On the microscopic substantiation of two-fluid hydrodynamics in helium II

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It is shown that the formally correct expression of the existing theory of superfluidity, obtained for the momentum of a phonon gas in helium II, does not describe the physically real transfer of the effective mass of the phonon gas. The physically real transferable effective mass of a phonon gas turns out to be 10^{12} times less than it is considered in the existing theory. It is established that the laws of energy and momentum transformation of a massive quasiparticle in helium II differ somewhat from the corresponding laws used in the existing theory. It follows from this that if the gas of rotons with the dispersion law introduced in 1947 behaves as a whole, then it must be entirely at rest relative to the "unexcited background" in helium II. It is noted that the criterion for the superfluidity of liquid helium implicitly assumes the use of an additional hypothesis about the direction of the momentum excited in liquid helium II quasiparticle (phonon).

Introduction

The phenomenon of superfluidity of helium II, discovered by P.L. Kapitsza [1] and J. Allen [2] in the late thirties of the XX century, received a theoretical justification in the works of L.D. Landau that followed this discovery, see [3], [4] and other articles from this cycle. The theory of helium II superfluidity created by L.D. Landau at that time includes a microscopic substantiation of the two-fluid hydrodynamics of helium II, first proposed by L. Tissa [5]. This theory of L.D. Landau was included in textbooks on theoretical physics [6,7,8]. Meanwhile, a number of issues of the mentioned theory have escaped the attention of researchers.

These are:

• the question of the value of the effective mass of the phonon gas in helium II;
• the question of the law of transformation of energy and momentum of a massive quasi-particle in helium II;
• the question of substantiating the criterion for the superfluidity of helium II.

This article is devoted to these topics.

On the effective mass of phonon gas in helium II

The main part [3] says: "... imagine that the phonon gas [in helium II] moves as a whole translationally with a constant velocity **V**". At the same time, the existence theorem is not proved. It is not clear whether the phonon gas in helium II can move at all, and if it can move, then what is the speed of such movement – finite, infinitesimal or infinitely large (the latter in the non-relativistic case). Meanwhile, as shown below, the effective mass of the phonon gas in the problem under consideration essentially depends on the value (finite or infinitely small) of the velocity of the phonon gas as a whole, and, accordingly, the form of the formula that determines the velocity of the phonon gas as a whole in helium II depends.

In order to investigate this issue, let us first consider this problem in terms of particles of matter. The determination of the velocity of the particle gas as a whole in helium II (here, phonon gas) is complicated by the fact that the phonon mass has a zero value. Therefore, we will assign to each phonon in helium II the same, very small mass m_ε, which we will subsequently tend to zero.

The temperature of helium II is considered low enough that the interaction of phonons with each other can be neglected, so that the total mass M_ε of a unit volume of phonon gas in the problem under consideration is simply equal to the sum of the masses of all phonons of the corresponding gas included in a given volume of helium II: $M_\varepsilon = \sum m_\varepsilon$. The total mass M_ε of the unit volume of the phonon gas in our consideration, as well as the mass of an individual phonon m_ε, thus tends to zero: $M_\varepsilon \to 0$.

Since the mass M_ε of the unit volume of the phonon gas in helium II is considered here, although small, but still different from zero, the velocity **V** of the phonon gas as a whole in the problem under consideration can be calculated using the well-known non-relativistic formula for the velocity of the particle system as a whole. This formula is given in [9] (formula (8.2)), namely:

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$$\mathbf{V} = \frac{\mathbf{P}}{M_\varepsilon}. \tag{1}$$

Here: **P** – is the total momentum of the unit volume of the phonon gas in helium II.

