Generalized Relativistic Transformations in Clifford Spaces and their Physical Implications

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Abstract

A brief introduction of the Extended Relativity Theory in Clifford Spaces (C-space) paves the way to the explicit construction of the generalized relativistic transformations of the Clifford multivector-valued coordinates in C-spaces. The most general transformations furnish a full mixing of the grades of the multivector-valued coordinates. The transformations of the multivector-valued momenta follow leading to an invariant generalized mass $\mathcal{M}$ in C-spaces which $differs$ from m. No longer the proper mass appearing in the relativistic dispersion relation $E^2 - \vec{p} \cdot \vec{p} = m^2$ remains *invariant* under the generalized relativistic transformations. It is argued how this finding might shed some light into the cosmological constant problem, dark energy, and dark matter. We finalize with some concluding remarks about extending these transformations to phase spaces and about Born reciprocal relativity. An appendix is included with the most general (anti) commutators of the Clifford algebra multivector generators.

Keywords : Clifford algebras; Extended Relativity Theory in Clifford Spaces; String theory; M-theory; Generalized geometries.

1 Introduction : The Extended Relativity Theory in Clifford Spaces

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed in [1], [11]. The Extended Relativity theory in

Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector/multivector valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. The theory has two fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes.

These multivector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background. C-space is parametrized not only by the vector coordinates x^μ but also by the bivector coordinates $x^{\mu\nu} = -x^{\nu\mu}$; trivector coordinates $x^{\mu\nu\rho}$ (antisymmetric in all of its indices); $\cdots$ called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes. By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string.

The Extended Relativity Theory in C-spaces (Clifford spaces) allows a unified formulation of point particles, strings, membranes and p-branes, moving in ordinary target spacetime backgrounds, within the description of a single *polyparticle* moving in C-spaces. The degrees of freedom of this polyparticle are provided by the Clifford multivector-valued coordinates (antisymmetric tensorial coordinates) and have a one-to-one correspondence with the number of vertices, edges, planes, facets of a simplex (the higher dimensional analog of a tetrahedron, a regular polytope). For example, a tetrahedron has 4 vertices, 6 edges, 4 faces, a three-dim bulk volume and a center of mass. The total count is $4 + 6 + 4 + 1 + 1 = 16$ which matches the $2^4 = 16$-dimensions of the $4D$ Clifford algebra.

Let $\mathbf{X}$ be the Clifford multivector-valued coordinate corresponding to the $Cl(3,1)$ algebra in four spacetime dimensions and which can be decomposed as

$$\mathbf{X} \;=\; x\;\mathbf{1} \;+\; x^\mu\;\gamma_\mu \;+\; x^{\mu\nu}\;\gamma_\mu \wedge \gamma_\nu \;+\; x^{\mu\nu\rho}\;\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \;+\; x^{\mu\nu\rho\tau}\;\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1.1)$$

where we have omitted combinatorial numerical factors for convenience in the expansion of eq.(1.1). If one imposes the lexicographic ordering of indices $\mu_1 < \mu_2 < \mu_3 < \cdots$ then it is not necessary to include combinatorial numerical factors in the eq.(1.1). To avoid introducing powers of a length parameter L (like the Planck scale L_p), in order to match physical units in the expansion of the multivector $\mathbf{X}$ in eq-(1.1), we can set it to unity to simplify matters after adopting the geometrical natural units $\hbar = c = G = 1$.

The component x in (1.1) is the Clifford scalar component of the multivector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval

$$(d\Sigma)^2 \;=\; (dx)^2 \;+\; dx_\mu\; dx^\mu \;+\; dx_{\mu\nu}\; dx^{\mu\nu} \;+\; \ldots \quad (1.2a)$$

that is *invariant* under $Cl(3,1)$ transformations and which are the Clifford-algebraic extensions of the $SO(3,1)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$ is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. The motion of the polyparticle in C-space is described by the C-space proper "time" Σ dependence of the multivector-valued coordinates

$$x = x(\Sigma), \quad x_\mu = x_\mu(\Sigma), \quad x_{\mu\nu} = x_{\mu\nu}(\Sigma), \quad x_{\mu\nu\rho}(\Sigma), \ \ldots \tag{1.2b}$$

and which is the generalization of the proper time dependence of the spacetime coordinates $x_\mu(\tau)$ in ordinary Minkowski space.

The concept of "photons" and generalized velocities in C-space was analyzed by [1]. One can have tachyonic (superluminal) behavior in ordinary spacetime while having non-tachyonic behavior in C-space. Hence from the C-space point of view there is no violation of causality nor the Clifford-extended Lorentz symmetry. The analog of "photons" in C-space are *tensionless* strings and branes [1].

Let us explicitly insert L and c to keep track of the units, and take the spacetime signature to be $(-,+,+,+,......,+)$ and factorize the C-space interval in eq.(1.2) as follows by bringing the $c^2(dt)^2$ factor outside the parenthesis

$$(d\Sigma)^2 \ = \ c^2(dt)^2 \ \left(\frac{L^2}{c^2} \ (\frac{dx}{dt})^2 \ - \ 1 \ + \ \frac{1}{c^2} \ (\frac{dx_i}{dt})^2 \ + \frac{1}{L^2c^2} \ (\frac{dx_{ij}}{dt})^2 \ - \ \frac{1}{L^2c^2} \ (\frac{dx_{0i}}{dt})^2 \ .... \right) \tag{1.3}$$

where the spatial index i range is $1, 2, ..., D-1$. The Clifford space associated with the Clifford algebra in $4D$ is 16-dimensional and has a neutral/split signature of $(8, 8)$ [1], [2]. For example, the terms $(dx_o)^2, (dx_{0i})^2, (dx_{0ij})^2, (dx_{0123})^2$ will appear with a negative sign, while the terms $(dx_i)^2, (dx_{ij})^2, (dx_{ijk})^2$ will appear with a positive sign.

