c}{b}\right) + c = 0$$

$$sp_{Y1[y]} = (x_{sp}, y_{sp}) = \left(0, -\frac{c}{b}\right)$$

6.2. Taylor shift for 2nd degree polynomials

$$Y2[y] = ay^2 + by + c$$
$$Y^{[1]}2[y] = 2ay + b$$
$$Y^{[2]}2[y] = 2a$$

Then,

$$Y2[y+h] = \frac{Y^{[2]}[h]}{2!}y^2 + Y^{[1]}[h]y + Y[h]$$
$$Y2[y+h] = \frac{2a}{2!}y^2 + (2ah + b)y + ah^2 + bh + c$$
$$Y2[y+h] = ay^2 + (2ah + b)y + ah^2 + bh + c$$

Doing $h = f = integer$, we have

$aº = a$	$a = aº$
$bº = b + 2ah$	$b = bº - 2ah$
$cº = ah^2 + bh + c$	$c = aºh^2 - bºh + cº$

Symmetry point:

$$y_{sp} = -\frac{a_{d-1}}{da_d} = -\frac{b}{2a}$$

$$x_{sp} = Y2\left[-\frac{b}{2a}\right] = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{-b^2 + 4ac}{4a}$$

$$sp_{Y2[y]} = (x_{sp}, y_{sp}) = \left(-\frac{b^2 - 4ac}{4a}, -\frac{b}{2a}\right)$$

6.3. Taylor shift for 3rd degree polynomials

$$Y3[y] = a_3y^3 + ay^2 + by + c$$
$$Y^{[1]}3[y] = 3a_3y^2 + 2ay + b$$
$$Y^{[2]}3[y] = 6a_3y + 2a$$
$$Y^{[3]}3[y] = 6a_3$$

$$Y3[y+h] = \frac{Y^{[3]}[h]}{3!}y^3 + \frac{Y^{[2]}[h]}{2!}y^2 + Y^{[1]}[h]y + Y[h]$$

$$Y3[y+h] = \frac{6a_3}{3!}y^3 + \frac{6a_3h + 2a}{2!}y^2 + (3a_3h^2 + 2ah + b)y + a_3h^3 + ah^2 + bh + c$$

$$Y3[y+h] = a_3y^3 + (3a_3h + a)y^2 + (3a_3h^2 + 2ah + b)y + a_3h^3 + ah^2 + bh + c$$

Doing $h = f = integer$, we have:

$$aº = 3a_3h + a$$
$$a = aº - 3a_3^{º}h$$
$$bº = 3a_3h^2 + 2ah + b$$
$$b = -3a_3h^2 - 2ah + b^{º}$$
$$b = -3a_3^{º}h^2 - 2(aº - 3a_3^{º}h)h + b^{º}$$
$$b = -3a_3^{º}h^2 - 2aºh + 6a_3^{º}h^2 + b^{º}$$
$$b = 3a_3^{º}h^2 - 2aºh + b^{º}$$
$$cº = a_3h^3 + ah^2 + bh + c$$
$$c = -a_3h^3 - ah^2 - bh + cº$$
$$c = -a_3^{º}h^3 - (aº - 3a_3^{º}h)h^2 - (3a_3^{º}h^2 - 2aºh + b^{º})h + cº$$
$$c = -a_3^{º}h^3 - aº\mathrm{h}^2 + 3a_3^{º}h^3 - 3a_3^{º}h^3 + 2aºh^2 - b^{º}h + cº$$
$$c = -a_3^{º}h^3 + aº\mathrm{h}^2 - b^{º}h + cº$$

$a_3^{º} = a_3$	$a_3 = a_3^{º}$
$aº = 3a_3h + a$	$a = aº - 3a_3^{º}h$
$bº = 3a_3h^2 + 2ah + b$	$b = 3a_3^{º}h^2 - 2aºh + b^{º}$
$cº = a_3h^3 + ah^2 + bh + c$	$c = -a_3^{º}h^3 + aºh^2 - b^{º}h + cº$

Symmetry point:

$$y_{sp} = -\frac{a_{d-1}}{da_d} = -\frac{a}{3a_3}$$

$$x_{sp} = Y3\left[-\frac{a}{3a_3}\right] = a_3\left(-\frac{a}{3a_3}\right)^3 + a\left(-\frac{a}{3a_3}\right)^2 + b\left(-\frac{a}{3a_3}\right) + c$$

$$x_{sp} = -\frac{a^3}{27a_3^2} + \frac{a^3}{9a_3^2} - \frac{ab}{3a_3} + c$$

$$sp_{Y3[y]} = \left(x_{sp}, y_{sp}\right) = \left(-\frac{a^3}{27a_3^2} + \frac{a^3}{9a_3^2} - \frac{ab}{3a_3} + c, -\frac{a}{3a_3}\right)$$