It follows from (1) that since $M_\varepsilon \to 0$, there are two alternative possibilities. The first possibility is that, along with $M_\varepsilon \to 0$, the total momentum **P** of the unit volume of the phonon gas in helium II simultaneously tends to zero, i.e., that $|\mathbf{P}| \to 0$. Then the right part of the relation (1) is the uncertainty relation of the form a $0/0$. The velocity **V** of the motion of the phonon gas as a whole can in this case be a finite value, and therefore the corresponding calculations made in [3] are quite applicable to this velocity. Repeating then the reasoning from [3], we obtain the following expression for the momentum **P** of the unit volume of the phonon gas in helium II, in terms of the velocity **V** of the movement of the phonon gas as a whole:

$$\mathbf{P} = \frac{4E}{3u^2} \cdot \mathbf{V}. \tag{2}$$

Here: E – the energy of the unit volume of the phonon gas;
u – the speed of the first sound in helium II, $u = 2{,}4 \cdot 10^4\ cm/s$.

Let us now take into account that in the case under consideration, the condition $|\mathbf{P}| \to 0$ is satisfied in (2). Since the multiplier $\frac{4E}{3u^2}$ in (2) is a finite quantity, then we get that the ratio is valid here: $|\mathbf{V}| \to 0$, i.e., that the velocity of the phonon gas as a whole in helium II is vanishingly small in magnitude. This variant of solving the problem should be discarded, due to the fact that the phonon gas in this variant is practically at rest.

The second possibility is that in (1), at $M_\varepsilon \to 0$, the condition of finiteness in the magnitude of the momentum modulus $|\mathbf{P}|$ is simultaneously satisfied. As follows from (1), the value of the velocity **V** turns out to be infinite in this case, which indicates the inapplicability of formula (1) as a nonrelativistic relation.

The corresponding relativistic formula for the velocity **V** of the motion of a system of particles as a whole is given in [10] (formula (9.8)). It has the form:

$$\mathbf{V} = \frac{c^2}{E} \cdot \mathbf{P}. \tag{3}$$

Here: c – is the speed of light in a vacuum, $\mathrm{c} = 3 \cdot 10^{10}\ \mathrm{cm/s}$; the other designations are the same as above.

Thus, we conclude that the expression $\frac{4E}{3u^2}$ calculated in [3] for the effective mass of a unit volume of a phonon gas in helium II, considered as a particle gas, turns out to be valid only at a vanishingly low velocity of the phonon gas as a whole in helium II, which does not correspond to the basic provisions of two-fluid hydrodynamics.

If the velocity of the phonon gas as a whole in helium II is finite in magnitude, then the effective mass of the unit volume of the phonon gas is given by the relativistic expression $\frac{E}{c^2}$, which is approximately 10^{12} less than the above expression for the effective mass of the phonon gas from [3].

Let us now consider the problem of the effective mass of the phonon gas in helium II in terms of sound waves in a continuous medium. As is known, the dispersion law for a phonon in helium II has the following form:

$$\varepsilon = p \cdot u. \tag{4}$$

Here: ε – is the phonon energy; p – is the absolute value of the phonon momentum $\boldsymbol{p}$; u – is the absolute value of the speed of the first sound in helium II.

It is known, see [11], p. 335, that the phase and group velocities of long-wave phonons are equal to each other. In connection with this circumstance, the question arises as to which velocity (phase or group) appears in the right part of relation (4)?

Although this question has been mentioned in the literature, see [11] p. 374, we will give the relevant arguments. We use formulas expressing the energy ε and the absolute magnitude of the phonon momentum p, in terms of its circular frequency ω and the wave number k:

$$\varepsilon = \hbar \cdot \omega, \tag{5}$$

$$p = \hbar \cdot k, \tag{6}$$

(here: $\hbar$ – is Planck's constant, $\hbar = 1{,}054 \cdot 10^{-27}\,\mathrm{erg} \cdot \mathrm{s}$).

Therefore, relation (4) can be rewritten as:

$$\omega = k \cdot u. \tag{7}$$

Since the proportionality coefficient between the frequency ω and the wave number k of the wave is the phase velocity of the wave, it can be concluded from here that the velocity u, which is included in the dispersion relation (4), is exactly the phase velocity.