There are many possible combination of numerical values for the 2^D terms inside the parenthesis in eq.(1.3). As explained in [1], [2], *superluminal* velocities in ordinary spacetime are possible, while retaining the null interval condition in C-space $(d\Sigma)^2 = 0$, associated with *tensionless* branes. For instance, let us set all the higher grade components beyond the bivectors in eq.(1.3) to zero, as well as setting $dx_{ij} = 0$, leaving only the following contributions

$$(d\Sigma)^2 \ = \ c^2(dt)^2 \ \left(\frac{L^2}{c^2} \ (\frac{dx}{dt})^2 \ - \ 1 \ + \ \frac{1}{c^2} \ (\frac{dx_i}{dt})^2 \ - \ \frac{1}{L^2c^2} \ (\frac{dx_{0i}}{dt})^2 \ \right) \tag{1.4}$$

One can then have the following combinations

$$\frac{L^2}{c^2} \ (\frac{dx}{dt})^2 \ - \ 1 \ = \ 0, \ \ \frac{1}{c^2} \ (\frac{dx_i}{dt})^2 \ - \ \frac{1}{L^2c^2} \ (\frac{dx_{0i}}{dt})^2 \ = \ 0 \tag{1.5}$$

as well as the superluminal condition on the standard velocity

$$\frac{1}{c^2} \ (\frac{dx_i}{dt})^2 \ = \ \frac{1}{c^2} \ \left(\ (\frac{dx_1}{dt})^2 \ + \ (\frac{dx_2}{dt})^2 \ + \ \ldots \ + \ (\frac{dx_{D-1}}{dt})^2 \ \right) \ > \ 1 \tag{1.6}$$

and still obey the null interval condition $(d\Sigma)^2 = 0$ in C-space.

In $4D$, the null interval condition in C-space $(d\Sigma)^2 = 0$ can be easily attained if each term inside the parenthesis in eq.(1.3) is ± 1, respectively. Since there are 8 positive $(+1)$

terms and 8 negative (-1) terms one has that $8-8=0$ and the null interval condition $(d\Sigma)^2 = 0$ is automatically satisfied. Hence in this case one would have a superluminal behavior such that

$$\frac{1}{c^2} \left((\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2 \right) = 1 + 1 + 1 = 3 > 1 \tag{1.7}$$

while still preserving the null interval condition in C-space. The three coordinates x_1, x_2, x_3 in (1.7) represent the center of mass coordinates of the tensionless 2-loop (a sphere S^2 encloses a three-dim bulk region).

A very different combination of numerical values, as compared to the previous ones, leading also to a null interval condition in C-space $(d\Sigma)^2 = 0$, as well as a null interval in ordinary Minkowski spacetime, occurs when

$$\frac{1}{c^2} \left((\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2 \right) = 1 \tag{1.8a}$$

$$\frac{1}{L^2c^2} \left((\frac{dx_{12}}{dt})^2 + (\frac{dx_{13}}{dt})^2 + (\frac{dx_{23}}{dt})^2 \right) =$$

$$\frac{1}{L^2c^2} \left((\frac{dx_{01}}{dt})^2 + (\frac{dx_{02}}{dt})^2 + (\frac{dx_{03}}{dt})^2 \right) \tag{1.8b}$$

$$\frac{1}{L^4c^2} \left((\frac{dx_{012}}{dt})^2 + (\frac{dx_{013}}{dt})^2 + (\frac{dx_{023}}{dt})^2 \right) = \frac{1}{L^4c^2} (\frac{dx_{123}}{dt})^2 \tag{1.8c}$$

$$\frac{1}{L^6c^2} (\frac{dx_{0123}}{dt})^2 = \frac{L^2}{c^2} (\frac{dx}{dt})^2 \tag{1.8d}$$

Another description of C-space "photons" can also be given in terms of an *effective* temporal variable T comprised of all the temporal coordinates in the interval of eq-(1.3). In order to simplify matters let us work with $D = 3$ instead of $D = 4$. The effective temporal variable T is defined as

$$c^2(dT)^2 \equiv c^2(dt)^2 + \frac{1}{c^2} (\frac{dx_{01}}{dt})^2 + \frac{1}{c^2} (\frac{dx_{02}}{dt})^2 + \frac{1}{L^2c^2} (\frac{dx_{012}}{dt})^2 \tag{1.9}$$

so that the C-space interval can be rewritten, after factoring out the $c^2(dT)^2$ term, as

$$(d\Sigma)^2 = - c^2(dT)^2 \left(1 - \frac{L^2}{c^2} (\frac{dx}{dT})^2 - \frac{1}{c^2} (\frac{dx_1}{dT})^2 - \frac{1}{c^2} (\frac{dx_2}{dT})^2 - \frac{1}{L^2c^2} (\frac{dx_{12}}{dT})^2 \right) \tag{1.10}$$

The last expression has the same functional form as the ordinary spacetime interval in Minkowski space. Namely one can write the C-space interval $(d\Sigma)^2$ in the form

$$(d\Sigma)^2 = - c^2(dT)^2 (1 - \frac{V^2}{c^2}) \tag{1.11}$$

where the generalization of the magnitude-squared of the spatial velocity divided by c^2 is

$$\frac{V^2}{c^2} \equiv \frac{L^2}{c^2} (\frac{dx}{dT})^2 + \frac{1}{c^2} (\frac{dx_1}{dT})^2 + \frac{1}{c^2} (\frac{dx_2}{dT})^2 + \frac{1}{L^2 c^2} (\frac{dx_{12}}{dT})^2 \quad (1.12)$$

When $V = c \Rightarrow (d\Sigma)^2 = 0$ in eq.(1.11), and once more it leads to a null interval in C-space.

The Extended Relativity Theory in Clifford Spaces (C-space) [3] leads to many interesting novel physical consequences like : (i) generalized dispersion relations, energy-dependent speed of light propagation, extended Lorentz transformations, relative locality, generalized Weyl-Heisenberg algebra and uncertainty relations, tensionless branes, superluminality, generalized velocities. (ii) Generalized areal, volume, $\cdots$ metrics and gravitational field equations in C-space. (iii) A unified description of particles, strings and branes. (iv) Clifford gravity based cosmology and dark energy. (v) Moyal deformations of Clifford gauge theories of gravity. (vi) N-ary algebras.

The results of this work are new to our knowledge. In **2.1** we write *explicitly* the extended relativistic transformations $\mathbf{X}' = \mathbf{R}\mathbf{X}\mathbf{R}^{-1}$ of the multivector-valued coordinates in C-spaces in a very special case when the exponential defining the rotor $\mathbf{R}$

$$\mathbf{R} = \exp(\theta + \theta_\mu \gamma^\mu + \theta_{\mu\nu\rho}\gamma^{\mu\nu\rho} + \theta_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}) \quad (1.13)$$

admits a factorization. One finds that a *mixing* of bivector/trivector and vector/quadvector coordinates occurs in the new frame of reference in C-space. In **2.2**, the transformations of the multivector-valued momenta are displayed leading to an invariant generalized mass $\mathcal{M}$ in C-spaces which differs from m. One finds that no longer the proper mass appearing in the relativistic dispersion relation $E^2 - \vec{p}\cdot\vec{p} = m^2$ is *invariant* under the extended transformations. It is argued how this finding might shed some light into the cosmological constant problem, dark energy, and dark matter. The most general transformations when the exponential (1.13) does *not* admit a factorization are displayed in **2.3**, and leading to a full mixing of all the grades of the multivector-valued coordinates. We finalize with some concluding remarks about how to extend these transformations to phase spaces and about Born reciprocal relativity. An appendix is included with the most general (anti) commutators of the Clifford algebra multivector generators.

