On the Zeta distribution $\zeta(s)$ *and the Riemann hypothesis*

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It seems very curious that such an important random variable $\zeta(s)$ is so little studied in probability theory and yet, it can be the basis for several discoveries in arithmetic.

Origin of the Zeta Law $\zeta(s)$

The Zeta law originates from the search for a uniform distribution in the set of integers N= {1,2,...}*
Such a uniform law does not exist, but there are asymptotic laws that tend towards this law.
And precisely, we will show that by tending s towards 1, we obtain almost the "same universe" as a uniform law.
Indeed, let X be a uniform random variable in the set {1,2,...,N}
Let (p) be the sequence of primes 2,3,5,...
According to the fundamental theorem of arithmetic, X is uniquely written as:

$$X = \prod_p p^{X_p} \text{ with } X_p \in \{0,1,2,\dots\}$$

Distribution of random variables X_p

$$P(X_p \geq k) = P(p^k \text{divise } X) = \frac{\left[\frac{N}{p^k}\right]}{N} \text{ with } k \in \{0,1,2,\dots\}$$

It can be seen that when $N \to +\infty$, the random variable X_p tends to a geometric random variable of parameter $1 - \frac{1}{p}$
Thanks to this very simple random variable, we can revisit the very famous formula, that of Legendre.
if X follows a uniform random variable in the set {1,2,...,N}, then E(ln(X)) is equal to:

$$\frac{1}{N}\sum_{n=1}^{N} \ln(n) = \frac{1}{N}\ln\left(\prod_{n=1}^{N} n\right) = \frac{1}{N}\ln(N!)$$

Likewise $\ln(X) = \sum_p X_p \ln(p) \Rightarrow E(\ln(X)) = \sum_p E(X_p)\ln(p)$

$$E(X_p) = \sum_{k=0}^{+\infty} k\left(\frac{\left[\frac{N}{p^k}\right]}{N} - \frac{\left[\frac{N}{p^{k+1}}\right]}{N}\right) = \sum_{k=0}^{+\infty} k\,\frac{\left[\frac{N}{p^k}\right]}{N} - \sum_{k=1}^{+\infty}(k-1)\,\frac{\left[\frac{N}{p^k}\right]}{N} = \sum_{k=1}^{+\infty} \frac{\left[\frac{N}{p^k}\right]}{N}$$

We can therefore write:

$$\frac{1}{N}\ln(N!) = \sum_p \sum_{k=1}^{+\infty} \frac{\left[\frac{N}{p^k}\right]}{N}\ln(p) = \frac{1}{N}\ln\left(\prod_p p^{\sum_{k=1}^{+\infty}\left[\frac{N}{p^k}\right]}\right)$$

Hence it is concluded that:

$$\boldsymbol{N}! = \prod_{\boldsymbol{p}} \boldsymbol{p}^{\sum_{k=1}^{+\infty}\left[\frac{\boldsymbol{N}}{\boldsymbol{p}^k}\right]}$$

better known as Legendre's formula.

In the same way, it can be shown that random variables (X_p) become independent when $N \to +\infty$
Now it is assumed that $X_p \sim G(1 - \frac{1}{p^s})$ and are independent.
To tender $s \to 1$ or $N \to +\infty$, brings us back to the same "universe", that of usual Arithmetic.
It is shown in this case that $X \sim \zeta(s)$ i.e. $P(X = x) = \frac{1}{\zeta(s)\ x^s}$ avec $x = 1,2,3,\dots$
Let's calculate the probability of choosing a number at random and that it is even:

$$P(X\ pair) = P(X_2 \geq 1) = \frac{1}{2^s}$$

$$P(X\ impair) = P(X_2 = 0) = 1 - \frac{1}{2^s}$$

When $s \to 1$, these two probabilities equalize and tend towards 1/2. This confirms the intuition that: "there is a one in two chance of drawing an even number at random."

Thanks to this zeta law, we will revisit many famous arithmetic results.