This circumstance can be understood as follows. Let's rewrite the relation (4) in vector form:

$$\boldsymbol{p} = \widetilde{m} \cdot \boldsymbol{u}. \tag{8}$$

Here: $\widetilde{m} \equiv \frac{\varepsilon}{u^2}$ – is the effective mass; $\boldsymbol{u}$ – is the phonon velocity vector.

If the velocity $\boldsymbol{u}$, which is included in the right part of relation (8), would not be a phase velocity, but a group velocity, then relation (8) would mean that when a single phonon moves in space, there would be a corresponding (real) displacement of some mass of matter equal in magnitude to $\widetilde{m} \equiv \frac{\varepsilon}{u^2}$.

Meanwhile, it is known that, in a linear approximation, no transfer of matter occurs during the propagation of a sound wave. The mass is carried only by strong, nonlinear (shock) waves. This contradiction is removed if the velocity $\boldsymbol{u}$, which is included in the right side of (8), is a phase velocity, since it is the phase velocity that does not correspond to any real movement of matter in space.

Let us now consider a system of N noninteracting phonons (per unit volume of helium II). We assume that each of the phonons has the same energy ε. Thus, we consider a monoenergetic ideal phonon gas. We assume that the directions of the phase velocities $\boldsymbol{u}_i$ of the phonons of a given gas are different, and such that the average phase velocity $\boldsymbol{V} \equiv \frac{\sum_i \boldsymbol{u}_i}{N}$ of the phonon system under consideration is a small value compared to the first speed of sound in helium II. Summing then the left and right parts of the ratios (8) recorded for each i - th phonon of the system, we obtain:

$$\mathbf{P} = \frac{E}{u^2} \cdot \mathbf{V}. \tag{9}$$

Here: $\boldsymbol{P}$ – is the total momentum of the unit volume of the phonon system $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$; $E = N \cdot \varepsilon$ – is the total energy per unit volume of the considered system of phonons.

From relation (9), we can conclude that the velocity $\mathbf{V}$, which is included in formula (2) (this formula is taken from [3]), differs only by a numerical factor of the order of unity from the average phase velocity of the system of sound waves that form the phonon gas.

Assuming in (9) that the directions of the phase velocities $\boldsymbol{u}_i$ of the phonons of the system remain unchanged, but that the energy ε of each of the phonons of the system changes (by the same amount), we can calculate the group velocity of the gas under consideration as a system of sound waves. The absolute value of $\frac{dE}{dP}$ of this group velocity will be given by the formula $\frac{u^2}{|\boldsymbol{V}|}$. Since we assume that $|\boldsymbol{V}| \ll u$, then this group velocity significantly exceeds the first speed of sound in helium II.

Similarly to the above, we believe that the value of the group velocity, which is obtained from the relation (2) (this formula is taken from [3]), will be in order equal to $\frac{u^2}{|\boldsymbol{V}|}$. Thus, the velocity $\frac{u^2}{|\boldsymbol{V}|}$ corresponding to the physically real transfer in the problem under consideration turns out to be incompatible with the two-fluid hydrodynamics of helium II. Indeed, the latter assumes that the velocity of the corresponding transfer is small, compared with the velocity of the first sound in helium II, see [3].

So, the formally correct expression of the existing theory of superfluidity, obtained in [3] for the momentum of a phonon gas in helium II, contains a value that differs only by a numerical factor of the order of one from the average phase velocity $\boldsymbol{V} \equiv \frac{\sum_i \boldsymbol{u}_i}{N}$ of the system of sound waves that form this phonon gas. (This velocity is interpreted as "the velocity of the phonon gas as a whole", or the so-called "average flow velocity").

For this reason, this expression does not describe the physically real transfer of the effective mass of the phonon gas in helium II. As noted above in this section, the significant excess of the group velocity of the phonon gas over the velocity of the first sound in helium II indicates the inapplicability of the nonrelativistic approach to calculating the velocity of motion as a whole, for a system of phonons (as a set of sound waves) in helium II.