2 Relativistic Transformations in C-spaces

2.1 Beyond Lorentz Transformations

In this section we shall be using the natural units $\hbar = c = G = L_P = 1$ and working in a $3+1$-dim spacetime. The time coordinate is $t = x_1$, and x_2, x_3, x_4 are the three spatial ones. The multivector valued coordinates

$$x, \quad x^\mu, \quad x^{\mu_1\mu_2} = -x^{\mu_2\mu_1}, \quad x^{\mu_1\mu_2\mu_3} = -x^{\mu_2\mu_1\mu_3}, \quad \ldots \quad (2.1)$$

are now linked to the basis generators given by $\mathbf{1}$, vectors γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element $\mathbf{1}$ (associated to a scalar coordinate).

We shall present examples of generalized Lorentz transformations in C-space. The standard Lorentz transformations involves bivector generators. For instance, given the bivector γ_{12}, the transformation effected by the rotor defined as

$$\mathbf{R} = \exp(\theta_{12}\gamma_{12}) = cosh(\theta_{12}) + \gamma_{12}\ sinh(\theta_{12}) \tag{2.2a}$$

corresponds to an ordinary Lorentz boost transformation along the x_2 direction and involving the temporal variable x_1. Under these Lorentz boosts the transformed multivector $\mathbf{X}'$ is given by

$$\mathbf{X}' \equiv \mathbf{R}\ \mathbf{X}\ \mathbf{R}^{-1} \tag{2.2b}$$

with

$$\mathbf{R}^{-1} = \exp(\ -\ \theta_{12}\gamma_{12}\ ) = cosh(\theta_{12}) - \gamma_{12}\ sinh(\theta_{12}) \tag{2.2c}$$

The vector coordinate components in eq.(2.2b) turn out to be given by the familiar expressions

$$t' = t\ cosh(2\theta_{12}) + x_2\ sinh(2\theta_{12}) \tag{2.3a}$$

$$x_2' = x_2\ cosh(2\theta_{12}) + t\ sinh(2\theta_{12}) \qquad (2.3b$$

$$x_3'' = x_3, \quad x_4'' = x_4 \tag{2.3c}$$

where $\xi = 2\theta_{12}$ is the Lorentz boost rapidity parameter ξ such that $tanh(\xi) = v$. Eqs.(2.3) yield the quadratic invariant

$$-(t')^2 + (x_2')^2 + (x_3')^2 + (x_4')^2 = -(t)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 \tag{2.3d}$$

The bivectors $\gamma_{\mu\nu}$ can also be expressed in terms of the commutators $[\gamma_\mu, \gamma_\nu]$, such that the the latter commutators implement a "rotation" along the $x_\mu - x_\nu$ directions. Hence, a Lorentz boost along the x_2 can be seen as a "rotation" along the $x_1 - x_2$ axes.

Generalized Lorentz transformations in flat C-spaces were discussed in [1]. In this work we shall write these transformations explicitly. Let us begin with the *reversion* involution operation † (some authors use the tilde notation for the reversion operation) that is defined by reversing the order of the wedge products $(\gamma_{12})^\dagger = \gamma_{21} = -\gamma_{12}$, $\gamma_{123}^\dagger = \gamma_{321} = -\gamma_{123}, \cdots$. The reversal of a product of two multivectors is $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$. There is also the Hermitian conjugation operation denoted by ‡ in both real and complex Clifford algebras [7] and defined by

$$\mathbf{X}^\ddagger = x^* + (x_\mu)^*\ (\gamma_\mu)^{-1} + (x_{\mu_1\mu_2})^*(\gamma_{\mu_1\mu_2})^{-1} + (x_{\mu_1\mu_2\mu_3})^*(\gamma_{\mu_1\mu_2\mu_3})^{-1} + \ \ldots \tag{2.4}$$

where $*$ is the complex conjugation and $(\gamma_{\mu_1\mu_2\cdots\mu_n})^{-1}$ denotes the inverse basis element $(\gamma_{\mu_1\mu_2\cdots\mu_n})^{-1}(\gamma_{\mu_1\mu_2\cdots\mu_n}) = \mathbf{1}$. There is also the grade inverse operation which involves changing the sign of all the basis elements and the Clifford conjugation comprised of a grade inversion followed by a reversion.

We shall study the most general transformations which leave invariant the quadratic form

$$\langle\, \mathbf{X}^\dagger\, \mathbf{X}\,\rangle \;=\; x^2 \;+\; x_\mu\, x^\mu \;+\; x_{\mu\nu}\, x^{\mu\nu} \;+\; \ldots \;=$$
$$\langle\, \mathbf{X}'^\dagger\, \mathbf{X}'\,\rangle \;=\; (x')^2 \;+\; x'_\mu\, x'^\mu \;+\; x'_{\mu\nu}\, x'^{\mu\nu} \;+\; \ldots \tag{2.5}$$

where the transformed multivector $\mathbf{X}'$ is defined as

$$\mathbf{X}' \;\equiv\; \mathbf{R}\,\mathbf{X}\,\mathbf{R}^\dagger \tag{2.6}$$

The bracket symbol $< \mathbf{X}^\dagger\mathbf{X} >$ denotes taking the scalar part of the Clifford geometric product of two multivectors. It is the analog of the trace of a product of matrices. Such scalar part can be obtained from the (anti) commutator relations of the Clifford algebra generators as displayed in the Appendix. For example

$$<\; \gamma_\mu\;\gamma^\nu\;> \;=\; \delta^\nu_\mu, \quad <\; \gamma_{\mu_1\mu_2}\;\gamma^{\nu_1\nu_2}\;> \;=\; -\;\delta_{\mu_1\mu_2}^{\;\;\nu_1\nu_2}$$
$$<\; \gamma_{\mu_1\mu_2\mu_3}\;\gamma^{\nu_1\nu_2\nu_3}\;> \;=\; -\,\delta_{\mu_1\mu_2\mu_3}^{\;\;\nu_1\nu_2\nu_3}, \quad <\; \gamma_{\mu_1\mu_2\mu_3\mu_4}\;\gamma^{\nu_1\nu_2\nu_3\nu_4}\;> \;=\; \delta_{\mu_1\mu_2\mu_3\mu_4}^{\;\;\nu_1\nu_2\nu_3\nu_4}, \;\; \ldots\ldots \tag{2.7}$$

One should note the presence of $\pm$ signs in the right hand side of eqs-(2.7). They are connected to the even/odd behavior of the reversal operation $(\gamma_C)^\dagger = \pm\gamma_C$.