6.4. Taylor shift for 4th degree polynomials

$$Y4[y] = a_4y^4 + a_3y^3 + ay^2 + by + c$$

$$Y^{[1]}4[y] = 4a_4y^3 + 3a_3y^2 + 2ay + b$$

$$Y^{[2]}4[y] = 12a_4y^2 + 6a_3y + 2a$$

$$Y^{[3]}4[y] = 24a_4y + 6a_3$$

$$Y^{[4]}4[y] = 24a_4$$

$$Y4[y+h] = \frac{Y^{[4]}[h]}{4!}y^4 + \frac{Y^{[3]}[h]}{3!}y^3 + \frac{Y^{[2]}[h]}{2!}y^2 + Y^{[1]}[h]y + Y[h]$$

$$Y4[y+h] = \frac{24a_4}{4!}y^4 + \frac{24a_4h + 6a_3}{3!}y^3 + \frac{12a_4h^2 + 6a_3h + 2a}{2!}y^2 + (4a_4h^3 + 3a_3h^2 + 2ah + b)y + a_4h^4 + a_3h^3 + ah^2 + bh + c$$

$$Y4[y+h] = a_4y^4 + (4a_4h + a_3)y^3 + (6a_4h^2 + 3a_3h + a)y^2 + (4a_4h^3 + 3a_3h^2 + 2ah + b)y + a_4h^4 + a_3h^3 + ah^2 + bh + c$$

$$a_3^{\circ} = 4a_4h + a_3$$

$$a_3 = -4a_4^{\circ}h + a_3^{\circ}$$

$$a^{\circ} = 6a_4h^2 + 3a_3h + a$$

$$a = a^{\circ} - 6a_4h^2 - 3a_3h$$

$$a = a^{\circ} - 6a_4^{\circ}h^2 - 3\left(a_3^{\circ} - 4a_4^{\circ}h\right)h$$

$$a = a^{\circ} - 6a_4^{\circ}h^2 - 3a_3^{\circ}h + 12a_4^{\circ}h^2$$

$$a = 6a_4^{\circ}h^2 - 3a_3^{\circ}h + a^{\circ}$$

$$b^{\circ} = 4a_4h^3 + 3a_3h^2 + 2ah + b$$

$$b = -4a_4h^3 - 3a_3h^2 - 2ah + b^{\circ}$$

$$b = -4a_4^{\circ}h^3 - 3(-4a_4^{\circ}h + a_3^{\circ})h^2 - 2(6a_4^{\circ}h^2 - 3a_3^{\circ}h + a^{\circ})h + b^{\circ}$$

$$b = -4a_4^{\circ}h^3 + 12a_4^{\circ}h^3 - 3a_3^{\circ}h^2 - 12a_4^{\circ}h^3 + 6a_3^{\circ}h^2 - 2a^{\circ}h + b^{\circ}$$

$$b = -4a_4^{\circ}h^3 + 3a_3^{\circ}h^2 - 2a^{\circ}h + b^{\circ}$$

$$c^{\circ} = a_4h^4 + a_3h^3 + ah^2 + bh + c$$

$$c = -a_4h^4 - a_3h^3 - ah^2 - bh + c^{\circ}$$

$$c = -a_4^{\circ}h^4 - \left(-4a_4^{\circ}h + a_3^{\circ}\right)h^3 - \left(6a_4^{\circ}h^2 - 3a_3^{\circ}h + a^{\circ}\right)h^2 - \left(-4a_4^{\circ}h^3 + 3a_3^{\circ}h^2 - 2a^{\circ}h + b^{\circ}\right)h + c^{\circ}$$

$$c = -a_4^{\circ}h^4 + 4a_4^{\circ}h^4 - a_3^{\circ}h^3 - 6a_4^{\circ}h^4 + 3a_3^{\circ}h^3 - a^{\circ}h^2 + 4a_4^{\circ}h^4 - 3a_3^{\circ}h^3 + 2a^{\circ}h^2 - b^{\circ}h + c^{\circ}$$

$$c = a_4^{\circ}h^4 - a_3^{\circ}h^3 + a^{\circ}h^2 - b^{\circ}h + c^{\circ}$$

$a_4^{\circ} = a_4$	$a_4 = a_4^{\circ}$

$\mathring{a}_3 = 4a_4h + a_3$	$a_3 = -4\mathring{a}_4h + \mathring{a}_3$
$a^{º} = 6a_4h^2 + 3a_3h + a$	$a = 6\mathring{a}_4h^2 - 3\mathring{a}_3h + a^{º}$
$b^{º} = 4a_4h^3 + 3a_3h^2 + 2ah + b$	$b = -4\mathring{a}_4h^3 + 3\mathring{a}_3h^2 - 2a^{º}h + b^{\circ}$
$c^{º} = a_4h^4 + a_3h^3 + ah^2 + bh + c$	$c = \mathring{a}_4h^4 - \mathring{a}_3h^3 + a^{º}h^2 - b^{\circ}h + c^{\circ}$