Let be the function $F(X) = \prod_p f(X_p)$ *and therefore* $E(F(X)) = \prod_p E(f(X_p)$

What gives

$$\sum_{n=1}^{+\infty} \frac{F(n)}{n^s} = \prod_p \sum_{k=0}^{+\infty} \frac{f(k)}{p^{sk}} \qquad (1)$$

I. Eulerian product

Cases where i.e. $f(k) = 1, \forall k \geq 0 F(n) = 1, \forall n \geq 1$

$$\sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \sum_{k=0}^{+\infty} \frac{1}{p^{sk}} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

II. Möbius function

Cases where i.e. $f(k) = 1_{k\leq 1}(-1)^k \quad \forall k \geq 0 F(n) = \mu(n), \forall n \geq 1$

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \prod_p \sum_{k=0}^{1} \frac{(-1)^k}{p^{sk}} = \prod_p (1 - \frac{1}{p^s}) = \frac{1}{\zeta(s)}$$

III. Probability of being square-free

Cases where i.e. $f(k) = 1_{k\leq 1} \quad \forall k \geq 0 F(n) = |\mu(n)|, \forall n \geq 1$

$$\sum_{n=1}^{+\infty} \frac{|\mu(n)|}{n^s} = \prod_p \sum_{k=0}^{1} \frac{1}{p^{sk}} = \prod_p (1 + \frac{1}{p^s}) = \frac{\zeta(s)}{\zeta(2s)}$$

$\sum_{n=1}^{+\infty} \frac{|\mu(n)|}{\zeta(s)\, n^s} = \frac{1}{\zeta(2s)}$ *refers to the probability of choosing a number at random and that it is square-free.*

$$When\ s \to 1, this\ probability\ tends\ to\ \frac{1}{\zeta(2)}\ c'est\grave{a}\ dire\ \frac{6}{\pi^2}$$

Proof of the Riemann hypothesis (yet another)

Now, we will be very optimistic and give what seems to be a probabilistic argument in favor of the Riemann hypothesis through Denjoy's version.

It seems that the Riemann hypothesis (RH) is very much related to the Möbius function and that ultimately RH is equivalent to the fact that $P(\mu(n) = +1) = P(\mu(n) = -1)$

We have $P(\mu(n) = +1) + P(\mu(n) = -1) = \frac{1}{\zeta(2s)}$

Similarly, i.e. $E(\mu(X)) = \frac{1}{\zeta(s)^2} P(\mu(n) = +1) - P(\mu(n) = -1) = \frac{1}{\zeta(s)^2}$

So we have and $P(\mu(n) = +1) = \frac{1}{2}(\frac{1}{\zeta(2s)} + \frac{1}{\zeta(s)^2}) P(\mu(n) = -1) = \frac{1}{2}(\frac{1}{\zeta(2s)} - \frac{1}{\zeta(s)^2})$

$When\ s \to 1, these\ two\ probabilities\ tend\ to\ \frac{3}{\pi^2}$

Suppose that X and Y are two random variables according to $\boldsymbol{\zeta(s)}$

Let Z= XY be the product of X and Y.

Z is defined in the set N, as X and Y.*

$P(Z = z) = P(XY = z)$

$$P\left(\prod_p p^{X_p+Y_p} = \prod_p p^{z_p}) = P\big(X_p + Y_p = z_p, \forall p\big) = \prod_p P(X_p + Y_p = z_p\right)$$

$$= \prod_p \sum_{x_p=0}^{z_p} P\big(X_p = x_p\big) P\big(Y_p = z_p - x_p\big)$$

$$= \prod_p \sum_{x_p=0}^{z_p} \frac{1}{p^{s\,x_p}}\left(1 - \frac{1}{p^s}\right) \frac{1}{p^{s\,(z_p - x_p)}}\left(1 - \frac{1}{p^s}\right) = \prod_p (z_p+1) \frac{1}{p^{sz_p}}\left(1 - \frac{1}{p^s}\right)^2 = \frac{\tau(z)}{z^s \boldsymbol{\zeta}(\mathbf{s})^2}$$