Invariance of the quadratic form (2.5) requires that the reversal of the rotor $\mathbf{R}$ obeys the key condition $\mathbf{R}^\dagger \;=\; \mathbf{R}^{-1}$ such that

$$\langle(\,\mathbf{R}\,\mathbf{X}\,\mathbf{R}^{-1})^\dagger\,(\mathbf{R}\,\mathbf{X}\,\mathbf{R}^{-1})\,\rangle \;=\; \langle\,\mathbf{R}\,\mathbf{X}^\dagger\,\mathbf{X}\,\mathbf{R}^{-1}\rangle \;=\; \langle\,\mathbf{R}^{-1}\,\mathbf{R}\,\mathbf{X}^\dagger\,\mathbf{X}\,\rangle \;=\; \langle\,\mathbf{X}^\dagger\,\mathbf{X}\,\rangle \tag{2.8}$$

resulting from the reversal operation and the *cyclic* property of the scalar part of the Clifford geometric product. In the case of complex Clifford algebras the real valued quadratic form is defined as

$$\langle\,\mathbf{X}^\ddagger\,\mathbf{X}\,\rangle \;=\; x^*\,x \;+\; x^*_\mu\,x^\mu \;+\; x^*_{\mu\nu}\,x^{\mu\nu} \;+\; \ldots \tag{2.9}$$

and the transformed multivector $\mathbf{X}'$ is given by

$$\mathbf{X}' \;\equiv\; \mathbf{R}\,\mathbf{X}\,\mathbf{R}^\ddagger \tag{2.10}$$

The quadratic form remains invariant if $\mathbf{R}^\ddagger = \mathbf{R}^{-1}$.

Let us choose now for simplicity the rotor operator in $D = 4$ given by

$$\mathbf{R} \;=\; e^{\theta_{12}\gamma_{12}\;+\;\theta_{123}\gamma_{123}} \;\Rightarrow\; \mathbf{R}^\dagger \;=\; \mathbf{R}^{-1} \tag{2.11}$$

where $\theta_{12}, \theta_{123}$ are the (antisymmetric) parameters associated with the $\gamma_{12}, \gamma_{123}$ generators, respectively. As mentioned above, the term $\theta_{12}\gamma_{12}$ in (2.11) represents a Lorentz boost along the spatial x_2 direction and whose magnitude is encoded in the θ_{12} parameter. We shall explain below the physical significance of the term $\theta_{123}\gamma_{123}$. Due to the commutativity $[\gamma_{12}, \gamma_{123}] = 0$ one can *factorize* $\mathbf{R}$ as

$$\mathbf{R} \ = \ e^{\theta_{12}\gamma_{12} \ + \ \theta_{123}\gamma_{123}} \ = \ e^{\theta_{12}\gamma_{12}} \ e^{\theta_{123}\gamma_{123}} \ =$$
$$(cosh(\theta_{12})\mathbf{1} \ + \ \gamma_{12} \ sinh(\theta_{12})) \ \ (cosh(\theta_{123})\mathbf{1} \ + \ \gamma_{123} \ sinh(\theta_{123})) \tag{2.12}$$

$$\mathbf{R}^{-1} \ = \ e^{-\theta_{12}\gamma_{12} \ - \ \theta_{123}\gamma_{123}} \ = \ e^{-\theta_{12}\gamma_{12}} \ e^{-\theta_{123}\gamma_{123}} \ =$$
$$(cosh(\theta_{12})\mathbf{1} \ - \ \gamma_{12} \ sinh(\theta_{12})) \ \ (cosh(\theta_{123})\mathbf{1} \ - \ \gamma_{123} \ sinh(\theta_{123})) \ =$$
$$(cosh(\theta_{123})\mathbf{1} \ - \ \gamma_{123} \ sinh(\theta_{123})) \ \ (cosh(\theta_{12})\mathbf{1} \ - \ \gamma_{12} \ sinh(\theta_{12})) \tag{2.13}$$

and which will simplify considerable the transformation defined as

$$\mathbf{X}'' \ = \ \mathbf{R} \ \mathbf{X} \ \mathbf{R}^{-1} \tag{2.14}$$

The use of double primes in (2.14) is due to the factorization of *two* separate transformations leading to a two step procedure $\mathbf{X} \rightarrow \mathbf{X}' \rightarrow \mathbf{X}''$. Setting $x_1 = t$ to be the temporal coordinate[1], and x_2, x_3, x_4 the three spatial coordinates with $\eta_{\mu\nu}$ =diag$(-1, +1, +1, +1)$ the flat metric in $3 + 1$-dim, after some lengthy algebra one ends up with

$$t'' \ = \ t \ cosh(2\theta_{12}) \ + \ x_2 \ sinh(2\theta_{12}) \tag{2.15}$$

$$x''_2 \ = \ x_2 \ cosh(2\theta_{12}) \ + \ t \ sinh(2\theta_{12}) \tag{2.16}$$

$$x''_3 \ = \ x_3, \tag{2.17}$$

$$x''_4 \ = \ x_4 \ cosh(2\theta_{123}) \ + \ x_{1234} \ sinh(2\theta_{123}) \tag{2.18}$$

$$x''_{13} \ = \ x_{13} \ cosh(2\theta_{12}) \ + \ x_{23} \ sinh(2\theta_{12}) \tag{2.19}$$

$$x''_{23} \ = \ x_{23} \ cosh(2\theta_{12}) \ + \ x_{13} \ sinh(2\theta_{12}) \tag{2.20}$$

$$x''_{12} \ = \ x_{12} \tag{2.21}$$

$$x''_{14} \ = \ x_{14} \ cosh(2\theta_{12}) \ cosh(2\theta_{123}) \ + \ x_{24} \ sinh(2\theta_{12}) \ cosh(2\theta_{123}) \ -$$
$$x_{134} \ sinh(2\theta_{12}) \ sinh(2\theta_{123}) \ - \ x_{234} \ cosh(2\theta_{12}) \ sinh(2\theta_{123}) \tag{2.22}$$