Symmetry point:

$$y_{sp} = -\frac{a_{d-1}}{da_d} = -\frac{a_3}{4a_4}$$

$$x_{sp} = Y4\left[-\frac{a_3}{4a_4}\right] = a_4\left(-\frac{a_3}{4a_4}\right)^4 + a_3\left(-\frac{a_3}{4a_4}\right)^3 + a\left(-\frac{a_3}{4a_4}\right)^2 + b\left(-\frac{a_3}{4a_4}\right) + c$$

$$x_{sp} = \frac{a_3^4}{256a_4^3} - \frac{a_3^4}{64a_4^3} + \frac{a_3^2 a}{16a_4^2} - \frac{a_3 b}{4a_4} + c$$

$$sp_{Y4[y]} = (x_{sp}, y_{sp}) = \left(\frac{a_3^4}{256a_4^3} - \frac{a_3^4}{64a_4^3} + \frac{a_3^2 a}{16a_4^2} - \frac{a_3 b}{4a_4} + c, -\frac{a_3}{4a_4}\right)$$

6.5. Taylor shift for 5th degree polynomials

$$Y5[y] = a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c$$
$$Y^{[1]}5[y] = 5a_5y^4 + 4a_4y^3 + 3a_3y^2 + 2ay + b$$
$$Y^{[2]}5[y] = 20a_5y^3 + 12a_4y^2 + 6a_3y + 2a$$
$$Y^{[3]}5[y] = 60a_5y^2 + 24a_4y + 6a_3$$
$$Y^{[4]}5[y] = 120a_5y + 24a_4$$
$$Y^{[5]}5[y] = 120a_5$$

$$Y5[y+h] = \frac{Y^{[5]}[h]}{5!}y^5 + \frac{Y^{[4]}[h]}{4!}y^4 + \frac{Y^{[3]}[h]}{3!}y^3 + \frac{Y^{[2]}[h]}{2!}y^2 + Y^{[1]}[h]y + Y[h]$$

$$\begin{aligned}Y5[y+h] = {} & \frac{120a_5}{5!}y^5 + \frac{120a_5h + 24a_4}{4!}y^4 + \frac{60a_5h^2 + 24a_4h + 6a_3}{3!}y^3 \\ & + \frac{20a_5h^3 + 12a_4h^2 + 6a_3h + 2a}{2!}y^2 \\ & + (5a_5h^4 + 4a_4h^3 + 3a_3h^2 + 2ah + b)y + a_5h^5 + a_4h^4 + a_3h^3 + ah^2 + bh \\ & + c\end{aligned}$$

$$\begin{aligned}Y5[y+h] = {} & a_5y^5 + (5a_5h + a_4)y^4 + (10a_5h^2 + 4a_4h + a_3)y^3 \\ & + (10a_5h^3 + 6a_4h^2 + 3a_3h + a)y^2 + (5a_5h^4 + 4a_4h^3 + 3a_3h^2 + 2ah + b)y \\ & + a_5h^5 + a_4h^4 + a_3h^3 + ah^2 + bh + c\end{aligned}$$

$\mathring{a}_5 = a_5$	$a_5 = \mathring{a}_5$
$\mathring{a}_4 = 5a_5h + a_4$	$a_4 = -5\mathring{a}_5h + \mathring{a}_4$
$\mathring{a}_3 = 10a_5h^2 + 4a_4h + a_3$	$a_3 = 10\mathring{a}_5h^2 - 4\mathring{a}_4h + \mathring{a}_3$
$a^{º} = 10a_5h^3 + 6a_4h^2 + 3a_3h + a$	$a = -10\mathring{a}_5h^3 + 6\mathring{a}_4h^2 - 3\mathring{a}_3h + a^{\circ}$
$b^{º} = 5a_5h^4 + 4a_4h^3 + 3a_3h^2 + 2ah + b$	$b = 5\mathring{a}_5h^4 - 4\mathring{a}_4h^3 + 3\mathring{a}_3h^2 - 2a^{\circ}h + b^{\circ}$
$c^{º} = a_5h^5 + a_4h^4 + a_3h^3 + ah^2 + bh + c$	$c = -\mathring{a}_5h^5 + \mathring{a}_4h^4 - \mathring{a}_3h^3 + a^{\circ}h^2 - b^{\circ}h + c^{\circ}$