Another way to translate the fundamental theorem into probabilistic terms is as follows:

Let X be a random variable according to $\boldsymbol{\zeta}(\boldsymbol{s})$

$X = \prod_p p^{X_p}$ *with* $X_p \in \{0,1,2,\ldots\}$
with that and are independent $X_p \sim G(1 - \frac{1}{p^s})$

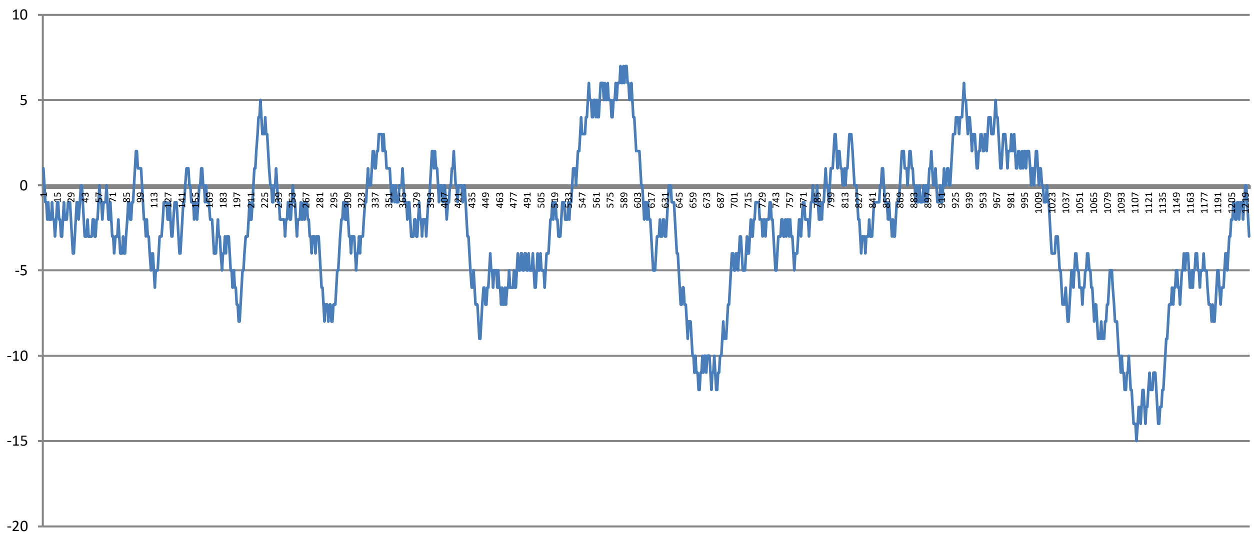
We can say that i.e. the entropy of X is the sum of the entropies of .$H(X) = \sum_p H(X_p)\, X_p$

After fairly simple calculations, we find $s \sum_p \ln(p) \frac{\frac{1}{p^s}}{1 - \frac{1}{p^s}} + \ln(\boldsymbol{\zeta}(\boldsymbol{s}))$ *that unfortunately tends towards* $+\infty$ *quand* $s \to 1$

Today, I just discovered "on the internet" a formula that gives the p-adic valuations of a number according to this number:

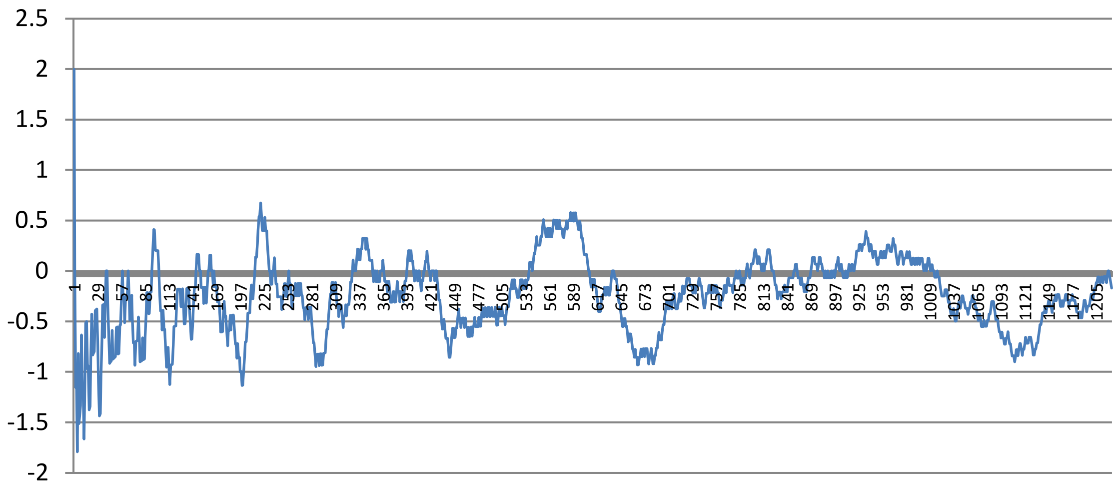
$$if\ \ X = \prod_p p^{X_p}\ \ then\ X_p = log_p(pgcd(X, p^{[log_p(X)]}))$$

Thanks to these formulas, I managed to make calculations on Excel of the function of Möbius or Mertens. This is my first graph of the Mertens function with the first 200 primes (up to p=1223)



I think we need to deepen the study of areas where the Mertens function "leaves the x-axis" to get lost, we speak of positive or negative peaks.

The graph of the function $\frac{M(x)}{\sqrt{x}}$



The Mertens situation states that $\forall x > 1, |M(x)| < \sqrt{x}$

If this conjuncture were true, it would have implied the Riemann hypothesis. Unfortunately, it turned out to be wrong and the counterexample is beyond 10 30

(see http://www.dtc.umn.edu/~odlyzko/doc/arch/mertens.disproof.pdf)

DGMP Act and MCPP

Suppose X and Y $\sim\zeta(s)$

Z=gcd(X,Y) and T=lcm(X,Y)

It is demonstrated without too great difficulty that:

$$P_{(Z,T)}(z,t) = \frac{1}{(zt)^s}\frac{1}{\zeta(s)^{\,2}}\prod_p (1 + signe(t_p - z_p))$$

Special case:

Choosing two numbers at random, what is the probability that they are prime to each other and that their smallest common multiple is equal to t?

We find this probability equal to with w(t) which denotes the number of primes that divide t. It is very clear that this probability tends towards 0 when s tends towards 1. This means that in real arithmetic, this probability is zero. $\frac{2^{w(t)}}{t^s}\frac{1}{\zeta(s)^{\,2}}$

The domain of definition of this probability distribution is as follows:

z t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1																				
2	0		0		0		0		0		0		0		0		0		0	
3	0	0		0	0		0	0		0	0		0	0		0	0		0	0
4	0	0	0		0	0	0		0	0	0		0	0	0		0	0	0	
5	0	0	0	0		0	0	0	0		0	0	0	0		0	0	0	0	
6	0	0	0	0	0		0	0	0	0	0		0	0	0	0	0			
7	0	0	0	0	0	0		0	0	0	0	0	0		0	0	0	0	0	0
8	0	0	0	0	0	0	0		0	0	0	0	0	0	0		0	0	0	0
9	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0		0	0
10	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	
11	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

We see that this function is defined if z is a divisor of t and is zero if z does not divide t. Which is trivial since by definition z (gcd) must necessarily divide t (lcm).

Can this distribution law bring something new about prime numbers?