[1]As a reminder, we chose the units $\hbar = c = G = 1$

$$x''_{24} \;=\; x_{24}\; cosh(2\theta_{12})\; cosh(2\theta_{123}) \;+\; x_{14}\; sinh(2\theta_{12})\; cosh(2\theta_{123}) \;-$$

$$x_{234}\; sinh(2\theta_{12})\; sinh(2\theta_{123}) \;-\; x_{134}\; cosh(2\theta_{12})\; sinh(2\theta_{123}) \tag{2.23}$$

$$x''_{34} \;=\; x_{34}\; cosh(2\theta_{123}) \;+\; x_{124}\; sinh(2\theta_{123}) \tag{2.24}$$

$$x''_{124} \;=\; x_{124}\; cosh(2\theta_{123}) \;+\; x_{34}\; sinh(2\theta_{123}) \tag{2.25}$$

$$x''_{234} \;=\; x_{234}\; cosh(2\theta_{12})\; cosh(2\theta_{123}) \;+\; x_{134}\; sinh(2\theta_{12})\; cosh(2\theta_{123}) \;-$$

$$x_{14}\; cosh(2\theta_{12})\; sinh(2\theta_{123}) \;-\; x_{24}\; sinh(2\theta_{12})\; sinh(2\theta_{123}) \tag{2.26}$$

$$x''_{134} \;=\; x_{134}\; cosh(2\theta_{12})\; cosh(2\theta_{123}) \;+\; x_{234}\; sinh(2\theta_{12})\; cosh(2\theta_{123}) \;-$$

$$x_{24}\; cosh(2\theta_{12})\; sinh(2\theta_{123}) \;-\; x_{14}\; sinh(2\theta_{12})\; sinh(2\theta_{123}) \tag{2.27}$$

$$x''_{1234} \;=\; x_{1234}\; cosh(2\theta_{123}) \;+\; x_{4}\; sinh(2\theta_{123}) \tag{2.28}$$

The transformation of the remaining multivector coordinate components are

$$x'' \;=\; x, \;\; x''_{123} \;=\; x_{123} \tag{2.29}$$

The most salient feature of the above transformations is that the rapidity parameter θ_{123} associated with the trivector generator γ_{123} induces a *mixing* among multivectors of different grade. Namely, it induces a mixing among the bivector/trivector ccordinates, and vector/quadvector coordinates. One can verify that the transformations in eqs.(2.15-2.29) leave invariant the quadratic form

$$\langle\; \mathbf{X}^\dagger\; \mathbf{X}\;\rangle \;=\; x^2 \;+\; x_\mu\; x^\mu \;+\; x_{\mu\nu}\; x^{\mu\nu} \;+\; \ldots \;=\; \langle\; \mathbf{X}''^\dagger\; \mathbf{X}''\;\rangle \;=$$
$$(x'')^2 \;+\; x''_\mu\; x''^\mu \;+\; x''_{\mu\nu}\; x''^{\mu\nu} \;+\; \ldots \tag{2.30}$$

In particular one finds that the combination

$$-(t'')^2 \;+\; (x''_2)^2 \;+\; (x''_3)^2 \;-\; (x''_{13})^2 \;-\; (x''_{12})^2 \;+\; (x''_{23})^2 \;=$$

$$-(t)^2 \;+\; (x_2)^2 \;+\; (x_3)^2 \;-\; (x_{13})^2 \;-\; (x_{12})^2 \;+\; (x_{23})^2 \tag{2.31}$$

remains invariant.

Also one has the following additional invariant combinations

$$-(x''_{14})^2 \;+\; (x''_{24})^2 \;+\; (x''_{34})^2 \;+\; (x''_{234})^2 \;-\; (x''_{134})^2 \;-\; (x''_{124})^2 \;=$$

$$- \ (x_{14})^2 \ + \ (x_{24})^2 \ + \ (x_{34})^2 \ + \ (x_{234})^2 \ - \ (x_{134})^2 \ - \ (x_{124})^2 \qquad (2.32)$$

$$(x''_4)^2 \ - \ (x''_{1234})^2 \ = \ (x_4)^2 \ - \ (x_{1234})^2; \ \ (x'')^2 \ = \ (x)^2; \ \ (x''_{123})^2 \ = \ (x_{123})^2 \qquad (2.33)$$

so that the net combination of all these invariants eqs.(2.31-2.33) leave invariant the full quadratic form (2.30) in a straightforward fashion involving a total of $2^4 = 16$ terms.

The quadratic form (2.30) is also invariant under the left/right isometry transformations [6]

$$\mathbf{X}' \ = \ \mathbf{R} \, \mathbf{X} \, \mathbf{L}^\dagger, \ \ \mathbf{R}^\dagger \, \mathbf{R} \ = \ 1, \ \ \mathbf{L}^\dagger \, \mathbf{L} \ = \ 1 \ \ \Rightarrow \ < \ \mathbf{X}'^\dagger \, \mathbf{X}' \ > \ = \ < \ \mathbf{X}^\dagger \, \mathbf{X} \ > \qquad (2.34)$$

due to the cyclic property of the scalar part projection

$$< \ \mathbf{X}'^\dagger \, \mathbf{X}' \ > \ = \ < \ \mathbf{L} \, \mathbf{X}^\dagger \, \mathbf{R}^\dagger \, \mathbf{R} \, \mathbf{X} \, \mathbf{L}^\dagger, \ > \ = \ < \ \mathbf{L} \, \mathbf{X}^\dagger \, \mathbf{X} \, \mathbf{L}^\dagger \ > \ =$$
$$< \ \mathbf{L}^\dagger \, \mathbf{L} \, \mathbf{X}^\dagger \, \mathbf{X} \ > \ = \ < \ \mathbf{X}^\dagger \, \mathbf{X} \ > \qquad (2.35)$$

where $\mathbf{R}, \mathbf{L}$ are Clifford-valued rotors acting on the right and left respectively. These left/right transformations lead to yet *another* set of multivector-valued coordinate transformations which differ from eqs.(2.15-2.29) and which we shall not write down. For instance, if one sets $\mathbf{R} = \mathbf{1}$, or $\mathbf{L} = \mathbf{1}$ it will considerably simplify the transformations.

If one sets $\theta_{123} = 0$ above in eqs-(2.15-2.29) one ends up with the well known Lorentz transformations involving the boost rapidity parameter $\xi = 2\theta_{12}$ along the x_2 direction such that $tanh(\xi) = \beta = v \leq 1$, and where *no mixing* of vector, bivector, trivector, quadvector coordinates occurs. Whereas, if one sets $\theta_{12} = 0$ in eqs.(2.25-2.28) one arrives at the following transformations leading to a *mixing* of bivector/trivector and vector/quadvector coordinates

$$x'_4 \ = \ x_4 \, cosh(2\theta_{123}) \ + \ x_{1234} \, sinh(2\theta_{123}) \qquad (2.36)$$

$$x'_{1234} \ = \ x_{1234} \, cosh(2\theta_{123}) \ + \ x_4 \, sinh(2\theta_{123}) \qquad (2.37)$$

$$x'_{14} \ = \ x_{14} \, cosh(2\theta_{123}) \ - \ x_{234} \, sinh(2\theta_{123} \qquad (2.38)$$

$$x'_{234} \ = \ x_{234} \, cosh(2\theta_{123}) \ - \ x_{14} \, sinh(\theta_{123}) \qquad (2.39)$$

$$x'_{24} \ = \ x_{24} \, cosh(2\theta_{123}) \ - \ x_{134} \, sinh(2\theta_{123}) \qquad (2.40)$$

$$x'_{134} \ = \ x_{134} \, cosh(2\theta_{123}) \ - \ x_{24} \, sinh(2\theta_{123}) \qquad (2.41)$$

$$x'_{34} \ = \ x_{34} \, cosh(2\theta_{123}) \ + \ x_{124} \, sinh(2\theta_{123}) \qquad (2.42)$$

$$x'_{124} \;=\; x_{124}\; cosh(2\theta_{123}) \;+\; x_{34}\; sinh(2\theta_{123}) \qquad (2.43)$$