Symmetry point:

$$y_{sp} = -\frac{a_{d-1}}{d a_d} = -\frac{a_4}{5a_5}$$

$$x_{sp} = Y5\left[-\frac{a_4}{5a_5}\right]$$

$$= a_5\left(-\frac{a_4}{5a_5}\right)^5 + a_4\left(-\frac{a_4}{5a_5}\right)^4 + a_3\left(-\frac{a_4}{5a_5}\right)^3 + a\left(-\frac{a_4}{5a_5}\right)^2 + b\left(-\frac{a_4}{5a_5}\right) + c$$

$$x_{sp} = -\frac{a_4^5}{3125a_5^4} + \frac{a_4^5}{625a_5^4} - \frac{a_4^3a_3}{125a_5^3} + \frac{a_4^2a}{25a_5^2} - \frac{a_4b}{5a_5} + c$$

$$sp_{Y5[y]} = (x_{sp}, y_{sp}) = \left(-\frac{a_4^5}{3125a_5^4} + \frac{a_4^5}{625a_5^4} - \frac{a_4^3a_3}{125a_5^3} + \frac{a_4^2a}{25a_5^2} - \frac{a_4b}{5a_5} + c, -\frac{a_4}{5a_5}\right)$$

6.6. The Taylor shift coefficients in the Pascal´s triangle

Because of the results above, see the summary of the Shaw and Traub method for the Taylor shift based on Pascal´s triangle.

We start from:

$$Yd[y] = a_d y^d + a_{d-1}y^{d-1} + a_{d-2}y^{d-2} + \cdots + a_4y^4 + a_3y^3 + ay^2 + by + c$$

And shift to:

$$Yd[y+h] = \overset{o}{a}_d y^d + \overset{o}{a}_{d-1}y^{d-1} + \overset{o}{a}_{d-2}y^{d-2} + \cdots + \overset{o}{a}_3y^3 + \overset{o}{a}_2y^2 + \overset{o}{a}_2y + \overset{o}{a}_0$$

The new coefficients come from Pascal´s triangle, such as:

	https://oeis.org/A000012	https://oeis.org/A256958	https://oeis.org/a000217	https://oeis.org/a000292	https://oeis.org/a000332	https://oeis.org/a000389	https://oeis.org/a000579	https://oeis.org/a000580	https://oeis.org/a000581	https://oeis.org/a000582
$a^{o}_0=$	1 a_0	1 ha_1	1 h^2a_2	1 h^3a_3	1 h^4a_4	1 h^5a_5	1 h^6a_6	1 h^7a_7	1 h^8a_8	1 h^9a_9
$a^{o}_1=$	1 a_1	2 ha_2	3 h^2a_3	4 h^3a_4	5 h^4a_5	6 h^5a_6	7 h^6a_7	8 h^7a_8	9 h^8a_9	10 h^9a_{10}
$a^{o}_2=$	1 a_2	3 ha_3	6 h^2a_4	10 h^3a_5	15 h^4a_6	21 h^5a_7	28 h^6a_8	36 h^7a_9	45 h^8a_{10}	55 h^9a_{11}
$a^{o}_3=$	1 a_3	4 ha_4	10 h^2a_5	20 h^3a_6	35 h^4a_7	56 h^5a_8	84 h^6a_9	120 h^7a_{10}	165 h^8a_{11}	220 h^9a_{12}
$a^{o}_4=$	1 a_4	5 ha_5	15 h^2a_6	35 h^3a_7	70 h^4a_8	126 h^5a_9	210 h^6a_{10}	330 h^7a_{11}	495 h^8a_{12}	715 h^9a_{13}
$a^{o}_5=$	1 a_5	6 ha_6	21 h^2a_7	56 h^3a_8	126 h^4a_9	252 h^5a_{10}	462 h^6a_{11}	792 h^7a_{12}	1287 h^8a_{13}	2002 h^9a_{14}
$a^{o}_6=$	1 a_6	7 ha_7	28 h^2a_8	84 h^3a_9	210 h^4a_{10}	462 h^5a_{11}	924 h^6a_{12}	1716 h^7a_{13}	3003 h^8a_{14}	5005 h^9a_{15}
$a^{o}_7=$	1 a_7	8 ha_8	36 h^2a_9	120 h^3a_{10}	330 h^4a_{11}	792 h^5a_{12}	1716 h^6a_{13}	3432 h^7a_{14}	6435 h^8a_{15}	11440 h^9a_{16}
$a^{o}_8=$	1 a_8	9 ha_9	45 h^2a_{10}	165 h^3a_{11}	495 h^4a_{12}	1287 h^5a_{13}	3003 h^6a_{14}	6435 h^7a_{15}	12870 h^8a_{16}	24310 h^9a_{17}
$a^{o}_9=$	1 a_9	10 ha_{10}	55 h^2a_{11}	220 h^3a_{12}	715 h^4a_{13}	2002 h^5a_{14}	5005 h^6a_{15}	11440 h^7a_{16}	24310 h^8a_{17}	48620 h^9a_{18}
$a^{o}_{10}=$	1 a_{10}	11 ha_{11}	66 h^2a_{12}	286 h^3a_{13}	1001 h^4a_{14}	3003 h^5a_{15}	8008 h^6a_{16}	19448 h^7a_{17}	43758 h^8a_{18}	92378 h^9a_{19}
$a^{o}_{11}=$	1 a_{11}	12 ha_{12}	78 h^2a_{13}	364 h^3a_{14}	1365 h^4a_{15}	4368 h^5a_{16}	12376 h^6a_{17}	31824 h^7a_{18}	75582 h^8a_{19}	167960 h^9a_{20}
$a^{o}_{12}=$	1 a_{12}	13 ha_{13}	91 h^2a_{14}	455 h^3a_{15}	1820 h^4a_{16}	6188 h^5a_{17}	18564 h^6a_{18}	50388 h^7a_{19}	125970 h^8a_{20}	293930 h^9a_{21}
$a^{o}_{13}=$	1 a_{13}	14 ha_{14}	105 h^2a_{15}	560 h^3a_{16}	2380 h^4a_{17}	8568 h^5a_{18}	27132 h^6a_{19}	77520 h^7a_{20}	203490 h^8a_{21}	497420 h^9a_{22}
$a^{o}_{14}=$	1 a_{14}	15 ha_{15}	120 h^2a_{16}	680 h^3a_{17}	3060 h^4a_{18}	11628 h^5a_{19}	38760 h^6a_{20}	116280 h^7a_{21}	319770 h^8a_{22}	817190 h^9a_{23}
$a^{o}_{15}=$	1 a_{15}	16 ha_{16}	136 h^2a_{17}	816 h^3a_{18}	3876 h^4a_{19}	15504 h^5a_{20}	54264 h^6a_{21}	170544 h^7a_{22}	490314 h^8a_{23}	1307504 h^9a_{24}

