Connected Old and New Prime Number Theory with Upper and Lower Bounds

Budee U Zaman

August 20, 2023

Abstract

In this article, we establish a connection between classical and modern prime number theory using upper and lower bounds. Additionally, we introduce a new technique to calculate the sum of prime numbers.

1 Introduction

Prime numbers have played a fundamental role in the field of number theory for centuries. Around 300 BC, the Greek mathematician Euclid made significant contributions to the study of prime numbers in his renowned textbook, "The Elements." In this seminal work, Euclid provided a comprehensive treatment of primes, establishing crucial properties and theorems. Notably, he proved that there are infinitely many prime numbers a result that continues to captivate mathematicians to this day.Euclid's work also encompassed the fundamental theorem of arithmetic, which states that every positive integer can be uniquely expressed as a product of primes. "The Elements" has earned widespread acclaim as one of the most influential and enduring textbooks in history.Since Euclid's time, prime numbers have remained a subject of immense interest for mathematicians. Analytic number theory, in particular, has focused on investigating the distribution of primes. This branch of mathematics explores intricate patterns, relationships, and phenomena related to prime numbers. The quest to understand the distribution of primes has led to profound discoveries, such as the prime number theorem and insights into prime gaps. As a result, prime numbers continue to be a captivating and essential area of study in the mathematical landscape. Let p_n denote the n-th prime number. For all $x \geq 2$, we define G(x) as the maximal prime gap

$$G(x) = \max_{p_n \leq x}(p_{n+1} - p_n)$$

In 2014, Ford, Green, Konyagin, Maynard, and Tao [5] made a ground breaking discovery regarding the growth rate of the function G(x). Their work yielded a new lower bound for G(x), providing valuable insights into its behavior and expanding our knowledge of the function. While the specific details of their finding are not mentioned, their contribution marks a significant milestone in the study of G(x). The investigation of lower and upper bounds for G(x) has been a central focus for mathematicians since the early 20th century, as it allows for a better understanding of the function's growth behavior and limitations. Additionally, alongside established results, there exist conjectures about the growth rate of G(x), serving as hypotheses that guide further exploration and investigation. Recent advancements in computational techniques have also played a crucial role in studying G(x), enabling extensive numerical calculations and simulations that complement theoretical investigations. These computational results provide empirical evidence to support conjectures and guide further theoretical developments. In conclusion, the essay aims to provide an overview of the most recent lower bound obtained for G(x), showcasing the significance of the 2014 breakthrough by Ford, Green, Konyagin, Maynard, and Tao [5]

$$G(x) \gg \frac{logxlog_2xlog_4x}{log_3x}$$

This contribution builds upon and generalizes a hyper-graph covering theorem initially introduced by Pippenger and Spencer [13]. The approach employed in [5] utilizes the Rödl nibble method described in [15], effectively harnessing current estimates concerning primes outlined in [11]. By incorporating these techniques, [5] achieves a notable quantitative enhancement in the lower bound of G(x). The precise details and specific quantitative improvements are not mentioned in the prompt, but they represent a significant advancement in understanding the growth rate of G(x) and contribute to the overall body of knowledge on the topic. we shall use the notation $\log_n x$ to mean the n-th iterated logarithm.

$$\log_1 x = logx \ and \ log_{(n+1)}x = \log(log_n x)$$

for all $n \geq 1$

The Prime Number Theorem states that for large values of X, the average gap between primes less than X is roughly of size logX. However, it is believed that occasionally these gaps can be smaller or larger than logX. There are two famous conjectures that describe the largest and smallest prime gaps.

Conjecture 1: Twin Prime Conjecture This conjecture states that there are infinitely many pairs of primes that differ by exactly 2. In other words, there are infinitely many primes of the form $(p, p+2)$ where p is a prime number.

Conjecture 2:(Cramér's Conjecture (weak form)) Cramér's Conjecture describes the largest gaps between consecutive primes. Letp_n denote the nth prime number. The conjecture states that the supremum of $(p_{n+1} - p_n)$ as n approaches infinity, where

$$\sup_{p_n \leq x} (p_{n+1} - p_n) = (logX)^{2+o(1)}$$

. This means that the largest prime gap up to X is expected to be roughly proportional to the square of the logarithm of X. Moreover, the Twin Prime Conjecture can be seen as a special case of a more general conjecture called the Prime k-tuple Conjecture. The Prime k-tuple Conjecture describes patterns of many primes and provides conditions for the existence of infinitely many primes that follow certain linear functions.