The above *mixing* of the different grades of multivectors in eqs.(2.36-2.43) due to the γ_{123} generator (via the θ_{123} parameter) can be easily understood from the following commutators leading to the γ_{123} generator in the right hand side

$$\theta_{123}\;[\gamma_{1234}, \gamma_4] \;\sim \theta_{123}\;\gamma_{123}, \quad \theta_{123}\;[\gamma_{14}, \gamma_{234}] \;\sim\; \theta_{123}\;\gamma_{123},$$

$$\theta_{123}\;[\gamma_{24}, \gamma_{134}] \;\sim\; \theta_{123}\;\gamma_{123}, \quad \theta_{123}\;[\gamma_{34}, \gamma_{124}] \;\sim\; \theta_{123}\;\gamma_{123} \qquad (2.44)$$

The first term of eq.(2.44) is associated to eqs.(2.36,2.37) and corresponds to a generalized boost along the spatial x_4 direction but involving now the temporal *quadvector* coordinate x_{1234} : namely, generalized "rotations" along the $x_4 - x_{1234}$ axes.

The second term of eq.(2.44) $\theta_{123}[\gamma_{14}, \gamma_{234}] \sim \theta_{123}\gamma_{123}$ is associated to eqs.(2.38,2.39) and corresponds to a generalized boost along the spatial *trivector* x_{234} direction but involving now the temporal *bivector* coordinate x_{14} : generalized "rotations" along the $x_{234} - x_{14}$ axes.

The third term $\theta_{123}[\gamma_{24}, \gamma_{134}] \sim \theta_{123}\gamma_{123}$ is associated to eqs.(2.40,2.41) and corresponds to a generalized boost along the spatial *bivector* x_{24} direction but involving now the temporal *trivector* coordinate x_{134} : generalized "rotations" along the $x_{24} - x_{134}$ axes.

And the last term $\theta_{123}[\gamma_{34}, \gamma_{124}] \sim \theta_{123}\gamma_{123}$ is associated to eqs.(2.42,2.43) and corresponds to a generalized boost along the spatial *bivector* x_{34} direction but involving now the temporal *trivector* coordinate x_{124} : generalized "rotations" along the $x_{34} - x_{124}$ axes. In this fashion one can find an intuitive physical interpretation of all of the above transformations involving the γ_{123} generator (via the θ_{123} parameter). Note also that the Lorentz boost transformations involving $\theta_{12}\gamma_{12}$ leave *inert* the values of x_{34}, x_{124} and this explains the form of the expressions in eqs.(2.24,2.25) which are only affected by the $\theta_{123}\gamma_{123}$ piece of the rotor $\mathbf{R}$.

In all of the above equations one used the identities of the hyperbolic functions

$$cosh^2(\xi) - sinh^2(\xi) \;=\; 1, \;\; cosh^2(\xi) + sinh^2(\xi) \;=\; cosh(2\xi), \;\; \sinh(2\xi) \;=\; 2\, sinh(\xi)\, cosh(\xi) \qquad (2.45)$$

2.2 Generalized Mass and Momentum in C-Space

The on-shell mass condition for a massive polyparticle moving in the 2^4-dimensional flat C-space, corresponding to a Clifford algebra in $D = 4$, can be written in terms of the multivector-valued momentum components as

$$p^2 \;+\; p_\mu\, p^\mu \;+\; p_{\mu_1\mu_2}\; p^{\mu_1\mu_2} \;+\; p_{\mu_1\mu_2\mu_3}\; p^{\mu_1\mu_2\mu_3} \;+\; p_{\mu_1\mu_2....\mu_4} p^{\mu_1\mu_2...\mu_4} \;=\; -\; \mathcal{M}^2 \qquad (2.46)$$

The scalar part of the momentum multivector can be absorbed into a redefinition of $\mathcal{M}$ as $\mathcal{M}^2 \to \mathcal{M}^2 + p^2 = \mathcal{M}'^2$. Both $\mathcal{M}$ and p are C-space invariants.

The C-space transformations of the multivector-valued momentum variables are the same as the multivector-valued coordinates transformations in eqs.(2.15-2.29). Hence, one has in particular that

$$p''_4 \ = \ p_4 \ cosh(2\theta_{123}) \ + \ p_{1234} \ sinh(2\theta_{123}) \tag{2.47}$$

$$p''_{1234} \ = \ p_{1234} \ cosh(2\theta_{123}) \ + \ p_4 \ sinh(2\theta_{123}) \tag{2.48}$$

$$E'' \ = \ E \ cosh(2\theta_{12}) \ + \ p_2 \ sinh(2\theta_{12}) \tag{2.49}$$

$$p''_2 \ = \ p_2 \ cosh(2\theta_{12}) \ + \ E \ sinh(2\theta_{12}) \tag{2.50}$$

$$p''_3 \ = \ p_3, \tag{2.51}$$

From eqs.(2.47-2.51) one finds that the combinations

$$-(E'')^2 \ + \ (p''_2)^2 \ + \ (p''_3)^2 \ + \ (p''_4)^2 \ - \ (p''_{1234})^2 \ = \ - E^2 \ + \ p_2^2 \ + \ p_3^2 \ + \ p_4^2 \ - \ p_{1234}^2 \tag{2.52}$$

remain invariant under the transformations. The most salient feature eq.(2.52) is that the m^2 appearing in the dispersion relation

$$-E^2 \ + \ p_2^2 \ + \ p_3^2 \ + \ p_4^2 \ = \ - \ m^2 \tag{2.53}$$

and p_{1234}^2 are *no* longer *invariant* when $\theta_{123} \neq 0$. What is now an invariant is the very specific *combination* of the five quantities displayed by eq.(2.52) and which can be rewritten as

$$-m^2 \ - \ p_{1234}^2 \ = \ - \ \kappa^2 \ \Rightarrow \ \kappa \ = \ \sqrt{m^2 \ + \ p_{1234}^2} \ = \ \sqrt{(m'')^2 \ + \ (p''_{1234})^2} \tag{2.54}$$

where κ^2 is the truly invariant quantity under the transformations. An immediate physical consequence of eq.(2.54) is that one may have in one frame of reference $m \neq 0; p_{1234} = 0$, while in another frame of reference one has $m'' < m$ and $p''_{1234} > 0$. Restoring physical constants, it yields $L_P^2 \ m^2 \ + \ L_P^8 \ p_{1234}^2 = \kappa^2 \Rightarrow L_P m \simeq \kappa$ after neglecting the contribution $L_P^8 p_{1234}^2$. The latter is not negligible when $p_{1234} \sim M_P^4$ which is the same order of magnitude as the ultraviolet-cutoff of the vacuum energy density.