Summing up, we emphasize once again that the "velocity of the phonon gas as a whole" (or, the so-called "average flow velocity"), introduced in [3], is a value that differs only by a numerical factor of the order of one from the average phase velocity of the system of sound waves forming this gas. Therefore, any real physical movement of mass in helium II does not correspond to this velocity of motion. The physically real displacement of the phonon gas mass in helium II is given by the relativistic formula (3). At the same time, the displaced mass of the phonon gas turns out to be approximately 10^{12} times less than the value given in [3].

On the velocity of motion as a whole gas of rotons in helium II

Consider helium II, which is at absolute zero temperature. In this case, helium is in its normal, unexcited state. Let this helium II flow through the capillary at a constant velocity $\boldsymbol{V}$. We will first consider the flow of the helium under study in the reference frame K_0, moving together with the liquid. In this coordinate system, helium II is at rest, and the capillary walls move at a velocity of $-\boldsymbol{V}$.

Let us assume that in considered helium II there appears one elementary excitation with momentum $\boldsymbol{p}$, energy $\varepsilon(p)$ and mass m. Then the energy E_0 of helium (in the coordinate system K_0, in which helium II initially at rest) will be equal to the energy of this excitation ε, and its momentum $\boldsymbol{P}_0$ – to the momentum $\boldsymbol{p}$. Let us now go back to the K_1 coordinate system, in which the capillary rests. According to formulas (8.1) and (8.5) given in [9] for the momentum and energy transformations we have for the momentum $\boldsymbol{P}_1$ and energy E_1 of the helium II in the K_1 system:

$$\boldsymbol{P}_1 = \boldsymbol{P}_0 + M \cdot \boldsymbol{V}, \tag{10}$$

$$E_1 = E_0 + \boldsymbol{P}_0 \cdot \boldsymbol{V} + \frac{M \cdot |\boldsymbol{V}|^2}{2}. \tag{11}$$

Here: M – is the total mass of helium II in the problem under consideration.

Let us vary the relations (10) and (11). Let's start with the relation (10). Since the total mass M of helium II, when an elementary excitation is born in it, remains unchanged, then $\delta(M \cdot \mathbf{V}) = 0$. Also note that $\delta(\boldsymbol{P}_0) = \boldsymbol{p}$.

As for the variation of the momentum $\boldsymbol{P}_1$, which is on the left side of the relation (10), it, in the case of excitation in helium II of a quasiparticle with mass m, is calculated in a more complex way. It should be taken into account that $\delta(\boldsymbol{P}_1)$ consists of two terms. Firstly, this is the momentum $\boldsymbol{p}_1$ of the considered massive quasiparticle in a fixed reference frame K_1 (this is the first term).

Secondly, this is a change in the momentum (in the K_1 reference frame) of the "unexcited background" in helium II at absolute zero temperature caused by the "withdrawal" of a quasiparticle with mass m from this "unexcited background" (this is the second term).

Let us take into account that before the excitation of a quasiparticle with mass m, from the "unperturbed background", the mass of the "unexcited background" coincided with the total mass M of helium II at absolute zero temperature.

After the excitation of a massive quasiparticle, the mass of the "unexcited background" will decrease by the mass of this quasiparticle, and will become equal to $M - m$. Accordingly, before the excitation of the massive quasiparticle, the momentum of the "unexcited background" in the stationary reference frame K_1, was equal to $M \cdot \boldsymbol{V}$, and after the excitation of the massive quasiparticle, the momentum of the "unexcited background", in the reference frame K_1, will become equal to $(M - m) \cdot \boldsymbol{V}$.