Figure 1 C001112 The Taylor shift coefficients $\overset{o}{a}_n$ in function of the original coefficients a_n, where $0 \leq n \leq d$.

We can get the value of each coefficient $\overset{o}{a}_n$ by adding up the terms in its row.

Each row has an infinite number of terms.

The number of terms to be added in each row is limited by the diagonal corresponding to the degree d of the polynomial. We sum the terms of a row up to the ladder-shaped line.

The number of terms of the sum of $\overset{o}{a}_n$ of a polynomial of degree d is given by:

$$number\ of\ terms = d - n + 1$$

Where n is the coefficient number, and d is the polynomial degree.

7. Symmetry and asymmetry detection.

Because we have proven the existence of a point of symmetry in polynomial sequences above, let us admit as a rule that every mathematical sequence has a symmetry point (sp) or point of symmetry or point of reference.

Because of that, we will always analyze the symmetry or the asymmetry of a sequence using the symmetry point as our referential.

Following this principle, the view of the sequence elements through the symmetry point does not change with the offset. That is, to see all the sequence of elements of any sequence, we will always see from the symmetry point as our referential.

Even preliminarily, every numerical sequence always has two directions. This is very evident in polynomial sequences because in all of them we have two recurrence equations. One direction is opposite to the other.

Because of this introduction, there is only three possibilities to check:

- If the symmetry point is in one of the elements of the sequence, then it is a symmetric sequence of type SUB. The name SUB comes from SUBmarine, one position in the battleship game.
- If the symmetry point is equidistant from all the elements of the sequence, then it is also a symmetric sequence, but of type DES. The name DES comes from DEStroyer, two positions in the battleship game.
- If the sequence is neither SUB nor DES, then it is an asymmetric sequence, and we call type ACC. The name ACC comes from ACC-AirCraft Carrier, three positions in the battleship game.

8. Bijection defines symmetry or asymmetry.

Let us assume that the reason for the symmetry or the asymmetry of any sequence is the result of the certainty or uncertainty of how to apply the bijection property.

This is independent of whether the sequence is finite or infinite.

This way of thinking help to explain how it is possible to detect Ramanujan's "equivocation" at https://www.mersenneforum.org/showpost.php?p=620141&postcount=19 simply by evaluating the shifts of the symmetry points of the polynomial sequences he created.

Because the bijection property always applies to pairs of elements of a sequence (one-to-one correspondence), then each of the pairs forms a duet.