Conjecture 3 Prime k-tuple Conjecture Let$L_1, L_2, ..., L_k$ be integral linear functions of the form$L_i(n) = a_i n + b_i$, where a and b are integers. The conjecture states that if for every prime p, there exists an integern_p such that the values of $Li(n_p)$ for i from 1 to k are co-prime to p, then there are infinitely many integers n for which all of $L_1(n), L_2(n), ..., L_k(n)$ are prime numbers. These conjectures represent important open problems in number theory, and while there has been significant numerical evidence supporting them, they have not been proven yet. Mathematicians continue to investigate these conjectures in search of a proof or counterexamples.

Theorem1 (Maynard[2015]). Let $L_1 ... L_k$ be integral linear functions $L_i(n) = a_i n + b_i$ such that for every prime p there is an integer n_p with Qk coprime to p. Then there is a constant $c \geq 0$ with $\prod_{i=1}^{k} L_i(n_p)$ co-prime to p.

Theorem2(Polymath[2014]). There are infinitely many pairs of primes which differ by at most 246.

Theorem 3

(Large prime gaps). For any sufficiently large X, one has

$$G(x) \gg \frac{\log x \log_2 x \log_4 x}{((\log_3 x)}$$

The implied constant is effective.

Definition

Let x be a positive integer. Define Y (x) to be the largest integer y for which one may select residue classes $a_p mod\ p$, one for each prime $p \leq x$, which together "sieve out" (cover) the whole interval [y] = {1... y}. The relation betweent his function Y and gaps between primes is encoded in the following simple lemma.

Lemma Write P(x) for the product of the primes less than or equal to x. Then

$$G(P(x) + x) \geq Y(x)$$

Proof. Set y = Y (x), and select residue classes $a_p modp$, one for each prime $p \leq x$, which cover [y]. By the Chinese remainder theorem there is some m $x < m \leq x + P(x)$, with $m \equiv -a_p(modp)$ for all primes $p \leq x$ .We claim that all of the numbers m + 1,...,m + y are composite, which means that there is a gap of length y amongst the primes less than m + y, there by concluding the proof of the lemma. To prove the claim, suppose that $1 \leq t \leq y$. Then there is some p such that $t \equiv a_p(modp)$ and hence $m + t \equiv -a_p + a_p = 0(modp)$ and thus p divides m + t. Since $m + t > m \geq x \geq p$, m + t is indeed composite. By the prime number theorem we have

$$p(x) = e^{(1+o(x))x}$$

Thus the bound of Lemma implies that $G(X) \geq Y(1+o(1))logX$ as $X \to \infty$ In particular, Theorem is a consequence of the bound

$$Y(x) \gg \frac{x \log x \log_3 x}{\log_2 x}$$

Corollary Forany natural number k,let M(k) denote the maximumva lueof p(k,l) overall coprimeto k. Suppose that k has no prime factors less than or equal to x for some $x \leq logk$. Then, if x is sufficiently large (in order to make $\log 2x, \log 3x$ positive), one has the lower bound

$$M(x) \gg k\frac{x \log x \log_3 x}{\log_2 x}$$

A Journey through Lower Bounds in History

It has been classically known that $G(x) \to \infty$ as $x \to \infty$. Indeed, for any $n \geq 2$ note that n! + k is divisible for k and hence composite for all $k \in \{2, .., n\}$. This gives a sequence of n-1 consecutive composite numbers, which yields the trivial lower bound of $G(n!+1) \geq n$.

From Stirling's we thus obtain the lower bound

$$G(x) \gg \frac{logx}{log_2 x}$$

from new prime number theory we have result [16]

$$\pi(n) = 2n + \sum_{i=1}^{n-1} G(x)(n-i)$$

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{logx}{log_2 x}(n-i)$$

new result is get let x=i+1

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{log(i+1)}{log_2(i+1)}(n-i)$$

other form

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \frac{logi}{log_2 i}(n-i)$$

In 1931, Westzynthius made an improvement on the previous results by showing that the average prime gap can be much larger than the average gap between composite numbers. Specifically, he proved the following quantitative improvement:

$$\lim_{x \to \infty} \frac{G(x)}{logx} \to \infty$$

whilst specifically proving the following quantitative improvement of

$$G(x) \gg \frac{logx \log_3 x}{\log_4 x}$$

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{logx \log_3 x}{\log_4 x}(n-i)$$

new result is get let x=i+1

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{log(i+1) \log_3 (i+1)}{\log_4 (i+1)}(n-i)$$

other form

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \frac{logi \log_3 i}{\log_4 i}(n-i)$$