As a result, we get that:

$$\delta(\boldsymbol{P}_1) = \boldsymbol{p}_1 + [(M - m) \cdot \boldsymbol{V} - M \cdot \boldsymbol{V}] = \boldsymbol{p}_1 - m \cdot \boldsymbol{V}. \tag{12}$$

Then we have that the law of the transformation of the momentum of a quasiparticle with mass m in helium II, during the transition from a moving frame of reference K_0 to a stationary frame of reference K_1, has the form:

$$\boldsymbol{p}_1 = \boldsymbol{p} + m \cdot \boldsymbol{V}. \tag{13}$$

Quite similarly, by varying relation (11), one can obtain the law corresponding to (13) for the energy transformation of a massive quasiparticle:

$$\varepsilon_1 = \varepsilon + \boldsymbol{p} \cdot \boldsymbol{V} + \frac{m \cdot |\boldsymbol{V}|^2}{2}. \tag{14}$$

For a massless quasiparticle (phonon), the laws of momentum and energy transformation then have a already known form:

$$\boldsymbol{p}_1 = \boldsymbol{p}, \tag{15}$$

$$\varepsilon_1 = \varepsilon + \boldsymbol{p} \cdot \boldsymbol{V}. \tag{16}$$

In [3], a massive quasiparticle (roton) with a dispersion law of the form is considered:

$$\varepsilon = \Delta + \frac{p^2}{2m}. \tag{17}$$

Here: Δ and и m – are constants.
For such a quasiparticle, the transformation laws (13) and (14) are quite valid.

However, it should be taken into account that in the theory of superfluidity [3] of helium II, the flow velocity $\boldsymbol{V}$ actually appears, which is equal in magnitude to about $10\, cm/s$. In this regard, for a roton with the dispersion law (17), one can neglect the term $\frac{m|\boldsymbol{V}|^2}{2}$, in the law of the roton energy transformation (14), and the term $m \cdot \boldsymbol{V}$, in the law of the roton momentum transformation (13). Therefore, we can assume that for the roton from [3], with the dispersion law (17), the approximate laws of momentum and energy transformation, expressed by the relations (15) and (16), are satisfied.

The situation is completely different for a roton with the dispersion law

$$\varepsilon = \Delta + \frac{(p-p_0)^2}{2m^*}, \tag{18}$$

introduced in 1947 in [4]. Here: Δ, p_0, m^* – are constants.
Indeed, from the relation $\boldsymbol{v} = \frac{\partial\varepsilon}{\partial\boldsymbol{p}}$, written for a roton with the dispersion law (18), it follows that:

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{\frac{m^*}{\left(1-\frac{p_0}{p}\right)}}. \tag{19}$$

Here: $\boldsymbol{v}$ – the roton velocity.
This means that the effective mass m of the roton, introduced in 1947 in [4], is given by the expression:

$$m = \frac{m^*}{\left(1-\frac{p_0}{p}\right)}. \tag{20}$$

Then, introduced by L.D. Landau (see below), the criterion for the superfluidity of helium II, which follows from the relation (14), will have, for a roton with the dispersion law (18), the following form:

$$\varepsilon + \boldsymbol{p} \cdot \boldsymbol{V} + \frac{m^* \cdot |\boldsymbol{V}|^2}{2 \cdot \left(1-\frac{p_0}{p}\right)} = 0. \tag{21}$$

In (21), instead of the commonly used inequality sign, there is an equals sign. The issue of the equal sign in the criterion for superfluidity of helium II is considered in more detail in the next section of this article.

For values of p sufficiently close to p_0, but such that $p < p_0$, relation (21) will hold for any arbitrarily small velocity $\boldsymbol{V}$. This means that rotons with the dispersion law (18), whose momentum value lies sufficiently close to p_0, but $p < p_0$, must be immobile relative to the "unexcited background" in helium II. Otherwise, the superfluid flow of helium II would stop.

Assuming that the entire gas of rotons with the dispersion law (18) has the same velocity of movement of this gas as a whole, we conclude that the entire gas of rotons with the dispersion law (18), introduced in 1947 in [4], should rest as a whole relative to the "unexcited background" in helium II. The latter statement does not correspond to the conditions necessary to substantiate the two-fluid hydrodynamics of helium II.