8.1. Definition 1: the bijection function.

Definition 1: The bijection function will occur if and only if exists a mathematical identity between the two elements of all the duets using the symmetry point as the reference.

Consequence 1.1: The bijection function does not depend on the signs of the elements.

The criterion to apply the bijection property (one-to-one correspondence) is always solely the absolute value of the elements of the sequence. For bijection, we only consider the absolute values of each element. This fully applies to polynomials of odd degree.

Consequence 1.2: The bijection function does not change when we offset a sequence.

Consequence 1.3: The bijection function does not change when we analyze a sequence in ascending or descending order, direct or reverse.

Consequence 1.4: See that in this kind of definition, there is bijection even if the two elements of the duets of the sequences are not of equal absolute value.

For example, the sequence of the positive divisors of the number 36 is a symmetric sequence of type SUB, and the sequence of the positive divisors of the number 24 is also a symmetric sequence, but of type DES.

8.2. Definition 2: the symmetric sequences.

Definition 2: We will define any finite or infinite sequence of numbers as a symmetric sequence, if and only if we can apply the bijection property (one-to-one correspondence) to all its elements precisely, **without ambiguity**.

Consequence 2.1: When we apply the bijection function to all pairs of elements with the same absolute value or same mathematical property, and it is impossible to leave out any element of the sequence, then this sequence is a DES type.

Examples: (1) the odd numbers, (2) the quadratics in the form of (oblong numbers $\pm$ an integer number), (3) the positive and negative divisors of the positive square numbers, (4) the repetend of the inverse of some primes, (5) etc.

Consequence 2.2: When we apply the bijection function to all pairs of elements with the same absolute value or same mathematical property, and it is imperative to leave out of the bijection a unique single element of the sequence, then this sequence is a SUB type.

Examples: (1) the number line, (2) the even numbers, (3) the quadratics in the form of (square numbers $\pm$ an integer number), (4) the positive divisors of the positive square numbers*, (5) etc.

* The number 1 is a square number with a single positive divisor and there is no duet of divisors formed. It is a case of a sequence of type SUB of a single element.

In the context of symmetric sequences of integers there are two kinds of certainties of application of the bijection function: DES type and SUB type.

8.3. Definition 3: the asymmetric sequences.

Definition 3: We will define any finite or infinite sequence of numbers as an asymmetric sequence if and only if we can apply the bijection property (one-to-one correspondence) to all its elements precisely, **with ambiguity**.

That is, in an asymmetric polynomial sequence we can apply the bijection property to all elements either leaving only a single element without bijection, or equally it is possible to apply the bijection property to all elements without leaving any element out.

There is no absolute value equality between all elements of the duets, and there is the possibility of applying a mathematical identity between the elements in more than one form.

In the case of asymmetric sequences of integers there is uncertainty about how to apply the bijection function. Asymmetric sequences are ACC type.

8.4. Example of an asymmetric sequence.

For an asymmetric example, let's take the sequence https://oeis.org/A079588 or https://oeis.org/A100147.

See C000446 https://www.facebook.com/groups/snypo/posts/287517449571868/ for the sequence of data for the positive and negative indices, as well as the sequence of its elements in the two possible directions.

See the summary here:

	https://oeis.org/A079588 https://www.facebook.com/groups/snypo/posts/287517449571868 C000446			
	Y_1 [y] = 8y^3−10y^2+3y	Y_2 [y] = −8y^3+10y^2−3y	Y_3 [y] = 8y^3+10y^2+3y	Y_4 [y] = −8y^3−10y^2−3y
f	0	0	0	0
Tally	1	2	3	4
a_4	0	0	0	0
a_3	8	-8	8	-8
a	-10	10	10	-10
b	3	-3	3	-3
c	0	0	0	0
-10	-9030	9030	-7030	7030
-9	-6669	6669	-5049	5049
-8	-4760	4760	-3480	3480
-7	-3255	3255	-2275	2275
-6	-2106	2106	-1386	1386
-5	-1265	1265	-765	765
-4	-684	684	-364	364
-3	-315	315	-135	135
-2	-110	110	-30	30
-1	-21	21	-1	1
0	0	0	0	0
1	1	-1	21	-21
2	30	-30	110	-110
3	135	-135	315	-315
4	364	-364	684	-684
5	765	-765	1265	-1265
6	1386	-1386	2106	-2106
7	2275	-2275	3255	-3255
8	3480	-3480	4760	-4760
9	5049	-5049	6669	-6669
10	7030	-7030	9030	-9030

C000446 Table study of the sequence A079588 or A100147 $x = 8y^3 - 10y^2 + 3y$.