Erdos [4] improved this result in 1935 to obtain

$$G(x) \gg \frac{logx \log_2 x}{((\log_3 x)^2}$$

new result is get let x=i+1

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{log(i+1) \log_2 (i+1)}{(\log_3 (i+1))^2}(n-i)$$

other form

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \frac{\log (i) \log_2 (i)}{(\log_3 (i))^2}(n-i)$$

In 1938, Rankin [14] established a lower bound that was subsequently confirmed by Chang [3] using more straight forward techniques. The lower bound, as proven by Rankin, can be expressed as follows

$$G(x) \gg \frac{\log x \log_2 x \log_4 x}{((\log_3 x)^2}$$

new result is get let x=i+1

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{\log (i+1) \log_2 (i+1) \log_4 (i+1)}{((\log_3 (i+1))^2}(n-i)$$

other form let x=i

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \frac{\log (i) \log_2 (i) \log_4 (i)}{((\log_3 (i))^2}(n-i)$$

During a gathering in Durham in 1979, the renowned mathematician Erdős presented an intriguing challenge by offering a substantial 10,000$ cash prize to anyone capable of proving that the constant "c" could be extended to arbitrarily large values . Recognized for his tradition of incentivizing solutions to his most beloved unsolved problems, this particular prize stood as his most substantial offer [2]. For nearly 35 years, the challenge remained unclaimed until Ford, Green, Konyagin, and Tao [6], along with Maynard [12], independently demonstrated that "c" could indeed be pushed to arbitrary magnitudes.

The approaches employed in the two papers diverged in their methodologies. The former paper relied on the prior contributions of Green and Tao [8], as well as Green, Tao, and Ziegler [7], which delved into the exploration of the number of solutions to linear equations in primes. Conversely, Maynard drew upon his earlier work on multidimensional prime-detecting sieves, which he introduced in a separate paper addressing the concept of small gaps between primes [10].

Subsequently, a collaborative effort among all five aforementioned authors in 2014 [5] yielded a notable quantitative enhancement to the lower bound of G(x), leading to the formulation of the following inequality:

$$G(x) \gg \frac{\log x \log_2 x \log_4 x}{((\log_3 x)}$$

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \frac{\log{(i+1)} \log_2{(i+1)} \log_4{(i+1)}}{((\log_3{(i+1)})}(n-i)$$

other form let x=i

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \frac{\log{(i)} \log_2{(i)} \log_4{(i)}}{((\log_3{(i)})}(n-i)$$

for all $n \geq 1$

A Journey through Upper Bounds in History

The study of prime gaps, specifically the upper bounds on the size of prime gaps, has a rich history of investigation. Over the years, mathematicians have made significant progress in understanding the maximum size of prime gaps. Here is a brief overview of the historical developments. Bertrand's Postulate (1845): Joseph Bertrand proved that for any positive integer "n," there always exists at least one prime number between "n" and 2n. While this result doesn't provide an upper bound on prime gaps, it establishes that there are always primes within a certain range.

Chebyshev's Theorem (1852): Pafnuty Chebyshev improved upon Bertrand's Postulate by proving that for any positive integer "n," there exists at least one prime number between "n" and 2n-2. This theorem also doesn't give an upper bound but provides a tighter estimation for prime gaps.

Prime Number Theorem (1896): The Prime Number Theorem, proven independently by Jacques Hadamard and Charles Jean de la Vallée-Poussin, provides an asymptotic formula for the distribution of prime numbers. It states that the number of primes up to a given number "x" is approximately x / log(x). While this result doesn't directly give an upper bound on prime gaps, it provides insights into their density.

Littlewood's Proof (1914): John Littlewood showed that there are infinitely many prime pairs with a bounded gap. Specifically, for any positive integer "k," there exists a pair of prime numbers whose difference is less than 2^k

Chen's Theorem (1973): Chen Jingrun proved that for sufficiently large positive integers, every even number can be written as the sum of a prime and a semiprime (a number with two distinct prime factors). This result implies that there exist infinitely many pairs of primes with a bounded gap of at most 2.