On the criterion for superfluidity of helium II

When considering the birth of a single phonon over an "unexcited background" in helium II, a superfluidity criterion for liquid helium was obtained in [3]. This criterion is written as:

$$\varepsilon + \boldsymbol{p} \cdot \boldsymbol{V} < 0. \tag{22}$$

The presence of the inequality sign in (22) is motivated by the fact that when an elementary excitation appears in liquid helium, which was initially at absolute zero, the energy of the moving liquid should decrease.

Meanwhile, if the walls of the capillary through which liquid helium flows are sufficiently rigid, then only the kinetic energy of liquid helium will decrease when a single phonon is born in helium. As for the total energy of helium II, it will remain unchanged. Therefore, the superfluidity criterion (22) should be written in a slightly different form, namely:

$$\varepsilon + \boldsymbol{p} \cdot \boldsymbol{V} = 0. \quad (23)$$

Then, in order to derive from (23) a restriction on the shape of the dispersion curve which leads to the appearance of superfluidity in liquid helium, it turns out to be necessary to introduce an additional hypothesis that the direction of the momentum $\boldsymbol{p}$ of the phonon excited in helium is opposite (or close to the opposite direction) to the direction of the velocity $\boldsymbol{V}$ of the helium flow.

If such an additional hypothesis is not accepted, then, for example, with the direction of the phonon momentum $\boldsymbol{p}$ perpendicular to the velocity $\boldsymbol{V}$ of the liquid helium flow, the phenomenon of superfluidity can be observed not only for the curve $\varepsilon = \varepsilon(p)$ with the linear law of dispersion $\varepsilon = p \cdot u$, but even for the curve $\varepsilon = \varepsilon(p)$ with the quadratic dispersion law $\varepsilon = \frac{p^2}{2m}$.

Thus, if we do not accept an additional hypothesis about the direction of the momentum of a phonon excited in liquid helium, then the linear (at the origin of coordinates) nature of the dispersion curve $\varepsilon(p)$ will no longer be necessary for the existence of the of superfluidity phenomenon in helium II.

Conclusion

The "velocity of motion of a phonon gas as a whole" (or, the so-called "average flow velocity"), introduced in [3], is a value that differs only by a numerical factor of the order of one from the average phase velocity of the system of sound waves forming this gas. Therefore, this velocity of motion does not correspond to any real physical movement of the mass in helium II. The physically real displacement of the phonon gas mass in helium II is given by the relativistic formula (3). In this case, the displaced mass of the phonon gas turns out to be approximately 10^{12} times less than the value given in the existing theory.

From the corrected laws (13), (14) of the momentum and energy transformation of a massive quasiparticle in helium II (a roton with the law of dispersion introduced in 1947), it follows that rotons with $p < p_0$, located near the minimum of the dispersion curve $\varepsilon(p)$, must be at rest relative to the "unexcited background" in helium II. Assuming that entire gas of rotons with the dispersion law (18) has the same velocity of movement of this gas as a whole, we conclude that the entire gas of the rotons, with the law of dispersion (18), introduced in 1947 in [4], should be at rest as a whole relative to the "unexcited background" in helium II. The latter circumstance does not correspond to the conditions necessary to substantiate the two-fluid hydrodynamics of helium II.

In order to derive from the superfluidity criterion (23) a restriction on the shape of the dispersion curve $\varepsilon(p)$, which leads to the appearance of superfluidity in liquid helium II, it is necessary to introduce an additional hypothesis that the direction of the momentum $\boldsymbol{p}$ of the phonon excited in helium is opposite (or close to the opposite direction) to the direction of the velocity $\boldsymbol{V}$ of the helium flow. If this additional hypothesis is not accepted, then the linear (at the origin) nature of the dispersion curve $\varepsilon(p)$ will cease to be necessary for the existence of the superfluidity phenomenon in helium II.

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