For each equation of the same cubic sequence, we have a shift of the symmetry point as below:

$Y_1[y] = 8y^3 - 10y^2 + 3y$ sequence $\{\dots, -315, -110, -21, 0, 1, 30, 135, \dots\}$.

$$y_{sp1} = -\frac{a}{3a_3} = -\frac{-10}{3 * 8} = \frac{5}{12} = 0.41\overline{6}$$

$$x_{sp1} = 8\left(\frac{5}{12}\right)^3 - 10\left(\frac{5}{12}\right)^2 + 3\left(\frac{5}{12}\right) = \frac{5}{54} = 0.0\overline{925}$$

$Y_2[y] = -8y^3 + 10y^2 - 3y$ sequence $\{\dots, 315, 110, 21, 0, -1, -30, -135, \dots\}$.

$$y_{sp2} = -\frac{a}{3a_3} = -\frac{10}{3 * (-8)} = \frac{5}{12} = 0.41\overline{6}$$

$$x_{sp} = -8\left(\frac{5}{12}\right)^3 + 10\left(\frac{5}{12}\right)^2 - 3\left(\frac{5}{12}\right) = -\frac{5}{54} = -0.0\overline{925}$$

$Y_3[y] = 8y^3 + 10y^2 + 3y$ sequence $\{\dots, -135, -30, -1, 0, 21, 110, 315, \dots\}$.

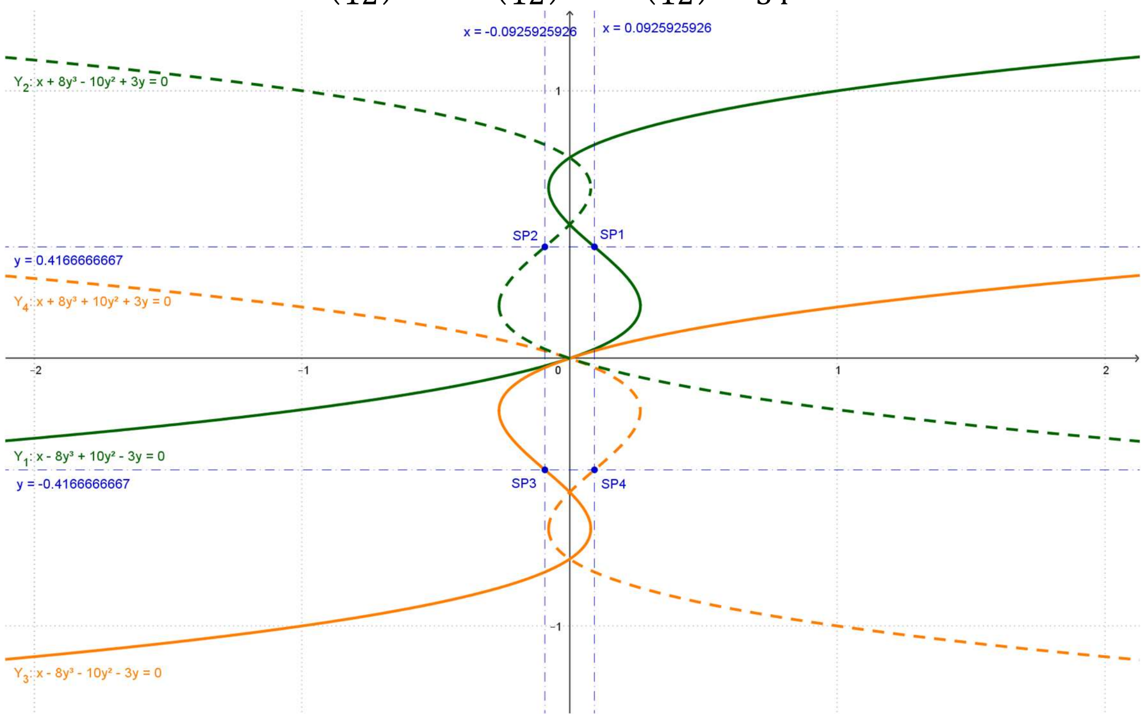
$$y_{sp3} = -\frac{a}{3a_3} = -\frac{10}{3 * 8} = -\frac{5}{12} = -0.41\overline{6}$$

$$x_{sp3} = 8\left(\frac{-5}{12}\right)^3 + 10\left(\frac{-5}{12}\right)^2 + 3\left(\frac{-5}{12}\right) = -\frac{5}{54} = -0.0\overline{925}$$

$Y_4[y] = -8y^3 - 10y^2 - 3y$ sequence $\{\dots, 135, 30, 1, 0, -21, -110, -315, \dots\}$.

$$y_{sp4} = -\frac{a}{3a_3} = -\frac{-10}{3 * (-8)} = -\frac{5}{12} = -0.41\overline{6}$$

$$x_{sp3} = -8\left(\frac{-5}{12}\right)^3 - 10\left(\frac{-5}{12}\right)^2 - 3\left(\frac{-5}{12}\right) = \frac{5}{54} = 0.0\overline{925}$$



C000446 Curves study of the sequence A079588 or A100147 $x = 8y^3 - 10y^2 + 3y$. The brown curves are reversal direction from green curves. The dashed curves are the negative values of the non-dashed curves.