One may note that p_{1234} has the same units (mass per unit volume) as the tension of a 3-brane (a lump). If one imagines the whole universe as the $3+1$-dim spacetime region spanned by the evolution of a 3-brane embedded in higher dimensions (like in the brane-world model), a zero value of p_{1234} (like the almost zero value of the observed vacuum energy density of our universe) would correspond to a tensionless 3-brane. Whereas a value of $p_{1234} \sim M_P^4$ corresponds to the ultraviolet-cutoff value of the 3-brane tension. These findings may shed some light into the cosmological constant problem, dark energy, and dark matter.

2.3 The Most General Transformations

In the most general case one cannot factorize the exponential $\exp(\theta_{ij}\gamma_{ij} + \theta_{ijk}\gamma_{ijk})$ into products of exponentials. The exponentials of generalized multivectors associated with real Clifford algebras have been found explicitly by [7], [8] (see also [9]). For simplicity we shall focus in $3D$ since the expressions in higher dimensions are very cumbersome. Given a multivector $\mathbf{A} = a_0 + a_i e_i + a_{ij} e_{ij} + a_{123} e_{123}$[2] its exponential $exp(\mathbf{A}) = \mathbf{B}$ is another multivector $\mathbf{B} = b_0 + b_i e_i + b_{ij} e_{ij} + b_{123} e_{123}$ whose components (coefficients) $b_0, b_i, b_{ij}, b_{123}$ are explicitly given in terms of $a_0, a_i, a_{ij}, a_{123}$. For instance, in the $Cl(2,1)$ algebra case corresponding to a $D = 2+1$ spacetime, with $e_1^2 = e_2^2 = 1; e_3^2 = -1$, the coefficients found by [7] are given by

$$b_0 = \frac{1}{2} e^{a_0} \left(e^{a_{123}} co(a_+^2) + e^{-a_{123}} co(a_-^2) \right) \quad (2.55a)$$

$$b_{123} = \frac{1}{2} e^{a_0} \left(e^{a_{123}} co(a_+^2) - e^{-a_{123}} co(a_-^2) \right) \quad (2.55b)$$

$$b_1 = \frac{1}{2} e^{a_0} \left(e^{a_{123}} (a_1 + a_{23}) si(a_+^2) + e^{-a_{123}} (a_1 - a_{23}) si(a_-^2) \right) \quad (2.55c)$$

$$b_2 = \frac{1}{2} e^{a_0} \left(e^{a_{123}} (a_2 - a_{13}) si(a_+^2) + e^{-a_{123}} (a_2 + a_{13}) si(a_-^2) \right) \quad (2.55d)$$

$$b_3 = \frac{1}{2} e^{a_0} \left(e^{a_{123}} (a_3 - a_{12}) si(a_+^2) + e^{-a_{123}} (a_3 + a_{12}) si(a_-^2) \right) \quad (2.55e)$$

$$b_{12} = \frac{1}{2} e^{a_0} \left(- e^{a_{123}} (a_3 - a_{12}) si(a_+^2) + e^{-a_{123}} (a_3 + a_{12}) si(a_-^2) \right) \quad (2.55f)$$

$$b_{13} = \frac{1}{2} e^{a_0} \left(- e^{a_{123}} (a_2 - a_{13}) si(a_+) + e^{-a_{123}} (a_2 + a_{13}) si(a_-^2) \right) \quad (2.55g)$$

$$b_{23} = \frac{1}{2} e^{a_0} \left(e^{a_{123}} (a_1 + a_{23}) si(a_+^2) - e^{-a_{123}} (a_1 - a_{23}) si(a_-^2) \right) \quad (2.55h)$$

where

$$a_+^2 = - (a_3 - a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2 \quad (2.56a)$$

$$a_-^2 = - (a_3 + a_{12})^2 + (a_2 + a_{13})^2 + (a_1 - a_{23})^2 \quad (2.56b)$$

$$si(a_\pm^2) = \frac{sinh(\sqrt{a_\pm^2})}{\sqrt{a_\pm^2}}, \ a_\pm^2 > 0; \quad si(a_\pm^2) = \frac{sin(\sqrt{-a_\pm^2})}{\sqrt{-a_\pm^2}}, \ a_\pm^2 < 0 \quad (2.57a)$$

$$co(a_\pm^2) = cosh(\sqrt{a_\pm^2}), \ a_\pm^2 > 0; \quad co(a_\pm^2) = cos(\sqrt{-a_\pm^2}), \ a_\pm^2 < 0 \quad (2.57b)$$

The condition imposed on the rotor $\mathbf{R}^\dagger = \mathbf{R}^{-1}$ forces to set the parameters $a_0 = a_1 = a_2 = a_3 = 0$ in eqs-(2.55-2.57), and such that in this particular case one has $a_+^2 = a_-^2$

[2] $I = e_{123}$ is a pseudo-scalar in $3D$. We are using the notation of [7] for the Clifford algebra generators

in eqs.(2.56). The components of $\mathbf{R}^{-1}$ are obtained from eqs-(2.55-2.57) by changing the signs of $a_{12}, a_{13}, a_{23}, a_{123}$.

The temporal direction in [7], [8] coincides with x_3 and the spatial ones are given by x_1, x_2. Thus the transformations involving the elements $a_{13}e_{13}; e_{23}e_{23}$ of the exponential of a multivector in $2+1$ spacetime dimensions correspond to Lorentz boosts along the spatial x_1, x_2 directions, respectively. While ordinary rotations along the $x_1 - x_2$ axes involve the element $a_{12}e_{12}$. If one sets $a_{12} = 0$ one ends up with the hyperbolic functions in eqs-(2.57) which are consistent with boosts. Whereas if one sets $a_{13} = a_{23} = 0$ one ends up with trigonometric functions which are consistent with rotations.