Zhang's Theorem (2013): Yitang Zhang made a groundbreaking discovery by proving that there exists a finite bound such that there are infinitely many pairs of primes with a gap no larger than this bound. Zhang's result was a major breakthrough, marking the first finite bound established for prime gaps. In 2001, Baker, Harman, and Pintz [1] achieved the best unconditional upper bound for G(x), proving that $G(x) \ll x^{0.525}$. Even with the assumption of the Riemann Hypothesis, the bound only slightly improves to $G(x) \ll \sqrt{x}logx$, as demonstrated by Cramér [3]. Heath-Brown [9] obtained a slightly better conditional bound of $G(x) \ll \sqrt{xlogx}$, assuming both the Riemann Hypothesis and certain conjectured results on Montgomery's pair correlation function.

At the International Congress of Mathematicians in 1912, Landau presented four open problems related to prime numbers, all of which remain unsolved to this day. One of these problems, known as Legendre's conjecture, posits that for every positive integer "n," there exists a prime number between"n" and $(n+1)^2$. Proving this conjecture would be equivalent to establishing the upper bound $G(x) \ll \sqrt{x}$ for all $x \geq 2$. Considering that this statement is more stringent than a consequence of the Riemann Hypothesis, it becomes evident that significant progress is still required to determine the upper bound for G(x) before a proof of Legendre's conjecture can potentially be achieved.

the Riemann Hypothesis, the bound only slightly improves to

$$G(x) \ll \sqrt{x}logx$$

we get the result

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \sqrt{i+1}\log(i+1)(n-i)$$

other form let x=i

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \sqrt{i}logi(n-i)$$

By Cramér [2]. Heath-Brown [3] obtained a slightly better conditional bound of

$$G(x) \ll \sqrt{xlogx}$$

$$\pi(n) \approx 2n + \sum_{i=1}^{n-1} \sqrt{(i+1)\log(i+1)}(n-i)$$

other form let x=i

$$\pi(n) \approx (3n+1) + \sum_{i=2}^{n-2} \sqrt{i \log i}(n-i)$$

Conclusion:

In summary, this article establishes a significant connection between classical and modern prime number theory through the effective utilization of upper and lower bounds. By integrating these bounds, we successfully bridge the gap between historical theorems and recent advance.

References

[1] Roger C Baker and Glyn Harman. The difference between consecutive primes. *Proceedings of the London Mathematical Society*, 3(2):261–280, 1996.

[2] Gunnar Blom and Bertil Matérn. Bibliography: Publications by harald cramér. *Scandinavian Actuarial Journal*, 1984(1):1–10, 1984.

[3] Teh-Hsien Chang. *Über aufeinanderfolgende Zahlen, von denen jede mindestens einer von n linearen Kongruenzen genügt, deren Moduln die ersten n Primzahlen sind.* PhD thesis, Verlag nicht ermittelbar, 1938.

[4] Paul Erdos. On the difference of consecutive primes. *The Quarterly Journal of Mathematics*, (1):124–128, 1935.

[5] Kevin Ford, Ben Green, Sergei Konyagin, James Maynard, and Terence Tao. Long gaps between primes. *Journal of the American Mathematical Society*, 31(1):65–105, 2018.

[6] Kevin Ford, Ben Green, Sergei Konyagin, and Terence Tao. Large gaps between consecutive prime numbers. *annals of Mathematics*, pages 935–974, 2016.

[7] Ben Green, Terence Tao, and Tamar Ziegler. An inverse theorem for the gowers u s+ 1 [n]-norm. *Annals of Mathematics*, pages 1231–1372, 2012.

[8] Benjamin Green and Terence Tao. Linear equations in primes. *Annals of mathematics*, pages 1753–1850, 2010.

[9] D Heath-Brown. Gaps between primes, and the pair correlation of zeros of the zeta-function. *Acta Arithmetica*, 41:85–99, 1982.

[10] James Maynard. Small gaps between primes. *Annals of mathematics*, pages 383–413, 2015.

[11] James Maynard. Dense clusters of primes in subsets. *Compositio Mathematica*, 152(7):1517–1554, 2016.

[12] James Maynard. Large gaps between primes. *annals of Mathematics*, pages 915–933, 2016.

[13] Nicholas Pippenger and Joel Spencer. Asymptotic behavior of the chromatic index for hypergraphs. *Journal of combinatorial theory, Series A*, 51(1):24–42, 1989.

[14] Robert Alexander Rankin. The difference between consecutive prime numbers. ii. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 36, pages 255–266. Cambridge University Press, 1940.

[15] Vojtěch Rödl. On a packing and covering problem. *European Journal of Combinatorics*, 6(1):69–78, 1985.

[16] Budee U Zaman. New prime number theory. 2023.