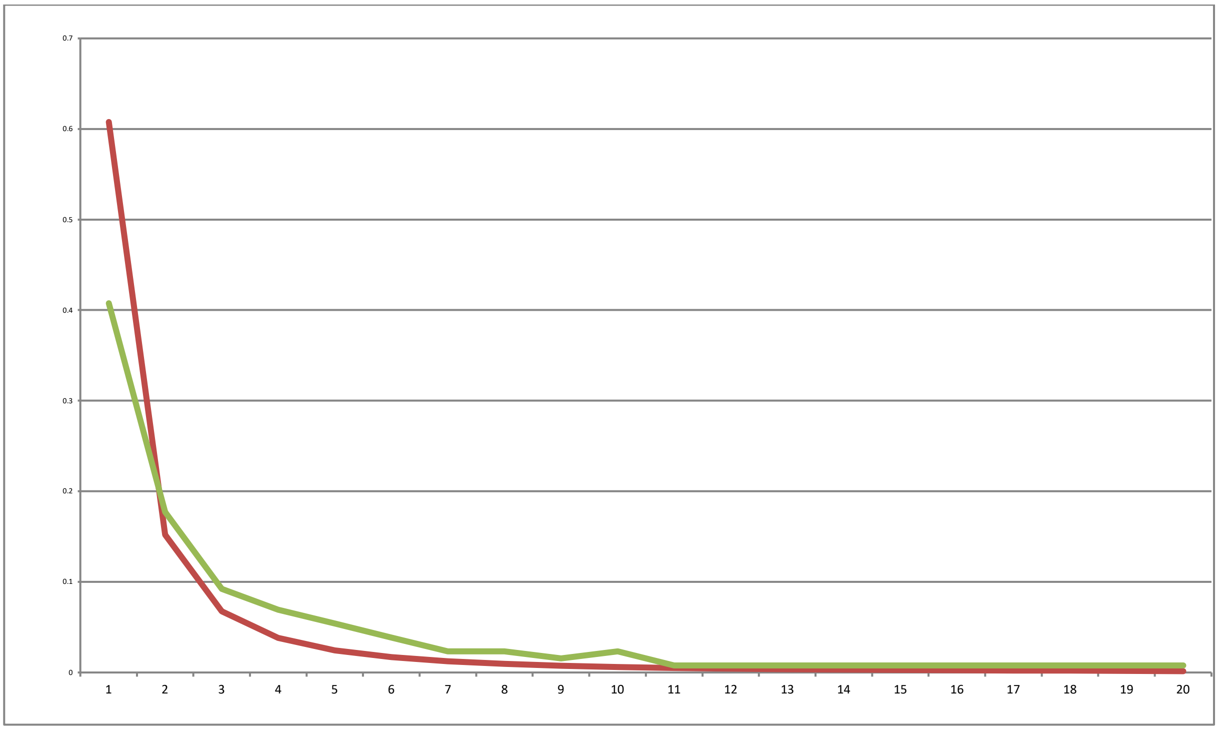
Marginal distributions are simple to find. Horizontally, the totals give a zeta distribution of parameter 2s.

Vertically, it's more complicated.