The most general transformations $\mathbf{X}' = \mathbf{R}\mathbf{X}\mathbf{R}^\dagger$ associated with the rotor

$$\mathbf{R} \ = \ \exp \left(\ a_{12} \ e_{12} + a_{13} \ e_{13} + a_{23} \ e_{23} + a_{123} \ e_{123} \ \right), \quad \mathbf{R}^\dagger = \mathbf{R}^{-1} \tag{2.58}$$

in $3D$ are extremely complex and lead now to a $full$ mixing of all the grades of the multivector coordinates $\mathbf{X}' = \mathbf{X}'(\mathbf{X})$ involving all the parameters $a_{12}, a_{13}, a_{23}, a_{123}$.

To find the explicit components of the exponential of a multivector associated with a Clifford algebra in $4D$ is a more difficult task, let alone writing down the most general multivector coordinate transformations. The authors [8] more recently have presented formulae to calculate multivector exponentials in a basis-free representation and orthonormal basis for an arbitrary Clifford geometric algebra in any dimension and signature. The formulae are based on the analysis of roots of the characteristic polynomial of a multivector. Elaborate examples of how to use the formulas in practice were presented. The results were generalized to *arbitrary* functions of a multivector, like the logarithm, hyperbolic and trigonometric functions and their inverses.

3 Concluding Remarks

The whole construction of the generalized Lorentz transformations in C-spaces presented in this work can be extended to phase spaces. The simplest way to attain this goal is to combine multivector-coordinates and momenta into complex variables as $\mathbf{Z} = \mathbf{X} + \mathrm{i}\ \mathbf{P}$, and recur to the Hermitian conjugation operation depicted in eq.(2.4), in order to define the complex transformations by

$$\mathbf{Z}' \ = \ \mathbf{R}\ \mathbf{Z}\ \mathbf{R}^\ddagger \ = \ \mathbf{R}\ \mathbf{Z}\ \mathbf{R}^{-1} \ \Rightarrow \ \langle \ \mathbf{Z}'^\dagger \ \mathbf{Z}' \ \rangle \ = \ \langle \ \mathbf{Z}^\dagger \ \mathbf{Z} \ \rangle \ \Rightarrow$$

$$(z^*)' \ (z)' \ + \ (z^*_\mu)' \ (z^\mu)' \ + \ (z^*_{\mu\nu})' \ (z^{\mu\nu})' \ + \ \ldots \ = \ z^* \ z \ + \ z^*_\mu \ z^\mu \ + \ z^*_{\mu\nu} \ z^{\mu\nu} \ + \ \ldots \tag{3.1}$$

with

$$z \ = \ x + ip, \quad z_\mu \ = \ x_\mu + ip_\mu, \quad z_{\mu\nu} \ = \ x_{\mu\nu} + ip_{\mu\nu}, \ \ldots, \tag{3.2}$$

The rotor is given by

$$\mathbf{R} \ = \ \exp \left(\ a_o \ + \ a_i \ e_i \ + \ a_{ij} \ e_{ij} \ + \ a_{ijk} \ e_{ijk} \ + \ \cdots \ \right) \tag{3.3}$$

where now $a_o, a_i, a_{ij}, a_{ijk}, \cdots$ are *complex*-valued. The Hermitian conjugate is

$$\mathbf{R}^\ddagger \ = \ \exp \Big(\ a^*_o \ + \ a^*_i \ e_i^{-1} \ + \ a^*_{ij} \ e_{ij}^{-1} \ + \ a^*_{ijk} \ e_{ijk}^{-1} \ + \ \cdots \ \Big) \tag{3.4}$$

The complex parameters in eq.(3.4) must be restricted in order to obey the key condition $\mathbf{R}^\ddagger = \mathbf{R}^{-1}$. The authors [7] have provided a general rule how to obtain the inverses of the Clifford multivector generators in a straightforward fashion and found that $e_I^{-1} = \pm e_I$ where I is a multivector index. Hence, the provision $\mathbf{R}^\ddagger = \mathbf{R}^{-1}$ yields

$$a_o^* = - a_o, \quad a^*_{i_1 i_2 \cdots i_n} \, e^{-1}_{i_1 i_2 \cdots i_n} = - \, a_{i_1 i_2 \cdots i_n} \, e_{i_1 i_2 \cdots i_n} \tag{3.5}$$

and one finds that the parameters $a_{i_1 i_2 \cdots i_n}$ are either real or purely imaginary, while a_o is purely imaginary. Under the transformations (3.1) the coordinates and momenta are entangled as it occurs in Born's reciprocal relativity theory [14], [15]. A phase space extension of C-space was advanced by [11].

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [12]. The principle behind Born's reciprocal relativity theory [14], [15] was based on the idea proposed long ago by [12] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A maximal speed limit (speed of light) must be accompanied with a maximal proper force (which is also compatible with a maximal and minimal length duality) [15]. The principle of maximal acceleration was advocated earlier on by [13].

We finalize by saying that one has not been trying to "squeeze" new physics out of Clifford algebras in this work. One the contrary, it was the physics behind string theory, p-branes that led us to Clifford space relativity in the first place.

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APPENDIX

The Clifford geometric product [4], [5] of the Clifford algebra multivector generators requires the evaluation of both commutators and anti-commutators. This is instrumental in deriving the extended Lorentz transformations in C-space described in this work. The evaluation of the commutators of the Clifford algebra generators can be found in [10]. In general for $pq = odd$ one has

$$[\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}] = 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \, \delta^{[a_1 a_2}_{[b_1 b_2} \, \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} + \frac{2p!q!}{4!(p-4)!(q-4)!} \, \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \, \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} - \ldots\ldots \tag{A.1}$$

for $pq = even$ one has

$$[\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}] = - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \, \delta^{[a_1}_{[b_1} \, \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1....a_3}_{[b_1....b_3}\ \gamma^{a_4....a_q]}_{b_4.....b_p]}\ +\ ...... \qquad (A.2)$$

The anti-commutators of the Clifford algebra generators can also be found in [10], and one has the reciprocal situation as in eqs-(2.2,2.3), one has instead that for $pq = even$

$$\{\gamma_{b_1 b_2.....b_p}, \gamma^{a_1 a_2......a_q}\} \ = \ 2\gamma^{a_1 a_2......a_q}_{b_1 b_2.....b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3....a_q]}_{b_3.....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1....a_4}_{[b_1....b_4}\ \gamma^{a_5....a_q]}_{b_5.....b_p]}\ -\ ...... \qquad (A.3)$$

And for $pq = odd$ one has

$$\{\gamma_{b_1 b_2.....b_p}, \gamma^{a_1 a_2......a_q}\} \ = \ -\ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3....a_q]}_{b_2 b_3.....b_p]}\ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1....a_3}_{[b_1....b_3}\ \gamma^{a_4....a_q]}_{b_4.....b_p]}\ +\ ...... \qquad (A.4)$$

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