The criterion for applying the bijection function to the four sequence possibilities must be absolutely the same.

That is, because of definition 1 above, the criterion for applying the bijection function between pairs of elements cannot change when we see a sequence in one direction or the other, nor changing the signal.

In each of these four sequences, no element has an absolute value repeated. That is, among the positive and negative elements, zero does not equally divide the two sides of the sequences.

This is the origin of the asymmetry and why we must define the symmetry point of any polynomial at https://www.mersenneforum.org/showthread.php?t=28269.

First, let us use element 0 in the application of the bijection function. We have two alternatives:

$$\{0; -21\}, \{1; \ -110\}, \{30; \ -315\}, \ldots$$

or else,

$$\{0; -1\}, \{21; \ -30\}, \{110; \ -135\}, \ldots$$

In either case, absolutely all elements of the sequences would correspond one-to-one. We call this bijection DES type.

So, it is a valid possibility.

But we could push a little further and think that we can leave only the element 0 out of the bijection. In this case we would have

$$\{1; -21), \{30; \ -110\}, \{135; \ -315\}, \ldots$$

or else,

$$\{-1; 21), \{-30; \ 110\}, \{-135; \ 315\}, \ldots$$

Because only one single element (in this case element 0) is out of the bijection, then, this bijection is a SUB type.

In this case, we can only think about the absolute values of the elements, then we can say that in this last SUB type case we have only a single possibility of bijection.

Finally, because of definition 3 above in this sequence we can have the bijection function applied between its elements in the two ways DES or SUB.

Because we do not know if we use DES or SUB, this is an asymmetric sequence, and there is an ambiguity or an uncertainty in the application of the bijection function.

So, we classify as being an asymmetric sequence ACC type.

9. Conclusions

Along this line of reasoning, we can classify a sequence of integers as being symmetric (or palindromic) if we can unambiguously apply the bijection property among all its elements.

Now, we can define the symmetry or asymmetry of a sequence as follows:

9.1. Definition of DES type of symmetric sequence

If we can perform the bijection directly and unambiguously by determining the one-to-one correspondence among all its infinite duet elements without exception, then the sequence is symmetric of type DES.

In this case, the symmetry point (sp) of the sequence is not an element of the sequence but lies exactly in the middle between two adjacent elements of the sequence.

Mathematically, we define polynomial DES type of symmetry by $Y[-y] = \pm Y[y+1]$.

In this case, the symmetry point (sp) of the polynomial curve $x = Y[y]$ has the Y-coordinate in the XY plane $y_{sp} = \pm \frac{odd}{2}$.

If the polynomial curve has offset $f = 0$, then $y_{sp} = \frac{1}{2}$.

9.2. Definition of SUB type of symmetric sequence

If we can perform the bijection directly and unambiguously by determining the one-to-one correspondence among all its infinite duet elements except one single element without bijection, then the sequence is also symmetric of type SUB.

In this case, the symmetry point (sp) of the sequence is an element of the sequence.

Mathematically, we define polynomial SUB type of symmetry by $Y[y] = \pm Y[-y]$.

In this case, the symmetry point (sp) of the polynomial curve $x = Y[y]$ has the Y-coordinate in the XY plane $y_{sp} = \pm integer$.

If the polynomial curve has offset $f = 0$, then $y_{sp} = 0$.

9.3. Definition of ACC type of asymmetry sequency

If we cannot perform the bijection directly and unambiguously by determining the one-to-one correspondence among all its infinite elements, this means we do not know if we can leave or not one single element without bijection. Consequently, the sequence is asymmetric of type ACC.

ACC type sequences are all the sequences that cannot be either DES type or SUB type sequences.

In this case, also the symmetry point (sp) of the sequence is not an element of the sequence but does not lie exactly in the middle between two adjacent elements of the sequence.

Mathematically, we define polynomial ACC type of symmetry by $Y[-y] \neq \pm Y[y+1]$ and $Y[y] \neq \pm Y[-y]$.

In this case, the symmetry point (sp) of the polynomial curve $x = Y[y]$ has the Y-coordinate in the XY plane $y_{sp} \neq \pm integer$ and $y_{sp} \neq \pm \frac{odd}{2}$.

If the polynomial curve has offset $f = 0$, then $0 < y_{sp} < 0.5$ and in the reversal direction $-0.5 < y_{sp} < 0$.