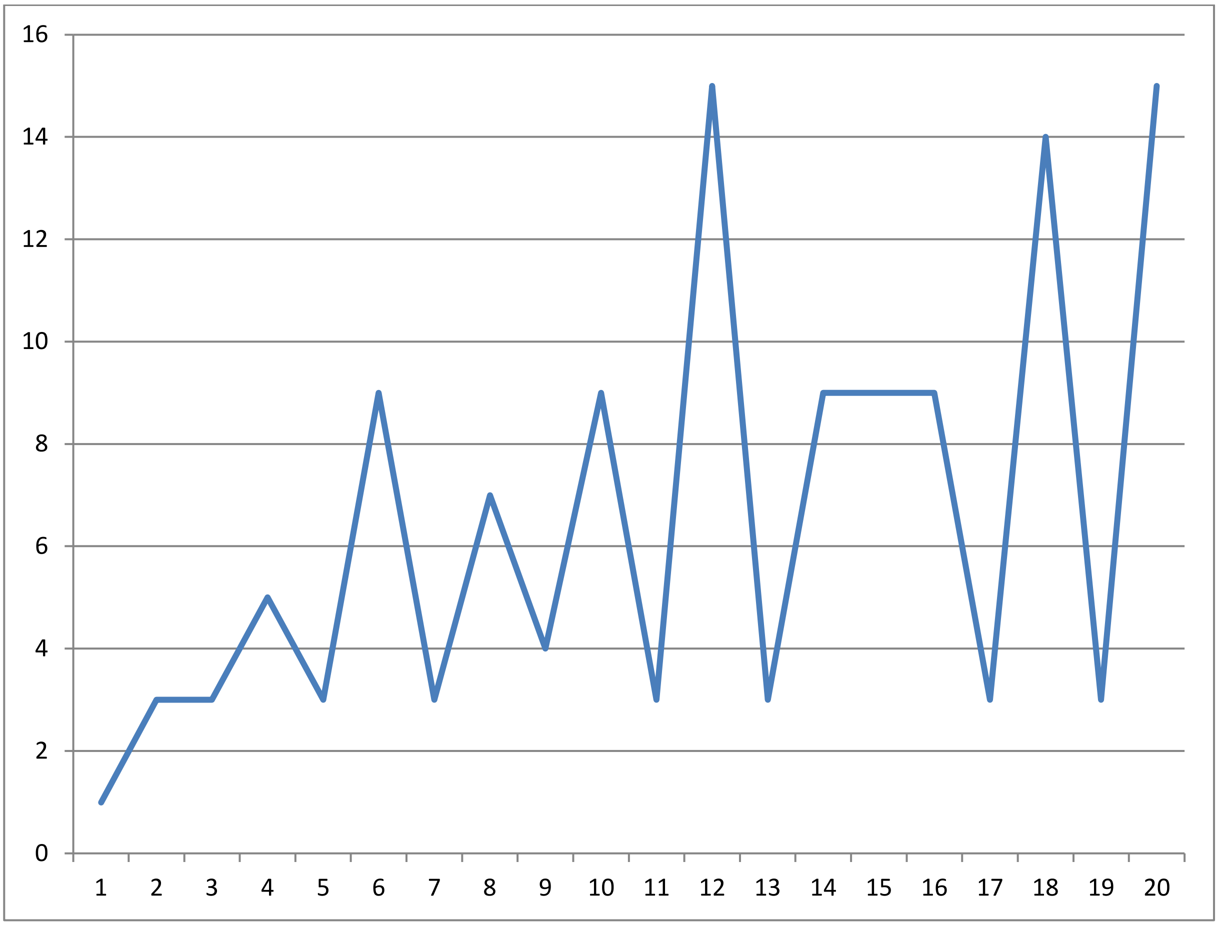
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	1	2	2	2	2	4	2	2	2	4	2	4	2	4	4	2	2	4	2	4	53
2	0	1	0	2	0	2	0	2	0	2	0	4	0	2	0	2	0	2	0	4	23
3	0	0	1	0	0	2	0	0	2	0	0	2	0	0	2	0	0	3	0	0	12
4	0	0	0	1	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0	2	9
5	0	0	0	0	1	0	0	0	0	2	0	0	0	0	2	0	0	0	0	2	7
6	0	0	0	0	0	1	0	0	0	0	0	2	0	0	0	0	0	2	0	0	5
7	0	0	0	0	0	0	1	0	0	0	0	0	0	2	0	0	0	0	0	0	3
8	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	2	0	0	0	0	3
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	2
10	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	2	3
11	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1

16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	1	3	3	5	3	9	3	7	4	9	3	15	3	9	9	9	3	14	3	15	130

We check that gcd follows a zeta(2) law but lcm is more irregular.



Distribution of the DGMP



MCPP Distribution

The distribution of lcm depends on the number of divisors of t.